

THE DIFFERENTIAL GEOMETRY OF THE  
FIBRES OF AN ALMOST CONTACT METRIC  
SUBMERSION

by

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submitted in accordance with the requirements  
for the degree of

**DOCTOR OF PHILOSOPHY**

in the subject

**MATHEMATICS**

at the

UNIVERSITY OF SOUTH AFRICA

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OCTOBER 2013

To  
Thérèse LUSAMBA TSHIMUANGA,  
my wife,  
and all our children in love.

# Acknowledgements

I have to confess that this thesis could not be finalized without the collaboration of some other persons.

Firstly, I would like to express my heartfelt thanks to Prof. T. A. BATUBENGE (University of South Africa), my Supervisor. He gave me, with all his best, an invaluable environment of scientific research, funding, a lot of encouragement and many times visited me in Kananga, at my fixed Institution for profound discussions. The finalization of this thesis is primarily the realization of his long dream to bring me to this high level of science. I remain deeply indebted to him.

Prof. F. MASSAMBA (University of KwaZulu-Natal), my Co-Supervisor has never hesitated to give me his remarkable counsels. From time to time he was available with an attentive ear and a vigilant eye on all of my writings; always stimulating me to pursue this work. His publications on totally contact umbilical submanifolds have been of a considerable help for this study of the fibres. I am grateful to him.

The Staff Members of the Department of Mathematical Science of UNISA (University of South Africa) were very hospitable to my person during the preparation of this work. I would like to express to each of them my sincere thanks.

To anyone who contributed directly or indirectly to the realization of this thesis, I express my gratitude.

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# Summary

Almost contact metric submersions constitute a class of Riemannian submersions whose total space is an almost contact metric manifold. Regarding the base space, two types are studied. Submersions of type *I* are those whose base space is an almost contact metric manifold while, when the base space is an almost Hermitian manifold, then the submersion is said to be of type *II*.

After recalling the known notions and fundamental properties to be used in the sequel, relationships between the structure of the fibres with that of the total space are established. When the fibres are almost Hermitian manifolds, which occur in the case of a type *I* submersions, we determine the classes of submersions whose fibres are Kählerian, almost Kählerian, nearly Kählerian, quasi Kählerian, locally conformal (almost) Kählerian,  $G_i$ -manifolds and so on. This can be viewed as a classification of submersions of type *I* based upon the structure of the fibres.

Concerning the fibres of a type *II* submersions, which are almost contact metric manifolds, we discuss how they inherit the structure of the total space.

Considering the curvature property on the total space, we determine its corresponding on the fibres in the case of a type *I* submersions. For instance, the cosymplectic curvature property on the total space corresponds to the Kähler identity on the fibres. Similar results are obtained for Sasakian and Kenmotsu curvature properties.

After producing the classes of submersions with minimal, superminimal or umbilical fibres, their impacts on the total or the base space are established. The minimality of the fibres facilitates the transference of the structure from the total to the base space. Similarly, the superminimality of the fibres facilitates the transference of the structure from the base to the total space. Also, it is shown to be a way to study the integrability of the horizontal distribution.

Totally contact umbilicity of the fibres leads to the asymptotic directions on the total space.

Submersions of contact CR-submanifolds of quasi- $K$ -cosymplectic and quasi-Kenmotsu manifolds are studied. Certain distributions of the under consideration submersions induce the CR-product on the total space.

**Key terms:** Differential Geometry, Riemannian submersions, almost contact metric submersions, CR-submersions, contact CR-submanifolds, almost contact metric manifolds, almost Hermitian manifolds, Riemannian curvature tensor, holomorphic sectional curvature, minimal fibres, superminimal fibres, umbilicity.

# Introduction

The theory of Riemannian submersions has been initiated by O'Neill [36] who wanted to relate the curvature tensor of the total space with that of the base space and the fibres. Replacing the total and the base space by almost Hermitian manifolds, Watson [52] studied almost Hermitian submersions.

