NONSTANDARD ANALYSIS, FRACTAL PROPERTIES AND BROWNIAN MOTION

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Abstract

In this paper I explore a nonstandard formulation of Hausdorff dimension. By considering an adapted form of the counting measure formulation of Lebesgue measure, I prove a nonstandard version of Frostman's lemma and find that Hausdorff dimension can be computed through a counting argument rather than by taking the infimum of a sum of certain covers. This formulation is then applied to obtain a simple proof of the doubling of the dimension of certain sets under a Brownian motion.

Keywords: Frostman's lemma, Nonstandard Hausdorff dimension, Brownian motion

1 INTRODUCTION

Using Loeb measure theory, it is possible to construct Lebesgue or even Wiener measure as a hyperfinite counting measure. In this paper I explore an extension of the idea to Hausdorff measure, which yields a hyperfinite formulation of Hausdorff dimension. In cases where the problem of dimension can be interpreted as an equivalent problem on a hyperfinite time line, this can lead to a simple and intuitively satisfying proof. For instance, I shall later consider certain properties of Brownian motion, and present nonstandard proofs which are somewhat easier than the original, and also seem to obey certain statistical "rules of thumb".

I now provide a short overview of the necessary nonstandard analysis as well as the standard formulation of Hausdorff dimension, since the nonstandard version will follow the same style and notation.

2 AN INTRODUCTION TO NONSTANDARD ANALYSIS

Before defining Loeb measures, we briefly introduce the nonstandard universe in which we will be working. Nonstandard analysis was introduced in the 1960s by Abraham Robinson [1]. This exposition is largely based on the very clear monograph of Cutland [2]. Although Loeb measures are standard measures, their construction involves nonstandard analysis (NSA).

2.1 The hyperreals

We construct a real line ${}^*\mathbb{R}$ which is richer than the standard reals $\mathbb{R}$. This is an ordered field which extends the real numbers to include non-zero infinitesimals; that is, numbers the absolute value of which is smaller than any real number; and also positive and negative "infinite" numbers. We now make these notions precise.

There are several ways of constructing the extended universe. Here we use an ultrapower construction. An axiomatic approach is also possible, as for instance in [3]. We rather use the ultrapower because it is pertinent to later constructions, and seems somewhat more satisfactory in a fractal context.

Definition 2.1. *A free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ that is closed under finite intersections and supersets (i.e. $A \subseteq B$ and $A \in \mathcal{U}$ implies $B \in \mathcal{U}$), contains no finite sets and for every $A \subseteq \mathbb{N}$ has either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.*

Given such a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we construct ${}^*\mathbb{R}$ as an ultrapower of the reals

$$^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{U}.$$

The set ${}^*\mathbb{R}$ that we obtain therefore consists of equivalence classes of sequences of reals under the equivalence relation $\equiv_{\mathcal{U}}$, where

$$(a_n) \equiv_{\mathcal{U}} (b_n) \Leftrightarrow \{n : a_n = b_n\} \in \mathcal{U}.$$

The equivalence class of a sequence (a_n) is denoted by either $(a_n)_{\mathcal{U}}$ or, in the sequel, by $\langle a_n \rangle_{\mathcal{U}}$. These are what we refer to as the *hyperreals*. It is clear that ${}^*\mathbb{R}$ is then an extension of $\mathbb{R}$, the usual real numbers represented by equivalence classes of constant sequences.

The usual algebraic operations such as $+, \times, <$ are extended in the above way, but shall be denoted in the usual way. For example, given $x = \langle x_n \rangle_{\mathcal{U}}$ and $y = \langle y_n \rangle_{\mathcal{U}}$ in ${}^*\mathbb{R}$, we can say that $x < y$ if the set of $n \in \mathbb{N}$ on which $x_n < y_n$ is a member of the ultrafilter $\mathcal{U}$.

We usually distinguish three important classes of hyperreals. Intuitively, the *infinitesimals* are equivalence classes of sequences converging to 0 (and therefore smaller than any real number), *bounded* (or *finite*) hyperreals are equivalence classes of convergent sequences (including the infinitesimals), and the *hyperfinite* or *infinite* hyperreals are equivalence classes of sequences diverging to ∞ or $-\infty$ (and thus, in absolute value, larger than any real number).

There exists a function (which we'll discuss in more detail when considering nonstandard topology) from the finite hyperreals to $\mathbb{R}$ which associates to each nonstandard number an element of $\mathbb{R}$, called the *standard part* of the number, as expressed in the following theorem:

Theorem 2.1. *If $x \in {}^*\mathbb{R}$ is finite (that is, $-R < x < R$ for some standard $R \in \mathbb{R}$), then there is a unique $r \in \mathbb{R}$ such that $x \approx r$. Any finite hyperreal is thus expressible as $x = r + \delta$ with $r \in \mathbb{R}$ and δ infinitesimal.*

We call r in the above theorem the *standard part* of x and denote it as either ${}^\circ x$ or as $\operatorname{st}(x)$. Both are used, sometimes in conjunction, to improve readability.

For infinitesimals the standard part is clearly 0; indeed, this suffices as a definition of infinitesimal for nonzero numbers. We say that $x, y \in {}^*\mathbb{R}$ are infinitely close whenever $x - y$ is infinitesimal and denote it by $x \approx y$. Thus, $x \approx y$ if for every $\varepsilon > 0$ in $\mathbb{R}$, $|x - y| < \varepsilon$. The set of all such y which are infinitesimally close to x is called the *monad* of x.

Functions are also defined by pointwise operations. Given a function $f : \mathbb{R} \to \mathbb{R}$, a nonstandard counterpart of f is given by the function $F : {}^*\mathbb{R} \to {}^*\mathbb{R}$, defined by

$$F(\langle a_n \rangle_{\mathcal{U}}) = \langle f(a_n) \rangle_{\mathcal{U}}.$$

Note that this is applicable to the characteristic functions of sets as well, giving us a way to extend $A \subset \mathbb{R}$ to its nonstandard counterpart ${}^*A \in {}^*\mathbb{R}$. (The imbedding ${}^* : \mathbb{R} \to {}^*\mathbb{R}$ so obtained is a Boolean homomorphism.) If f is a real function defined on a set A, we call the function F defined on ${}^*A$ by the above a *lifting* of f, and denote it by ${}^*f$. Note that if $t \in \mathbb{R}$, then $f(t) = {}^\circ F(t)$, and if $\tau \in {}^*\mathbb{R}$ then ${}^*f(\tau) \approx F(\tau)$.

Exactly which properties of ${}^*\mathbb{R}$ are inherited from $\mathbb{R}$ is specified in the following theorem, a restricted version of the more general *transfer principle*:

Theorem 2.2. *Let φ be any first order statement. Then φ holds in $\mathbb{R}$ if and only if ${}^*\varphi$ holds in ${}^*\mathbb{R}$.*

A first order statement φ (or ${}^*\varphi$ in ${}^*\mathbb{R}$) is one referring to elements (fixed or variable) of $\mathbb{R}$ (respectively, ${}^*\mathbb{R}$) and to fixed functions and relations on $\mathbb{R}$ (respectively, ${}^*\mathbb{R}$), that uses the usual logical connectives *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$) and *not* ($\neg$). Quantification may be done over elements but not over relations or functions; i.e., $\forall x$, $\exists y$ are allowed, but $\forall f$, $\exists R$ are not. As an example, the density of the rationals in the reals can be written as

$$\forall x \forall y (x < y \rightarrow \exists z (z \in \mathbb{Q} \wedge (x < z < y))),$$

an expression meaning, "between every two reals is a rational". From the transfer principle we can therefore immediately conclude that the statement is true in ${}^*\mathbb{R}$, i.e. that the hyperrationals are dense in the hyperreals. The following theorem relates the concepts of convergence and being infinitely close.

Theorem 2.3. *Let (s_n) be a sequence of real numbers and let $l \in \mathbb{R}$. Then*

$$s_n \to l \text{ as } n \to \infty \iff {}^*s_K \approx l \text{ for all infinite } K \in {}^*\mathbb{N}.$$

Proof. [2] Suppose that $s_n \to l$ and let $K \in {}^*\mathbb{N}$ be a fixed infinite number. We must show, for all real $\varepsilon > 0$, that $|{}^*s_K - l| < \varepsilon$. From ordinary real analysis we know that there exists some $n_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}\,[n \geq n_0 \to |s_n - l| < \varepsilon].$$

According to the transfer principle, the following is true in ${}^*\mathbb{R}$:

$$\forall N \in {}^*\mathbb{N}\,[N \geq n_0 \to |{}^*s_N - l| \leq \varepsilon].$$

In particular, $|{}^*s_K - l| < \varepsilon$ as required.

Conversely, suppose that ${}^*s_K \approx l$ for all infinite $K \in {}^*\mathbb{N}$. For any given real $\varepsilon > 0$ we have

$$\exists K \in {}^*\mathbb{N}\,\forall N \in {}^*\mathbb{N}\,[N \geq K \to |{}^*s_N - l| < \varepsilon].$$

By transferring this "down" to $\mathbb{R}$, we get

$$\exists k \in \mathbb{N} \;\; \forall n \in \mathbb{N}\,[n \geq k \to |s_n - l| < \varepsilon].$$

This implies convergence to l. □

2.2 The nonstandard universe

The principles of the previous section can be used in a much broader context than just real analysis. Given any mathematical object $\mathcal{M}$ (whether it is a group, ring, vector space, etc.), we can construct a nonstandard version ${}^*\mathcal{M}$. We consider a somewhat more economical construction however, by starting with a working portion of the mathematical universe $\mathbb{S}$ and ending up with a ${}^*\mathbb{S}$ which will contain ${}^*\mathcal{M}$ for every $\mathcal{M} \in \mathbb{S}$. This has the added advantage of preserving some of the relations between structures through the more general transfer principle.

We start with the superstructure over $\mathbb{R}$, denoted by $\mathbb{S} = S(\mathbb{R})$. It is defined as follows:

$$\begin{aligned} S_0(\mathbb{R}) &= \mathbb{R} \\ S_{n+1}(\mathbb{R}) &= S_n(R) \cup \mathcal{P}(S_n(\mathbb{R})), \quad n \in \mathbb{N} \\ \mathbb{S} &= \bigcup_{n \in \mathbb{N}} S_n(\mathbb{R}). \end{aligned}$$

($\mathcal{P}(A)$ denotes the power set of the set A.)

If a larger (or simply different) universe is required, start the same process with a more suitable set than $\mathbb{R}$. However, one usually only needs the first few levels of this construction.

Next one must construct a mapping ${}^* : S(\mathbb{R}) \to S({}^*\mathbb{R})$ associating to an $\mathcal{M} \in \mathbb{S}$ a nonstandard extension ${}^*\mathcal{M} \in S({}^*\mathbb{R})$. The nonstandard universe can now be constructed by means of an ultrapower

$$\mathbb{S}^{\mathbb{N}}/\mathcal{U}.$$

This is somewhat more complicated to do than in the case of ${}^*\mathbb{R}$ and we do not go into details here. It is sufficient to consider the nonstandard universe as the set of objects

$${}^*\mathbb{S} = \{x : x \in {}^*\mathcal{M} \text{ for some } \mathcal{M} \in \mathbb{S}\}.$$

Sets in ${}^*\mathbb{S}$ are called *internal* sets. It should be noted that ${}^*\mathbb{S} \in S({}^*\mathbb{R})$, but that $S({}^*\mathbb{R})$ contains sets which are not internal.

We now also have a Transfer Principle which specifies which statements may be moved from one structure to the other (see [4], for instance). A *bounded quantifier statement* differs from a first-order statement only in that quantifiers must range over a fixed set - which may mean a set of functions or relations, not just elements of $\mathbb{S}$ (the functions and relations are of course merely elements of the superstructure). Thus, quantifiers like $\forall x \in A$ or $\exists y \in B$ are allowed, but not unbounded quantifiers such as $\forall x$ and $\exists y$. Note that boundedness of the quantifier is often implied in the exposition and is not always specifically indicated in the statement.

Theorem 2.4. *A bounded quantifier statement φ holds in $\mathbb{S}$ if and only if ${}^*\varphi$ holds in ${}^*\mathbb{S}$.*

We show as a first application that the concept of supremum transfers:

Proposition 2.5. *Every nonempty internal subset of ${}^*\mathbb{R}$ with an upper bound has a least upper bound.*

Proof. [2] The notation used in this proof refers back to our construction of the nonstandard universe. We express the fact that any nonempty internal subset has a least upper bound by the statement

$$\Phi({}^*\mathbb{R}, {}^*S_2(\mathbb{R})) = \forall A \in {}^*S_2(\mathbb{R})[A \neq \emptyset \wedge (\exists x \in {}^*\mathbb{R}\,(\forall y \in A(y < x))) \rightarrow \\ \exists z \in {}^*\mathbb{R}\,(\forall y \in A(y < x) \wedge \forall u \in {}^*\mathbb{R}\,\forall y \in A\,(y \leq u \rightarrow z \leq u))].$$

Since the condition $\Phi(\mathbb{R}, S_2(\mathbb{R}))$ is true in $S(\mathbb{R})$, the transferred condition is true in $S({}^*\mathbb{R})$. □

The Transfer Principle can after some consideration be seen to apply only to *internal* sets. For instance, the concept of supremum implies that each bounded set will have a least upper bound. However, $\mathbb{N}$ seen as a member of ${}^*\mathbb{R}$ is bounded by each element of ${}^*\mathbb{N} \setminus \mathbb{N}$, but has no supremum. It is therefore an *external* (i.e., non-internal) set. It is sufficient to think of internal sets as sets in ${}^*\mathbb{S}$ that are images under the homomorphism *.