Considering that both the total and the base space are almost contact metric manifolds, D. Chinea, [11]-[12] initiated the study of almost contact metric submersions. Independently from Chinea, Watson [52], also studied two types of such a class of submersions. Submersions of type *I* being those whose base space is an almost contact metric manifold while those of type *II* have almost Hermitian manifolds as base space. Almost contact metric submersions continue to fascinate a number of differential geometers. For instance, we can cite [14], [25] and [26] among many others.

By a Riemannian submersion, one understands a submersion

$$F \longrightarrow M \xrightarrow{\pi} M'$$

between two Riemannian manifolds such that  $\pi_{*|(Ker\pi_*)^\perp}$  is a linear isometry [36]. In this definition,  $M$  and  $M'$  are called the total and the base space respectively,  $F$  denoting the fibres of the submersion, which are the closed embedded submanifolds of the total space.

The main purpose of this thesis is to interrelate the geometry of the fibres with that of the total and the base space. In other words, we would like to know how the properties on the fibres are related, the total and the base space. In a natural way, the first question should be

*“What geometric properties can be induced on the fibres of an almost contact metric submersion?”*

As a first tentative of responses, we say that the fibres can have the same structure or a structure other than that of the total space. The above response leads to the following series of research directions:

Suppose that the fibres have the same structure as that of the total space. In what class can they lie? This question is of interest in the sense that the fibres of an almost contact metric submersion of type *I* are almost Hermitian manifolds. It is well-known that there are 4.096 classes of almost contact metric structures and 16 classes of almost

Hermitian ones (see [15] and [22] for more details). Thus, how can these structures be interrelated?

If the fibres have another structure than that of the total space. What kind of properties do they have to influence the geometry of the total or the base space? As responses to this question, the fibres can be minimal, superminimal or umbilical. From the responses to this question, other questions then occur such as:

- (1) Under what conditions the fibres can be minimal, superminimal or umbilical?
- (2) If the fibres are minimal, superminimal or umbilical, what implications do they have on the geometry of the total or of the base space?

Among the properties to be considered, we have settled the structure of the total space and the curvature property in the one hand, the minimality, superminimality and umbilicity on the other hand.

To this end, the thesis is organized as follows.

Chapter 1 is devoted to the background notions on almost Hermitian and almost contact metric manifolds to be used throughout the thesis. Examples of almost contact metric manifolds are given. Using the warped product, some illustrations of the various classes of Kenmotsu structures are constructed. Let  $M^{2m+1}$  be an almost contact metric manifold, it is shown that the direct product  $\mathbb{R}^2 \times M^{2m+1}$  is a way to characterize the Kenmotsu structure on  $\mathbb{R}^{2m+1}$ .

In Chapter 2, we shall recall some fundamental properties of almost contact metric submersions, then we shall determine the structure of the fibres according to that of the total space. We have to know how to interrelate these structures. For instance, we have proved that if the total space is endowed with a quasi Sasakian, a cosymplectic, a Kenmotsu, a  $C_8$ ,  $C_9$ ,  $C_{10}$ ,  $C_{11}$  or a  $C_{12}$ -structure of a type  $I$  almost contact metric submersion, then the fibres are endowed with the Kähler structure. Many similar results are obtained with other almost contact metric structures on the total space.

Chapter 3 is devoted to the curvature relation. The main objective of this work is to establish relationships between properties of the fibres with those of the total and the base space. It seems interesting to examine the curvature property for the following reason.

Let  $\alpha$  be a real number. In [28], Janssens and Vanhecke have defined the  $C(\alpha)$ -curvature tensors on almost contact metric manifolds while  $K_i$ -curvature properties are already studied on almost Hermitian manifolds. Then the following problem intertwines the geometry of the fibres with that of the total space.

*Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type  $I$ . Suppose the total space satisfying a  $C(\alpha)$ -curvature property. What is the corresponding curvature property on the base space and what kind of  $K_i$ -identity do have the fibres?*

In order to obtain the desired interrelations, the  $\varphi$ -linearity of the O'Neill configuration tensors  $T$  and  $A$  appears to be an important tool. Then, in Section 3.1, we have

determined the defining relations of some almost contact metric structures for which the  $\varphi$ -linearity of  $T$  and  $A$  is obtained. Also, we have settled the classes of almost contact metric structures which satisfy the cosymplectic curvature property. Then, we have shown that, for such classes, the cosymplectic curvature property on the total space transfers to the base space while the fibres have, as corresponding, the Kähler identity for a type  $I$  submersion. Under supplementary conditions on the configuration tensors  $T$  and  $A$ , we have obtained that the Kenmotsu and Sasakian curvature properties are related to the  $K_2$  and  $K_3$ -curvature identities respectively.

Chapter 4 deals with the notions of minimality, superminimality and umbilicity of the fibres. We first fixed the classes of submersions enjoying these properties, and carried on establishing their implications on the total or the base space. For instance, in the case of the minimality, it is shown that if the configuration tensor  $T$  is  $\varphi$ -linear in the second variable, which means that  $T_U\varphi V = \varphi T_U V$ , then the fibres are minimal. As a consequence of minimality, it is proved that if the total space of a type  $I$  submersion is defined by the codifferential of the fundamental 2-form  $\phi$  or that of the contact 1-form,  $\eta$ , then the base space inherits the structure of the total space if and only if the fibres are minimal.

Concerning the superminimality and among many other results, we have:

*Assume that the base space of an almost contact metric submersion of type  $I$  is  $G_1$ -Sasakian,  $G_1$ -semi-Sasakian,  $G_1$ -semi cosymplectic,  $G_1$ -Kenmotsu,  $G_1$ -semi Kenmotsu, and that the fibres are superminimal. Then the total space inherits the structure of the base space.*

Furthermore, the property of being superminimal occurs only in the case of a type  $I$  submersion. If the fibres of a type  $II$  submersion are superminimal, then the total space is either cosymplectic or an almost- $K$ -contact manifold.

Concerning the case of umbilicity, we have

*Let  $\pi : M^{2m+1} \longrightarrow B^{2m'}$  be an almost contact metric submersion of type  $II$ . Then the following statements are true.*

- (i) *If the fibres are totally contact umbilic, then the characteristic vector field  $\xi$  of the total space defines an asymptotic direction.*
- (ii) *Suppose that the total space is equipped with the  $K$ -contact structure. If the fibres are totally umbilic, then they are totally geodesic.*

In Chapter 5, we have extended the study to the case where the total space is a contact CR-submanifold of an almost contact metric manifold. Following [27], who studied the case where the total space is nearly trans-Sasakian manifold, we have focused our study on quasi- $K$ -cosymplectic and quasi-Kenmotsu manifolds; some remarkable distributions  $\mathcal{D}^\perp$ , vertical and  $\mathcal{D} \oplus \{\xi\}$ , horizontal are of interest and the following results established.

*\* Under a certain condition, the base space is a  $(1, 2)$ -symplectic manifold.*

\* *Integrability and parallelism of the distributions under consideration induce the property of CR-product on the total space.*

Considering that the fibres of a submersion give rise to a foliation, the following is of interest.

*If  $\pi : M \longrightarrow M'$  is a submersion of a contact CR-submanifold of a quasi-Kenmotsu manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$  such that the holomorphic sectional curvatures  $\overline{H}$  and  $H^*$  of  $\overline{M}$  and  $M'$ , respectively, coincide on  $\mathcal{D} \oplus \{\xi\}$ . Then,  $M$  is locally a product  $M^* \times C$ , where  $M^*$  is a totally geodesic leaf of  $\mathcal{D} \oplus \{\xi\}$  and  $C$  is a curve tangent to the distribution  $\mathcal{D}^\perp$ .*

We end our study with a conclusion where some new directions for future research are suggested.

# Chapter 1

## Preliminaries

In this chapter, we recall fundamental notions on manifolds which will be used in the sequel.

Concerning the manifolds, the aim is not to discuss their geometric or topological properties. We just present their defining relations in order to be used as fibres, as total or as base space of fibration. Noting by  $K(D, \xi)$  the holomorphic  $\varphi$ -sectional curvature of a 2-plan containing the characteristic vector field  $\xi$ , it is known that almost contact metric manifolds are grouped into three classes, namely,

- (1) the Sasakian type if  $K(D, \xi) \geq 0$ ;
- (2) the cosymplectic type if  $K(D, \xi) = 0$ ;
- (3) the Kenmotsu type if  $K(D, \xi) \leq 0$ .

We shall begin by examine the case of almost Hermitian before looking at the almost contact metric structures.

Next, three diagrams of strict inclusions of the main known structures are drawn. Furthermore, examples of these structures are constructed with the emphasis on the Kenmotsu case.

Finally, the product of Riemannian manifolds, which prepare examples of two types of almost contact metric submersions are treated.

### 1.1 Almost Hermitian manifolds

By an almost Hermitian manifold, one understands a Riemannian manifold,  $(M, g)$ , of even dimension  $2m$ , furnished with a tensor field  $J$ , of type  $(1, 1)$  satisfying the following two conditions:

- (i)  $J^2D = -D$ , and

(ii)  $g(JD, JE) = g(D, E)$ , for all  $D, E \in \Gamma(M)$ .

The tensor field  $J$  is called almost complex structure. A differentiable manifold, equipped with an almost complex structure is called an almost complex manifold. The above condition (ii) means that the Riemannian metric  $g$  is compatible with the almost complex structure  $J$ . In this case,  $g$  is an almost Hermitian metric. Then,  $(M^{2m}, g, J)$ , is an almost Hermitian manifold.

Any almost Hermitian manifold admits a differential 2-form,  $\Omega$ , called the fundamental form or the Kähler form, defined by

$$\Omega(D, E) = g(D, JE).$$

A local  $J$ -basis of an open subset of  $M$  is

$$\{E_1, \dots, E_m, JE_1, \dots, JE_m\}.$$

Extending the Levi-Civita connection  $\nabla$  to all tensorial algebra of  $M$ , one obtains many tensor fields such as  $\nabla_D J$ ,  $\nabla_D \Omega$  and so on which occur in the defining relations of various classes of almost Hermitian manifolds obtained by Gray and Hervella in [22].

Let us recall some remarkable identities obtained by the use of the following known Koszul formula

$$2g(\nabla_E G, D) = E.g(G, D) + Gg(D, E) - Dg(E, G) - g(E, [G, D]) + g(G, [D, E]) + g(D, [E, G]), \quad (1.1)$$

$$(\nabla_D J)E = \nabla_D JE - J\nabla_D E, \quad (1.2)$$

$$(\nabla_D \Omega)(E, G) = -g((\nabla_D J)E, G) = g(E, (\nabla_D J)G), \quad (1.3)$$

$$3d\Omega(D, E, G) = \mathcal{G}\{(\nabla_D \Omega)(E, G)\}, \quad (1.4)$$

$$(\nabla_D \Omega)(E, G) = (\nabla_D \Omega)(JE, JG), \quad (1.5)$$

where  $\mathcal{G}$  denotes the cyclic sum over  $D$ ,  $E$  and  $G$ .

The codifferential,  $\delta$ , of  $\Omega$  is given by

$$\delta\Omega(D) = - \sum_{i=1}^m \{(\nabla_{E_i} \Omega)(E_i, D) + (\nabla_{JE_i} \Omega)(JE_i, D)\} \quad (1.6)$$

Let us recall that the Lee form of an almost Hermitian manifold is a 1-form  $\theta$ , given by

$$\theta(D) = \frac{1}{m-1} \delta\Omega(JD). \quad (1.7)$$

The Nijenhuis tensor,  $N_J$ , of the almost complex structure  $J$  is a tensor field of type  $(1, 2)$  given in [6, p. 47] or [7, p. 63] by

$$N_J(D, E) = J^2[D, E] + [JD, JE] - J[JD, E] - J[D, JE]. \quad (1.8)$$

When  $N_J(D, E) = 0$ , the almost complex structure  $J$  is said to be integrable; in this case, the almost Hermitian manifold is called Hermitian.