Another important property of any nonstandard universe constructed as an ultrapower is that of $\aleph_1$-saturation:

Definition 2.2. *If $(A_m)_{m\in\mathbb{N}}$ is a countable decreasing sequence of nonempty internal sets, then $\cap_{m\in\mathbb{N}} A_m \neq \emptyset$.*

A useful reformulation of this is known as *countable comprehension*: Given any sequence $(A_n)_{n\in\mathbb{N}}$ of internal subsets of an internal set A, there is an *internal* sequence $(A_n)_{n\in{}^*\mathbb{N}}$ of subsets of A that extends the original sequence (that is, agrees with the original sequence on the standard natural numbers, and inherits any properties that can be phrased as a bounded quantifier statement). This property is used in the construction of Loeb measure. Two other important properties are those of *overflow* and *underflow*: the first states that an internal set containing arbitrarily large finite numbers must contain an infinite number, and the second that an internal set containing arbitrarily small infinite numbers must contain a finite number. (Considering the reciprocals of the numbers, one can state similar results for infinitesimals.)

We observe as well that the notion of the cardinality of a set transfers. For a finite set the cardinality function simply gives the number of elements of a set (more formally and generally, the cardinality of a set X is the least ordinal α such that there is a bijection between X and α). In the sets I will consider in the case of nonstandard Hausdorff dimension, all will be finite within the nonstandard context. One may therefore still intuitively regard the transferred cardinality function as giving the "size" of a set. The cardinality of a set X will be denoted by $|X|$ in both standard and nonstandard cases; which is meant should be clear.

2.3 Nonstandard topology

Since Brownian motion, and hence continuous functions, is later considered, a knowledge of nonstandard topology is required. Firstly, we see that the concept of being infinitely close, and therefore the idea of a *monad*, can be extended:

Definition 2.3. *Let (X, τ) be a topological space.*

(i) For $a \in X$ the monad of a is

$$monad(a) = \bigcap_{a \in U \in \tau} {}^*U.$$

(ii) For $x \in {}^*X$, we write $x \approx a$ if $x \in monad(a)$.

(iii) $x \in {}^*X$ is said to be nearstandard if $x \approx a$ for some $a \in X$.

(iv) For any $Y \subseteq {}^*X$, we denote the nearstandard points in Y by $ns(Y)$.

(v) $st(Y) = \{a \in X : x \approx a \text{ for some } x \in Y\}$ is called the standard part of Y (also denoted by ${}^\circ Y$).

The following result allows us to generalise the pointwise standard part mapping:

Proposition 2.6. *A topological space X is Hausdorff if and only if*

$$monad(a) \cap monad(b) = \emptyset \text{ for } a \neq b, \quad a, b \in X.$$

This means we can define the function

$$\mathrm{st} : \mathrm{ns}({}^*X) \to X$$

as

$$\mathrm{st}(x) = \text{ the unique } a \in X \text{ with } a \approx x.$$

The notation ${}^\circ x = \mathrm{st}(x)$ is again used interchangeably.

We mention some general topological results.

Proposition 2.7. *[2] Let (X, τ) be separable and Hausdorff. Suppose $Y \subseteq^* X$ is internal and $A \subseteq X$. Then*

(i) $st(Y)$ is closed,

(ii) if X is regular and $Y \subseteq ns({}^*X)$, then $st(Y)$ is compact,

(iii) $st({}^*A) = \overline{A}$ (closure of A),

(iv) if X is regular, then A is relatively compact iff ${}^*A \subseteq ns({}^*X)$.

Since we will be dealing almost exclusively with continuous functions, we should introduce corresponding notions in the nonstandard universe.

Definition 2.4. *Let Y be a subset of ${}^*X$ for some topological space X and let $F : {}^*X \to {}^*\mathbb{R}$ be an internal function. Then F is said to be* S-continuous on Y *if for all $x, y \in Y$ we have*

$$x \approx y \Rightarrow F(x) \approx F(y).$$

The following result allows us to switch from the one notion of continuity to another.

Theorem 2.8. *[2] If $F : {}^*\mathbb{R} \to {}^*\mathbb{R}$ is S-continuous on an interval ${}^*[a, b]$ for real a, b and $F(x)$ is finite for some $x \in^* [a, b]$, then the standard function defined in $[a, b]$ by*

$$f(t) =^\circ F(t)$$

is continuous and ${}^*f(\tau) \approx F(\tau)$ for all $\tau \in^* [a, b]$.

2.4 Loeb measure

A Loeb measure is a standard measure, but constructed from a nonstandard one. That is, the Loeb measure exists on a σ-algebra and obeys all the usual rules for a measure, e.g. countable additivity.

We start with a given internal set Ω and an algebra $\mathcal{A}$ of internal subsets of Ω. Let μ be a finitely additive finite internal measure on $\mathcal{A}$. Thus μ is a function from $\mathcal{A}$ to $^*[0,\infty)$ such that $\mu(\Omega) < \infty$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint $A, B \in \mathcal{A}$. (We focus only on bounded Loeb measures; infinite ones shall not concern us in the sequel). We can then define the mapping

$$^\circ\mu : \mathcal{A} \to [0,\infty)$$

by $^\circ\mu(A) = ^\circ(\mu(A))$. This is finitely additive and therefore $(\Omega, \mathcal{A}, ^\circ\mu)$ is a standard finitely additive measure space. This is usually not a measure, since $^\circ\mu$ is usually not σ-additive. We shall see shortly, however, that it is *almost* a measure. The following crucial theorem was proved by Loeb [5].

Theorem 2.9. *There is a unique σ-additive extension of $^\circ\mu$ to the σ-algebra $\sigma(A)$ generated by $\mathcal{A}$. The completion of this measure is the Loeb measure associated with μ, denoted by μ_L. The completion of $\sigma(A)$ is the Loeb σ-algebra, denoted by $L(\mathcal{A})$.*

The more straightforward proof depends on the notion of a Loeb null set:

Definition 2.5. *Let $B \subseteq \Omega$, where B is not necessarily internal. We call B a* Loeb null set *if for each real $\varepsilon > 0$ there is a set $A \in \mathcal{A}$ with $B \subseteq A$ and $\mu(A) < \varepsilon$.*

This allows us to make precise the notion that $\mathcal{A}$ is almost a σ-algebra.:

Lemma 2.10. *[2] Let $(A_n)_{n\in\mathbb{N}}$ be an increasing family of sets, with each $A_n \in \mathcal{A}$ and let $B = \bigcup_{n\in\mathbb{N}} A_n$. Then there is a set $A \in \mathcal{A}$ such that*

(i) $B \subseteq A$

(ii) $^\circ\mu(A) = \lim_{n\to\infty} {}^\circ\mu(A_n)$ *and*

(iii) $A \setminus B$ *is null*

Proof. Let $\alpha = \lim_{n\to\infty} {}^\circ\mu(A_n)$. For any finite n,

$$\mu(A_n) \leq^\circ \mu(A_n) + \frac{1}{n} \leq \alpha + \frac{1}{n}.$$

Let $(A_n)_{n\in^*\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$ extending the sequence $(A_n)_{n\in\mathbb{N}}$, possible by $\aleph_1$ saturation. The overflow principle then guarantees an infinite N such that

$$\mu(A_n) \leq \alpha + \frac{1}{N}.$$

If we now let $A = A_N$, (i) will hold because $A_n \subseteq A$ for each n. Also, $\mu(A_n) \leq \mu(A)$ for finite n, so $^\circ\mu(A_n) \leq \mu(A) \leq \alpha$ and therefore $^\circ\mu(A) = \alpha$. This gives (ii). For (iii), note that $A \setminus B \subseteq A \setminus A_n$ and $^\circ\mu(A \setminus A_n) =^\circ \mu(A) -^\circ (A_n) \to 0$. □

Thus $\mathcal{A}$ is a σ-algebra modulo null sets. We can now define the concepts *Loeb measurable* and *Loeb measure* exactly:

Definition 2.6. *(i) Let $B \subseteq \Omega$. We say that B is* Loeb measurable *if there is a set $A \in \mathcal{A}$ such that $A \triangle B$ (the symmetric difference of A and B) is Loeb null. The collection of all the Loeb measurable sets is denoted by $L(\mathcal{A})$. $L(\mathcal{A})$ is known as the* Loeb algebra.