Almost Hermitian structures have been completely classified by A. Gray and L.M. Hervella [22]. We just recall the defining relations of some classes which will be used in this study.

An almost Hermitian manifold  $(M^{2m}, g, J)$  is said to be :

(1) *Kählerian* if  $d\Omega(D, E, G) = 0$  and  $N_J = 0$ , where  $N_J$  denotes the Nijenhuis tensor of  $J$ ;

(2) *almost Kählerian* (or  $W_2$ -manifold) if  $d\Omega(D, E, G) = 0$ ;

(3) *nearly Kählerian* (or  $W_1$ -manifold) if  $(\nabla_D\Omega)(D, E) = 0$ ;

(4)  $W_3$ -manifold if  $(\nabla_D\Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0 = \delta\Omega$ ;

(5) *semi-Kählerian* (or  $W_1 \oplus W_2 \oplus W_3$  - manifold) if  $\delta\Omega = 0$ ;

(6)  $W_1 \oplus W_3$ -manifold if  $(\nabla_D\Omega)(D, E) - (\nabla_{JD}\Omega)(JD, E) = 0 = \delta\Omega$ ;

(7)  $G_1$ -manifold if  $(\nabla_D\Omega)(D, E) - (\nabla_{JD}\Omega)(JD, E) = 0$ ;

(8) *Hermitian* or ( $W_3 \oplus W_4$ -manifold) if  $N_J = 0$  or equivalently

$$(\nabla_D\Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0;$$

(9) a  $G_2$ -manifold or ( $W_2 \oplus W_3 \oplus W_4$ -manifold) if

$$\mathcal{G}\{(\nabla_D\Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G)\} = 0 \text{ or } \mathcal{G}\{g(N_J(D, E), JG)\} = 0;$$

(10) *quasi Kählerian* or ( $W_1 \oplus W_2$ -manifold) if

$$(\nabla_D\Omega)(E, G) + (\nabla_{JD}\Omega)(JE, G) = 0;$$

(11)  $W_2 \oplus W_3$ -manifold if

$$\mathcal{G}\{(\nabla_D\Omega)(D, E) - (\nabla_{JD}\Omega)(JD, E)\} = 0 = \delta\Omega;$$

(12) *locally conformal almost Kähler* ( $W_2 \oplus W_4$ -manifold) if

$$d\Omega = \Omega\Lambda\theta \text{ or } \mathcal{G}\left\{(\nabla_D\Omega)(E, G) - \frac{1}{m-1}\Omega(D, E)\delta\Omega(JG)\right\} = 0;$$

(13) *locally conformal Kähler* ( $W_4$ -manifold) if

$$\begin{aligned} (\nabla_D\Omega)(E, G) &= \frac{-1}{2(m-1)} \{g(D, E)\delta\Omega(G) - g(D, G)\delta\Omega(E)\} \\ &+ \frac{-1}{2(m-1)} \{-g(D, JE)\delta\Omega(JG) + g(D, JG)\delta\Omega(JE)\}; \end{aligned}$$

(14)  $W_1 \oplus W_4$ -manifold if

$$(\nabla_D \Omega)(D, E) = \frac{-1}{2(m-1)} \{g(D, D)\delta\Omega(E) - g(D, E)\delta\Omega(D) - g(JD, E)\delta\Omega(JD)\};$$

(15)  $W_1 \oplus W_2 \oplus W_4$ -manifold if

$$\begin{aligned} (\nabla_D \Omega)(E, G) + (\nabla_{JD} \Omega)(JE, G) &= \frac{-1}{m-1} \{g(D, E)\delta\Omega(G) - g(D, G)\delta\Omega(E)\} \\ &+ \frac{-1}{m-1} \{-g(D, JE)\delta\Omega(JG) + g(D, JG)\delta\Omega(JE)\}; \end{aligned} \quad (1.9)$$