(ii) For $B \in L(\mathcal{A})$ define

$$\mu_L(B) =^\circ\mu(A)$$

for any $A \in \mathcal{A}$ with $A \triangle B$ null. We call $\mu_L(B)$ the Loeb measure *of B.*

This brings us to the central theorem of Loeb measure theory [5].

Theorem 2.11. *$L(\mathcal{A})$ is a σ-algebra and μ_L is a complete σ-additive measure on $L(A)$.*

The measure space $\mathbf{\Omega} = (\Omega, L(\mathcal{A}), \mu_L)$ is called the Loeb space associated with $(\Omega, \mathcal{A}, \mu)$. If $\mu(\Omega) = 1$, we refer to $\mathbf{\Omega}$ as a *Loeb probability space.*

2.5 Loeb counting measure

I devote a short but separate section to the idea of counting measures, since they are prominent throughout the sequel, being useful in the construction of nonstandard Hausdorff dimension.

Let $\Omega = \{1, 2, \ldots, N\}$, where $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. The set Ω is internal. Define the counting probability ν on Ω by

$$\nu(A) = \frac{|A|}{N},$$

for $A \in^* \mathcal{P}(\Omega) = \mathcal{A}$. The cardinality function $|\cdot|$ transfers, so $|A|$ can be interpreted as denoting the number of elements in A. The Loeb counting measure ν_L is the completion of the extension to $\sigma(A)$ of the finitely additive measure ${}^\circ\nu$.

The following definition will be used repeatedly in the sequel:

Definition 2.7. *Fix $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $\triangle t = N^{-1}$. The hyperfinite time line for the interval $[0, 1]$ based on the infinitesimal $\triangle t$ is the set*

$$\mathbf{T} = \{0, \triangle t, 2\triangle t, 3\triangle t, \ldots, 1-\triangle t\}.$$

The following theorem provides an intuitive construction of Lebesgue measure.

Theorem 2.12. *Let ν_L be the Loeb counting measure on $\mathbf{T}$. Define*

(i) $\mathcal{M} = \{B \subseteq [0, 1] : st_{\mathbf{T}}^{-1}(B)$ is Loeb measurable$\}$, where $st_{\mathbf{T}}^{-1}(B) = \{t \in \mathbf{T} :^\circ t \in B\}$

(ii) $\lambda(B) = \nu_L(st_{\mathbf{T}}^{-1}(B))$ for $B \in \mathcal{M}$

Then $\mathcal{M}$ is the completion of the Borel sets $\mathcal{B}[0, 1]$ and λ is Lebesgue measure on $\mathcal{M}$.

A sketch of the proof can be found in [2]

3 HAUSDORFF DIMENSION

Given a compact set A on the unit interval (or any bounded subset of $\mathbb{R}$) and $\epsilon > 0$, consider all coverings of the set by open balls B_n of diameter smaller than or equal to ϵ. For each cover, form the sum

$$\sum_{n=0}^{\infty} \|B_n\|^\alpha,$$

where $\|\cdot\|$ denotes the diameter of a set (i.e., the maximum distance between any two points of the set). For each A and $\epsilon > 0$, take the infimum over all such sums, as $\{B_n\}$ ranges over all possible covers of A of diameter $\leq \epsilon$:

$$S_\alpha^\epsilon(A) = \inf_{\{B_n\}} \sum_n \|B_n\|^\alpha.$$

As ϵ decreases to 0, $S_\alpha^\epsilon(B)$ increases to a limit $\text{meas}_\alpha(A)$ (which might be infinite) which is called the α-Hausdorff measure of A, or the Hausdorff measure of A in dimension α (I will refer to this as just "the measure" when the context is clear). Since meas_α is σ-subadditive but otherwise satisfies the requirements of a measure, it is an outer measure.

Definition 3.1. *The* Hausdorff dimension*, dimA, of a compact set* $A \subseteq [0,1]$ *is the supremum of all the* $\alpha \in [0,1]$ *for which, for any cover* B *of* A*,* $meas_\alpha(B) = \infty$*. This is equal to the infimum of all* $\beta \in [0,1]$ *for which there exists a cover* C *of* A *such that* $meas_\alpha(B) = 0$*.*

To see that the supremum of the one set of values is indeed equal to the infimum of the other, let $0 < \alpha < \beta \leq 1$ and consider the following:

$$\sum_n \|B_n\|^\beta \leq \sup_n \|B_n\|^{\beta-\alpha} \sum_n \|B_n\|^\alpha.$$

Hence, if $\mathrm{meas}_\alpha(A) < \infty$, $\mathrm{meas}_\beta(A) = 0$, or equivalently, $\mathrm{meas}_\alpha(A) = \infty$ if $\mathrm{meas}_\beta(A) > 0$.

From Hausdorff's original paper [6] it may inferred that his intention was somewhat akin to some of the motivation behind the creation of nonstandard analysis (which shall soon be using in this context). In this paper, he states:

> In this way, the dimension becomes a sort of characteristic measure of graduality similar to the 'order' of convergence to zero, the 'strength' of convergence, and related concepts.

Although I work almost exclusively with compact sets in one (topological) dimension, it is possible to do so in any number of dimensions. The principles remain inviolate and the Hausdorff dimension of a set is the same whether we consider it as a subset of $\mathbb{R}$ or $\mathbb{R}^n$.

4 NONSTANDARD HAUSDORFF DIMENSION

In this section and those following I show that a formulation of Hausdorff measure as a nonstandard counting measure, similar to Loeb's formulation of Lebesgue measure, is possible and prove some well-known theorems using these nonstandard techniques. It turns out that some interesting dimensional properties of Brownian paths become quite easy to prove using hyperfinite counting arguments.

Before we start the proof, we need a nonstandard version of the following result [7]. For notational convenience, the diameter of a set is denoted by $\|\cdot\|$ and the finite cardinality function or its transfer by $|\cdot|$, although which is intended should be clear. The following definition is included for clarity.

Definition 4.1. *A Radon measure is a measure defined on a* σ*-algebra of Borel sets of a set* X *which is both inner regular (the measure of a set is the supremum of the measures of the compact sets contained therein) and locally finite (every point has a neighbourhood of finite measure).*

Theorem 4.1. *(Frostman's lemma) Let* A *be a compact subset of* $[0,1]$ *and* $\beta \in (0,1)$*. Then* $meas_\beta A > 0$ *if and only if there exists a nonzero Radon measure* μ *on* A *such that* $\mu(B) \leq C\|B\|^\beta$ *for each interval* $B \subseteq [0,1]$ *and some positive* C*.*

I will now prove a nonstandard analogue of Frostman's lemma, in which I use the hyperfinite time line, as introduced in Section 2.5. Instead of considering the time line as a set of points in ${}^*[0,1]$, it will be useful to regard it instead as a subdivision of the nonstandard real line between 0 and 1 into equal parts of infinitesimal size. The "time line based on the number N" indicates that these intervals have length N^{-1}. If we base the time line on a sequence $\{a_n\}$, this implies that ${}^*[0,1]$ is divided into $N = \langle a_n \rangle_{\mathcal{U}}$ equal intervals of the form $[k\,\Delta t, (k+1)\,\Delta t]$, $0 \leq k < N$.

(Note that I abuse the notation slightly in the following by using ${}^\circ\left(\frac{|A'|}{2^{N\alpha}}\right) > 0$ to mean either that the standard part of the expression in brackets exists and is larger than 0, or that the expression is infinite.)

Theorem 4.2. *Let* A *be a compact subset of* $[0,1]$*. Suppose* $\mathbf{T}$ *is a hyperfinite time line on* $[0,1]$*, based on the dyadic sequence* $\{2^n\}_{n\geq 1}$*, and* A' *is any internal subset of* $\mathbf{T}$*, such that its standard part is* A*. If* ${}^\circ\left(\frac{|A'|}{2^{N\alpha}}\right) > 0$*, there exists a nonstandard measure* μ *on* $\mathbf{T}$ *such that the (nonzero) Loeb measure* μ_L *associated to* μ *has the property that for an absolute constant* C *and an arbitrary interval* $B \subseteq [0,1]$*, it is true that* $\mu_L(B) \leq C\|B\|^\alpha$*.*

Proof. The measure in question is not quite as simple as, for instance, the counting measure used to generate Lebesgue measure. In this case we have to take into account how "close" elements of A are to each other and a uniform counting measure cannot provide that information. Thus the construction of the measure is not generic but will depend specifically on the nature of A.

We use a time line $\mathbf{T}$ based on the hyperfinite number 2^N, where $N = \langle 1, 2, 3, ... \rangle_{\mathcal{U}}$. We say a dyadic interval is of order m if it has length 2^{-m}. Let A' be any internal subset of $\mathbf{T}$ such that ${}^{\circ}A' = A$. On all intervals of order N, we distribute the mass $2^{-N\alpha}$ if the interval is also in A'. Clearly the desired inequality holds trivially for intervals of order N, but may not hold for any intervals of lower order. Because of the condition on the cardinality of A', we also have a total mass which is larger than some positive (standard) real number. We use this mass to normalise, dividing each interval of A' by this total. Hence the inequality continues to hold. We now consider order $N-1$. If the inequality continues to hold on such an interval, we leave it be, and it retains its original measure. If now, however, two intervals of order N are both contained in an interval of order $N-1$, the inequality will be violated, and we need to multiply each by a factor of $2^{\alpha}/2$. Once this is accomplished for all intervals of order $N-1$, we have that the inequality is satisfied for all these intervals and for those of order N (since their individual masses cannot increase during this procedure) and again obtain a total measure larger than some standard positive number. We normalise the measure thus obtained, and move on to the next level. At each subsequent level we refine the masses all the way up to the Nth level, so that the inequality will continue to hold. Since we adjust by at most a finite factor at each stage, and the total mass is larger than some real number, we never have to normalise by dividing by an infinitesimal. After N steps, we obtain a finite internal finitely additive nonstandard measure μ supported by A' for which there is an absolute (real) constant C such that the inequality $\mu(B) \leq C\|B\|^{\alpha}$ holds for dyadic intervals of any order.

(If we only have that $|A'|/2^{N\alpha} \approx 0$, this construction does not hold. Indeed, in the proofs of Theorems 4.2 and 4.3 it is shown (independently from this result) that in such a case no such probability will exist. This construction will fail in such a case because we require the number C in the statement of the theorem to be a standard real. If we normalise by dividing by an infinitesimal, the inequality will not hold for any real number at order N. If we normalise by dividing by an infinite number there is no such objection, since it will then hold for any real number C.)

In the same way as with Lebesgue measure, we now obtain a measure on the Borel subsets of $[0,1]$ by taking the standard part of μ and performing the necessary completions, obtaining the measure μ_L. The inequality on dyadic intervals in $[0,1]$ will then also hold for μ_L. An arbitrary interval D will always be contained in two such dyadic intervals and therefore

$$\mu_L(D) \leq C\|D\|^{\alpha}.$$

□

We prove the main result of this section in two separate theorems. The first guarantees the existence of a subset of a time line from which we can compute the dimension and the second shows that the choice of set is not very important. It is proved for subsets of $[0,1]$ only, but note that it can easily be extended to any compact interval and arbitrary (finite) dimension.

Theorem 4.3. *Given a compact subset A of $[0,1]$, there is a subset $A_{\mathbf{T}}$ on the hyperfinite time line $\mathbf{T}$ and a hyperfinite number $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that ${}^{\circ}A_{\mathbf{T}} = A$ and*

$$\begin{aligned} {}^{\circ}\left(\frac{|A_{\mathbf{T}}|}{N^{\beta}}\right) &= \infty \text{ for } \beta < \alpha \\ {}^{\circ}\left(\frac{|A_{\mathbf{T}}|}{N^{\beta}}\right) &= 0 \text{ for } \beta > \alpha \end{aligned}$$

if and only if $\dim A = \alpha$.

Proof. Suppose that $\beta < \dim A$. We know that the sum diverges to infinity as the sizes of the intervals decrease. Thus there will be some $N \in \mathbb{N}$ such that the β-Hausdorff sum will be larger than 1 for covers constituting of sets with diameter smaller than 2^{-K}, for all $K > N$.

We will now state, as a bounded quantifier statement, that this will hold for any cover and that such a cover always exists, a seemingly trivial point in the standard case, but not as obvious in the nonstandard.

Let $S = S(A, X, K, J)$ be the following statement, where $X \subset \mathbb{N} \times \mathbb{N}$:

$$S = \quad \forall x \in A \exists (i,j) \in X \left(x \in \left(\tfrac{i}{2^K}, \tfrac{j}{2^K} \right] \right) \wedge \forall (i,j) \in X \exists x \in A \left(x \in \left(\tfrac{i}{2^K}, \tfrac{j}{2^K} \right] \right)$$

$$\wedge \forall (i,j) \in X \left(2^{-K} \le (j-i) 2^{-K} \le 2^{-J} \right)$$

$$\wedge \left[(i,j) \in X \Rightarrow \neg \left(\exists k \in \{0, 1, \ldots, 2^K - 1\} ((j,k) \in X) \right) \right].$$

The statement S encapsulates the idea that there is a cover of A by intervals no smaller than 2^{-K}, such that no member of the cover is redundant (i.e., does not contain a member of A). What is more, S states that the largest interval is no larger than 2^{-J}, which we may assume because A is compact, and that no two intervals border in each other, because then the inequality $2 \cdot 2^{-K\beta} > (2 \cdot 2^{-K})^\beta$ implies that the Hausdorff sum may be decreased.