Notice that nearly Kählerian manifolds are also called *almost Tachibana* [54] and [55]. On the other hand,  $W_3$ -manifolds are both semi-Kählerian and Hermitian; for this reason, they are called *special Hermitian*. From the defining relations, it is clear that a Hermitian manifold is a  $G_1$ -manifold. Furthermore, a  $W_1 \oplus W_3$ -manifold is both a  $G_1$ -manifold and semi-Kählerian.

Many examples of almost Hermitian manifolds are given by Gray and Hervella in [22].

Now, let us focus on almost contact metric manifolds.

## 1.2 Almost contact metric manifolds

Let  $M$  be a differentiable manifold of odd dimension  $(2m + 1)$ . An almost contact structure on  $M$  is a triple  $(\varphi, \xi, \eta)$  where:

- (1)  $\xi$  is a characteristic vector field,
- (2)  $\eta$  is a 1-form such that  $\eta(\xi) = 1$ , and
- (3)  $\varphi$  is a tensor field of type  $(1, 1)$  satisfying

$$\varphi^2 = -\mathbb{I} + \eta \otimes \xi, \quad (1.10)$$

where  $\mathbb{I}$  is the identity transformation.

If  $M$  is equipped with a Riemannian metric  $g$  such that

$$g(\varphi D, \varphi E) = g(D, E) - \eta(D)\eta(E), \quad (1.11)$$

then  $(g, \varphi, \xi, \eta)$  is called an *almost contact metric structure*. Therefore, the quintuple  $(M^{2m+1}, g, \varphi, \xi, \eta)$  is an almost contact metric manifold. As in the case of almost Hermitian manifolds, any almost contact metric manifold admits a fundamental 2-form,  $\phi$ , defined by

$$\phi(D, E) = g(D, \varphi E). \quad (1.12)$$

Furthermore,  $\varphi\xi = 0$  and  $\eta(D) = g(D, \xi)$ .

As in the case of almost Hermitian manifolds, we recall some of the important identities.

$$(\nabla_D\phi)(E, G) = g(E, (\nabla_D\varphi)G) = -g((\nabla_D\varphi)E, G), \quad (1.13)$$

$$(\nabla_D\eta)E = g(E, \nabla_D\xi) = (\nabla_D\phi)(\xi, \varphi E), \quad (1.14)$$

$$(\nabla_D\phi)(E, G) + (\nabla_D\phi)(\varphi E, \varphi G) = \eta(G)(\nabla_D\eta)\varphi E - \eta(E)(\nabla_D\eta)\varphi G, \quad (1.15)$$

$$2d\eta(D, E) = D\eta(E) - E\eta(D) - \eta([D, E]), \quad (1.16)$$

$$3d\phi(D, E, G) = \mathcal{G}\{(\nabla_D\phi)(E, G)\}. \quad (1.17)$$

Let  $\{E_1, \dots, E_m, \varphi E_1, \dots, \varphi E_m, \xi\}$  be a local  $\varphi$ -basis of an open subset of M, then the coderivative,  $\delta$ , is given by

$$\begin{aligned} \delta\phi(D) &= -\sum_{i=1}^m \{(\nabla_{E_i}\phi)(E_i, D) + (\nabla_{\varphi E_i}\phi)(\varphi E_i, D)\} - (\nabla_\xi\phi)(\xi, D), \\ \delta\eta &= -\sum_{i=1}^m \{(\nabla_{E_i}\eta)E_i + (\nabla_{\varphi E_i}\eta)\varphi E_i\}. \end{aligned}$$

The analogous of the Lee form is the 1-form,  $\omega$ , defined by

$$\omega(D) = \frac{1}{m}(\delta\phi(\varphi D) - \eta(D)\delta\eta).$$

In [41], S. Sasaki and Y. Hatakeyama have defined two tensors fields  $N^{(1)}$  and  $N^{(2)}$  of type  $(0, 2)$  by setting

$$(1) \quad N^{(1)}(D, E) = N_\varphi(D, E) + 2d\eta(D, E)\xi,$$

$$(2) \quad N^{(2)}(D, E) = (\mathcal{L}_{\varphi D}\eta)E - (\mathcal{L}_{\varphi E}\eta)D$$

where  $N_\varphi$  is the Nijenhuis tensor of  $\varphi$  while  $\mathcal{L}$  denotes the Lie derivative.