Let $T = T(X, K, \beta)$ be the statement

$$\sum_{(i,j) \in X} \left(\frac{j-i}{2^K} \right)^\beta > 1,$$

which captures the idea that since the Hausdorff sum diverges, it will eventually be larger than 1.

We then express $\beta < \dim A$ as:

$$\begin{gathered} [\exists N \in \mathbb{N} \forall J > N \forall K \ge J \forall X \subseteq \{0, 1, \ldots, 2^K - 1\} \times \{0, 1, \ldots, 2^K - 1\} \\ (S \Rightarrow T)] \quad \wedge \\ [\exists N \in \mathbb{N} \forall J > N \forall K \ge J \exists X \subseteq \{0, 1, \ldots, 2^K - 1\} \times \{0, 1, \ldots, 2^K - 1\} \\ (S \Rightarrow T)]. \end{gathered}$$

The first part of the above statement states that any cover (indexed by the set $X \subseteq \mathbb{N} \times \mathbb{N}$) of A which satisfies the conditions stipulated in S will yield a Hausdorff sum larger than 1, and the second part states that such a cover will indeed exist.

The transferred statement now reads as

$$\begin{gathered} [\exists N \in {}^*\mathbb{N} \forall J > N \forall K \ge J \forall X \subseteq \{0, 1, \ldots, 2^K - 1\} \times \{0, 1, \ldots, 2^K - 1\} \\ ({}^*S \Rightarrow {}^*T)] \quad \wedge \\ [\exists N \in {}^*\mathbb{N} \forall J > N \forall K \ge J \exists X \subseteq \{0, 1, \ldots, 2^K - 1\} \times \{0, 1, \ldots, 2^K - 1\} \\ ({}^*S \Rightarrow {}^*T)], \end{gathered}$$

where ${}^*S$ and ${}^*T$ are the transferred versions of the statements S and T. Note that this necessitates replacing only A with ${}^*A$ in the original S and T.

We now choose any sufficiently large $J \in {}^*\mathbb{N} \setminus \mathbb{N}$. The statement will still hold if we set $K = J$. This results in a "cover" of ${}^*A$ by intervals of diameter 2^{-K}. Set

$$A_{\mathbf{T}} = \left\{ \frac{j}{2^K} : (j-1, j) \in X \right\},$$

where X is the set the existence of which is guaranteed in the second line of the previous transferred statement.

By the transferred statement we know that $\sum_{(i,j) \in X} \left(\frac{j-i}{2^K} \right)^\beta > 1$, but $j - i = 1$ because of the choice of K — all the infinitesimal intervals are now of the same size. Also, $|A_{\mathbf{T}}| = |X|$; therefore $\frac{|A_{\mathbf{T}}|}{2^{K\beta}} > 1$. Thus,

$$\text{meas}_\beta A > 0 \Rightarrow \exists A_{\mathbf{T}} \subseteq \mathbf{T}, K \in {}^*\mathbb{N} \setminus \mathbb{N} \text{ such that } {}^\circ A_{\mathbf{T}} = A \text{ and } {}^\circ \left(\frac{|A_{\mathbf{T}}|}{2^{K\beta}} \right) > 0.$$

Since the converse holds by the nonstandard Frostman lemma, the theorem is proved. □

We now show that any set which satisfies certain of the above properties is rich enough to yield Hausdorff dimension.

Theorem 4.4. *Consider a hyperfinite time line* $\mathbf{T}$ *based on the infinitesimal* 2^{-N}*, for a given* $N \in {}^*\mathbb{N} \setminus \mathbb{N}$*. Suppose that an internal subset* A' *of the time line is such that* ${}^\circ(A') = A$ *and for some* $\alpha > 0$

$$\begin{aligned} {}^\circ\left(\frac{|A'|}{2^{N\beta}}\right) &> 0 \text{ for } \beta < \alpha \text{ and} && (4.1) \\ {}^\circ\left(\frac{|A'|}{2^{N\beta}}\right) &= 0 \text{ for } \beta > \alpha. && (4.2) \end{aligned}$$

Then $\alpha = dimA$*.*

Proof. Given (4.1), the nonstandard version of Frostman's lemma immediately implies that $\dim A \geq \alpha$. For the converse inequality, notice that the second condition implies that for each $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$,

$$\frac{|A'|}{2^{N\beta}} < \varepsilon,$$

which implies the following nonstandard statement for each positive $\varepsilon \in \mathbb{R}$:

$$\exists N \in {}^*\mathbb{N} \exists Y \subseteq \{0, 1, \ldots, 2^N - 1\} \forall x \in A' \exists i \in Y \left(x \in (i2^{-N}, (i+1)2^{-N}]\right) \wedge \left(\frac{|Y|}{2^{N\beta}} < \varepsilon\right).$$

The statement merely affirms the existence of an indexing set Y for intervals of length 2^{-N} which form a cover of A' and for which the term $|Y|2^{-N\beta}$ is smaller than any real number.

Transferring down to the standard case, we find that for each $\varepsilon > 0$,

$$\exists n \in \mathbb{N} \exists y \subseteq \{0, 1, \ldots, 2^n - 1\} \forall x \in {}^\circ A' \exists i \in y \left(x \in (i2^{-n}, (i+1)2^{-n}]\right) \wedge \left(\frac{|y|}{2^{n\beta}} < \varepsilon\right).$$

This implies that $\text{meas}_\beta A = 0$ and therefore that $\dim A \leq \alpha$. □

For computational purposes it is therefore enough to find a set in the time line with standard part A that satisfies the conditions in the above theorem. This fact will be used in subsequent sections.

In the sequel I refer to $|A_{\mathbf{T}}| \vartriangle t^\beta$ (where $\vartriangle t = 1/N$) as nonstandard β-Hausdorff measure and to $\text{meas}_\beta A$ as just β-Hausdorff measure.

Several of the properties of the standard β-Hausdorff measure can easily be seen to be valid in the nonstandard case, such as its outer measure properties, invariance under translation (and rotation, in the multidimensional case) and homogeneity of degree β with respect to dilation.

To illustrate some applications of this formulation, I first turn to the perennial example of a set of non-integer dimension, the triadic Cantor set. The "base-infinitesimal" of the construction is $\langle 1, 3^{-1}, 3^{-2}, \ldots, 3^{-k}, \ldots \rangle_{\mathcal{U}} = \vartriangle t = 1/N$. The cardinality of the NS Cantor set $|A_{\mathbf{T}}|$ is given by $\langle 1, 2/3, 4/9, \ldots, (2/3)^k, \ldots \rangle_{\mathcal{U}} N$. The NS β-Hausdorff measure of A is then given by

$$\begin{aligned} |A_{\mathbf{T}}| \vartriangle t^\beta &= \langle (2/3)^k \rangle_{\mathcal{U}} N \langle (1/3)^{k\beta} \rangle_{\mathcal{U}} \\ &= \langle (2/3^\beta)^k \rangle_{\mathcal{U}}, \end{aligned}$$

where I have used the obvious notation, $\langle a^k \rangle_{\mathcal{U}}$ instead of $\langle a, a^2, \ldots, a^k, \ldots \rangle_{\mathcal{U}}$. The above expression then has value 1 for $\beta = \log 2 / \log 3$, which is then $\dim A$ by our previous theorems. Since the standard β-Hausdorff sum for the triadic Cantor set is also 1 for $\beta = \dim A$, I suspect that the standard parts of the nonstandard sum will be equal to the standard sum at $\dim A$ for other sets as well. This remains to be proved.