If  $N^{(1)} = 0$ , the manifold is said to be normal and in this case  $N^{(2)} = 0$ .

Following Gray and Hervella [22], in the classification of almost Hermitian structures, D. Chinea and C. Gonzalez [15] have obtained a classification of almost contact metric manifolds. They have shown that there are 4,096 classes of almost contact metric manifolds.

Note that, among the 4,096 classes of these structures, only a few of them have been identified. We recall here the defining relations of those manifolds which are already identified and are susceptible to be used in the sequel.

An almost contact metric manifold is said to be :

$$(1) \quad \textit{cosymplectic} \text{ if } d\phi = 0, \quad N^{(1)} = 0, \text{ and } d\eta = 0;$$

- (2) *almost cosymplectic* if  $d\phi = 0$  and  $d\eta = 0$ ;
- (3) *semi-cosymplectic normal* if  $\delta\phi = 0 = \delta\eta = N^{(1)}$ ;
- (4) *Sasakian* if  $\phi = d\eta$ , and  $N^{(1)} = 0$ ;
- (5) *quasi-Sasakian* if  $d\phi = 0$ , and  $N^{(1)} = 0$ ;
- (6) *semi-Sasakian* if  $\eta = \frac{1}{2m}\delta\phi$ ;
- (7) *Kenmotsu* if  $d\phi(D, E, G) = \frac{2}{3}\mathcal{G}\{\eta(D)\phi(E, G)\}$ ,  $d\eta = 0$  and  $N^{(1)} = 0$ ;
- (8)  $G_1$ -*Sasakian* if  $(\nabla_D\phi)D - (\nabla_{\varphi D}\phi)\varphi D + \eta(D)(\nabla_{\varphi D}\xi) = 0$ ;
- (9)  $G_1$ -*semi-Sasakian* if it is  $G_1$ -*Sasakian* and  $\frac{\delta\phi}{2m} = \eta$ ;
- (10)  $G_1$ -*Kenmotsu* if

$$(\nabla_D\phi)(D, E) - (\nabla_{\varphi D}\phi)(\varphi D, E) - \eta(D)\phi(E, D) = 0 = d\eta;$$

- (11)  $G_1$ -*semi-Kenmotsu* if it is  $G_1$ -*Kenmotsu* and  $\delta\phi = 0$ ;
- (12)  $G_1$ -*semi-cosymplectic* if it is  $G_1$ -*Sasakian* and  $\delta\phi = 0 = \delta\eta$ .
- (13)  $G_2$ -*Sasakian* if

$$\mathcal{G}\{(\nabla_D\phi)(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) - \eta(E)(\nabla_{\varphi D}\eta)G\} = 0;$$

- (14)  $G_2$ -*semi-Sasakian* if it is  $G_2$ -*Sasakian* and  $\frac{\delta\phi}{2m} = \eta$ ;
- (15)  $G_2$ -*Kenmotsu* if

$$\mathcal{G}\{(\nabla_D\phi)(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) - \eta(D)\phi(E, G)\} = 0 = d\eta;$$