4 THE FRACTAL GEOMETRY OF BROWNIAN MOTION

In this section I briefly discuss a nonstandard version of Brownian local time, level sets and the effect of a Brownian motion on a set with a given dimension. Although these results are not original, the proofs using a nonstandard version of Hausdorff dimension are very simple and intuitive. We start with a discussion on Brownian motion in the nonstandard context, with emphasis on Anderson's simple and beautiful construction [8].

4.1 Anderson's construction of Brownian motion

The idea is to construct Brownian motion as a hyperfinite random walk, instead of, as is often done, a limit of random walks. We start with a hyperfinite time line $\mathbf{T}$, based on a fixed $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. We let $\Omega = \{-1, +1\}^{\mathbf{T}}$. If $\omega \in \Omega$, we define the hyperfinite random walk as a polygonal path, filled in linearly between time points $t \in \mathbf{T}$ with $B(\omega, 0) = 0$ and

$$B(\omega, t + \triangle t) - B(\omega, t) = \triangle B(t) = \omega(s)\sqrt{\triangle t},$$

where $\omega(s) = \pm 1$. We let $\mathcal{C}_N$ be the set of all such paths, $\mathcal{A}_N = {}^*\mathcal{P}(\mathcal{C})_N$ and W_N the counting probability on $\mathcal{C}_N$ (where $\mathcal{P}(\mathcal{C})_N$ denotes the power set of $\mathcal{C}$. This gives us the internal probability space

$$(\mathcal{C}_N, \mathcal{A}_N, W_N)$$

which in turn gives us the Loeb space

$$\mathbf{\Omega} = (\mathcal{C}_N, L(\mathcal{A}_N), P_N = (W_N)_L).$$

The following theorem is due to Anderson [2]. Recall that an internal function F is S-continuous if, whenever arguments x and y are infinitesimally close, the corresponding function values $F(x)$ and $F(y)$ are infinitesimally close as well.

Theorem 4.1. 1. *For almost all $B \in \mathcal{C}_N$, B is S-continuous and gives a continuous path $b = {}^\circ B \in \mathcal{C}$.*

2. *For Borel $D \subseteq \mathcal{C}$, $W(D) = P_N(st^{-1}(D))$ is Wiener measure.*

3. *The following process is a Brownian motion on the space* $\mathbf{\Omega}$*:*

$$b(t, \omega) = {}^\circ B(w, t) : [0, 1] \times \Omega \to \mathbb{R}.$$

For a proof, as well as a nonstandard version of the central limit theorem, see [4].

4.2 Brownian local time

The local time of a Brownian motion is a measure of the time a Brownian motion spends at x, giving an indication of how many times the path returns to a certain value. The Lebesgue measure of this set is 0, but it can be described using Hausdorff measure, as we shall see shortly.

Heuristically, we define the local time $l(t, x)$ as

$$l(t, x) = \int_0^t \delta(x - b(s))ds,$$

where b is a Brownian motion and δ the delta function. A more precise definition and a discussion of local time can be found in [9]. The integral therefore "counts" how many times the Brownian path visits x up until the time t. The standard approach (which can be found in detail in, for example, [9]) is to show there exists a jointly continuous process $l(t, x)$ such that

$$l(t, x) = \frac{d}{dx} \int_0^t I_{(-\infty, x]}(b(s))ds,$$

for almost all $(t, x) \in [0, 1] \times \mathbb{R}$, where I_A is the characteristic function of the set A. Note that although the definition is valid for a time line $[0, \infty)$ as well as $[0, 1]$, I use a bounded interval throughout. The nonstandard approach, due to Perkins [10], is clearer and more intuitive. We think of the Brownian path b as the standard part of a hyperfinite random walk. The following exposition follows [4]. We start by approximating $l(t, x)$ by

$$(\triangle x)^{-1} \int_0^t I_{[x, x + \triangle x]}(b(s))ds.$$

Now replace the time line $[0,1]$ by a discrete hyperfinite time line $\mathbf{T}$ and the space $\mathbb{R}$ by $\Gamma = \{0, \pm\sqrt{\triangle t}, \ldots, \pm n\sqrt{\triangle t}, \ldots, \pm N\sqrt{\triangle t}\}$ and define the internal process $L : \mathbf{T} \times \Gamma \to {}^*\mathbb{R}$ by

$$L(t,x) = \sum_{s<t} I_x(B(s))(\triangle t)^{1/2},$$

where $I_x = I_{\{x\}}$. Perkins showed that $L(t,x)$ has a standard part which is Brownian local time. He used the nonstandard formulation to prove the following global characterisation of local time, which was previously known to hold only for each x separately: Let $\lambda(t,x,\delta)$ be the Lebesgue measure of the set of points within a distance of $\delta/2$ of $\{s \leq t | b(s) = x\}$. Then for almost all $\omega \in \Omega$ and each $t_0 > 0$,

$$\lim_{\delta \to 0^+} \sup_{t \leq t_0, x \in \mathbb{R}} |\lambda(t,x,\delta)\delta^{-1/2} - 2(2/\pi)^{1/2} l(t,x)| = 0.$$

It is shown in [9] that local time for $t = 1$ is the same as $\frac{1}{2}$-dimensional Hausdorff measure on a level set of Brownian motion; that is, for fixed x, $l(1,x) = \text{meas}_{\frac{1}{2}}(b^{-1}(x))$. From the nonstandard formulation, however, it is immediately clear. If we define the set A as the set of all $t \in [0,1]$ such that $b(t) = x$, the nonstandard local time becomes simply $|A_{\mathbf{T}}| \triangle t^{1/2}$, where $A_{\mathbf{T}}$ is the nonstandard version of the set A encountered in the proof of Theorem 3.3. But this is exactly the quantity whose standard part is the same as $\frac{1}{2}$-dimensional Hausdorff measure (up to a finite constant factor — which depends on which author one reads). We must now show that level sets have dimension $1/2$. We show this for $x = 0$ only, since the level sets all have similar dimension. Denote the zero set of a Brownian path $b(\omega)$ by Z_ω (or just Z when possible). The required nonstandard version of the set is denoted by $Z_{\omega,\mathbf{T}}$. We now turn to a standard property of local time to show that the dimension of this set is $1/2$. It can be shown (as for instance in [9]) that local time is identical in law to the function

$$M_\omega(t) = \max_{s \leq t} b(s).$$

This implies that $P[l(1,0) > 0] = P[\max_{s \leq 1} b(s) > 0] = 1$. By the nonstandard formulation of local time, this immediately implies that ${}^\circ(Z_{\omega,\mathbf{T}}) > 0$, which implies that $\dim Z \leq 1/2$, almost surely. By the same token, however, $l(1,0)$ is almost certainly finite, implying that ${}^\circ(Z_{\omega,\mathbf{T}}) < \infty$ and therefore $\dim(Z) \geq 1/2$.

The following lemma will be used in the subsequent section. In this case the standard approach is easier than the hyperfinite, by using the Hölder condition for Brownian motion. The proof is akin to the proof of the dimension of the level set found in [11]. We will use the fact that $Y(t) = M(t) - b(t)$ (where M is as defined above and b is a standard Brownian motion) has the same distribution as b [12]. Note also that the zeroes of Y correspond to the global maxima (from the left) of b; these are known as *record times*.

Lemma 4.2. *If Z is a level set and $A \subseteq Z$, then either D has dimension $1/2$ or $\dim Z/A = 1/2$.*

Proof. Since $M(t)$ is an increasing function, it can be considered to be the distribution of some measure μ, which has its support on the set of record times. Let Z be the zeroset for Y. (Because of the similar distributions, dimensional results for this set will hold for any Brownian level set.) We therefore have that $\mu(a,b] = M(b) - M(a)$. By the Hölder condition for Brownian motion we get

$$M(b) - M(a) \leq \max_{0 \leq h \leq b-a} b(a+h) - b(a) \leq C_\alpha (b-a)^\alpha$$

for some constant C_α, for all $\alpha < 1/2$.

Consider now a subset A of Z and let μ' be the restriction of the measure μ to A; that is,

$$\mu'(a,b] = \mu\{(a,b] \cap A\}.$$

Suppose that μ' is not 0 on every interval. Then,

$$0 < \mu'(a,b] \leq M(b) - M(a) \leq C_\alpha(b-a),$$

as above, for some $(a,b]$. By normalising the measure μ' we find a measure ν such that $\nu(a,b] \leq D_\alpha(b-a)$ for some D_α, for each $0 < \alpha < 1/2$. By Frostman's lemma, $\dim A \geq 1/2$.

If there exists no interval $(a,b]$ such that $\mu'(a,b] > 0$, then we can safely leave out A without changing the Hausdorff dimension of Z, as in [11], and $\dim Z/A = 1/2$. $\square$

Corollary 4.3. *If* $\dim A < 1/2$*, then the inverse image of any element in* $B(A)$ *(where* B *is a Brownian motion) has dimension* 0.

4.3 The image of a set under Brownian motion

A very interesting property of Brownian motion is its effect on sets of a certain Hausdorff dimension. If a compact subset of $[0,1]$ has dimension $\alpha < 1/2$, its image under Brownian motion is a set of dimension 2α. A set of dimension $\alpha > 1/2$ will have dimension 1 and will almost surely contain an interval. As for sets of dimension $1/2$, we have seen above that they may have an image of dimension 0. No hard and fast rule exists for such sets. We now look at nonstandard proofs of these results. The advantage of this approach is a more intuitive (counting) argument. A Fourier analytical proof of the following can be found in [13].

Theorem 4.4. *Let* $A \subset [0,1]$ *be a compact set. If* $dim A = \alpha < 1/2$ *and* b *is a Brownian motion,* $dim\, b(A) = 2\alpha$.

Proof. The basis for the time line of the image is no longer $\triangle t$, but $\sqrt{\triangle t}$. Let B denote a nonstandard Brownian motion which has b as standard part. We let $A_{\mathbf{T}}$ be the nonstandard counterpart of A constructed in Theorem 3.3. Since $|B(A_T)| \leq |A_T|$ and we know that $|A_T| \triangle t^\beta \approx 0$ for $\beta > \alpha$, we will have that $|B(A_T)| \triangle t^\beta \approx 0$ for any $\beta > \alpha$. Therefore, $|B(A_T)|(\sqrt{\triangle t})^\gamma \approx 0$ for $\gamma > 2\alpha$ and we conclude that $\dim b(A) \leq 2 \dim A$ by Theorem 3.4, since ${}^\circ B(A_{\mathbf{T}}) = b(A)$ (because of the S-continuity of the functions involved). It is left to show that $\dim b(A) \geq 2 \dim A$. This is not quite as simple as the previous proof, since the matter of possible level sets complicates the question of the cardinality of the image. We overcome this by considering only one element of each level set and discarding the rest. The remaining set will have the same dimension as the original and the image will have the same cardinality. This is made possible because the set A has a dimension of less than $1/2$. Any subsets of level sets in A are small enough to be left out (mostly) without affecting the dimension (see Lemma 4.2). For any $x \in b(A)$, pick one representative $x' \in B(A_{\mathbf{T}})$ such that ${}^\circ x' = x$ and some $t \in A_{\mathbf{T}}$ such that $B(t) = x'$. Let X' be the set of all such representatives x'. We denote by $L_{x',\mathbf{T}}$ the subset of $A_{\mathbf{T}}$ for which ${}^\circ B(L_{x',\mathbf{T}}) = x$.We want to show now that the standard parts of the sums

$$\sum_{x' \in X'} \frac{1}{N^\alpha}, \quad \sum_{x' \in X'} \frac{|L_{x',\mathbf{T}}|}{N^\alpha}$$

are 0 and ∞ for the same values of α. To do this, all that is necessary is to show that the first one is infinite whenever the second one is. So suppose that

$${}^\circ\left(\sum_{x' \in X'} \frac{|L_{x',\mathbf{T}}|}{N^\alpha} \right) = \infty.$$

We know that

$$\frac{|L_{x',\mathbf{T}}|}{N^\alpha} = s_x^\beta \approx 0$$

for any $\beta > 0$. This implies that

$$\sum_{x' \in X'} \frac{s_x^\beta N^\beta}{N^\alpha} = \sum_{x' \in X'} \frac{s_x^\beta}{N^{\alpha-\beta}} \leq \sum_{x' \in X'} \frac{1}{N^{\alpha-\beta}} = \infty,$$

for β arbitrarily close to 0. Thus we may conclude that the number of level sets is important and not the cardinality of each. But the number of level sets is equal to the cardinality of the range, thus the standard parts of

$$\frac{|B(A_T)|}{N^\alpha} \text{ and } \frac{|A_T|}{N^\alpha}$$

are 0 and ∞ for the same values of α. Keeping in mind that the time line of the image is based on $\sqrt{\triangle t}$ and not $\triangle t$, we can conclude that the dimension has doubled. □

As has been shown, this formulation of Hausdorff dimension proves to be useful in studying the fractal properties of Brownian motion, according to Anderson's construction. One may speculate that it might also aid in other contexts where the phenomenon in question has a satisfactory formulation in terms of the hyperfinite time line. Specifically, the approach given here is based on a *counting measure* approach and may therefore apply to systems where the measure used may be constructed as a Loeb measure. (For more interesting applications of Loeb measure to stochastic fluid mechanics, stochastic calculus of variations and mathematical finance theory, see [2].) The method has already been used in the study of the points of rapid growth of Brownian motion, as well as those of so-called complex oscillations [14], a constructive version of Brownian motion, to be presented in a forthcoming paper by the author.

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