- (16)  $G_2$ -*semi-Kenmotsu* if it is  $G_2$ -*Kenmotsu* and  $\delta\phi = 0$ ;
- (17)  $G_2$ -*semi-cosymplectic* if it is  $G_2$ -*Sasakian* and  $\delta\phi = 0 = \delta\eta$ .
- (18)  $C_7$ -*manifold* if  $(\nabla_D\phi)(E, G) = \eta(G)(\nabla_E\eta)\varphi D + \eta(E)(\nabla_{\varphi D}\eta)E$ , and  $\delta\phi = 0$ ;
- (19)  $C_8$ -*manifold* if  $(\nabla_D\phi)(E, G) = -\eta(G)(\nabla_E\eta)\varphi D + \eta(E)(\nabla_{\varphi D}\eta)G$ , and  $\delta\eta = 0$ ;
- (20)  $C_9$ -*manifold* if  $(\nabla_D\phi)(E, G) = \eta(G)(\nabla_E\eta)\varphi D - \eta(E)(\nabla_{\varphi D}\eta)G$ ;
- (21)  $C_{10}$ -*manifold* if  $(\nabla_D\phi)(E, G) = -\eta(G)(\nabla_E\eta)\varphi D - \eta(E)(\nabla_{\varphi D}\eta)G$ ;
- (22)  $C_{11}$ -*manifold* if  $(\nabla_D\phi)(E, G) = -\eta(D)(\nabla_\xi\phi)(\varphi E, \varphi G)$ ;
- (23)  $C_{12}$ -*manifold* if  
 $(\nabla_D\phi)(E, G) = \eta(D)\eta(G)(\nabla_\xi\eta)\varphi E - \eta(D)\eta(E)(\nabla_\xi\eta)\varphi G$ .

- (24) *nearly cosymplectic* if  $(\nabla_D\varphi)D = 0$ ;
- (25) *semi-cosymplectic* if  $\delta\phi = 0$  and  $\delta\eta = 0$ ;
- (26) *quasi-K-cosymplectic* if  $(\nabla_D\varphi)E + (\nabla_{\varphi D}\varphi)\varphi E - \eta(E)(\nabla_{\varphi D}\xi) = 0$ ;
- (27) *closely cosymplectic* if  $(\nabla_D\varphi)D = 0$  and  $d\eta = 0$ ;
- (28) *nearly-K-cosymplectic* if  $(\nabla_D\varphi)E + (\nabla_E\varphi)D = 0$  and  $\nabla_D\xi = 0$ ;
- (29) *nearly Sasakian* if  $(\nabla_D\varphi)E + (\nabla_E\varphi)D = 2g(D, E)\xi - \eta(D)E - \eta(E)D$ ;
- (30) *semi-Sasakian normal* if  $\eta = \frac{1}{2m}\delta\phi$  and  $N^{(1)} = 0$ ;
- (31) *contact* if  $\phi = d\eta$ ;
- (32) *K-contact* if  $\phi = d\eta$  and  $(\nabla_D\eta)E + (\nabla_E\eta)D = 0$ ;
- (33)  $C_3$ -*manifold* if  $(\nabla_D\phi)(E, G) = (\nabla_{\varphi D}\phi)(\varphi E, G)$  and  $\delta\phi = 0$ ;
- (34) *locally conformal almost cosymplectic* if  $d\phi = -2\phi \wedge \omega$  and  $d\eta = \eta \wedge \omega$ ;
- (35) *locally conformal cosymplectic* if  $d\phi = -2\phi \wedge \omega$ ,  $d\eta = \eta \wedge \omega$  and  $N_\varphi = 0$ ;
- (36) *generalized Kenmotsu* if  $(\nabla_D\phi)(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) = \eta(E)\phi(G, D)$  and  $d\eta = 0$ ;
- (37) *nearly Kenmotsu* if  $(\nabla_D\varphi)D = -\eta(D)\varphi D$  and  $d\eta = 0$ ;
- (38) *almost Kenmotsu* if  $d\phi(D, E, G) = \frac{2}{3}\mathcal{G}\{\eta(D)\phi(E, G)\}$ ;
- (39) *semi-Kenmotsu normal* if  $(\nabla_D\phi)(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) = \eta(E)\phi(G, D)$ ,  $\delta\phi = 0$  and  $d\eta = 0$ ;
- (40) *quasi-Kenmotsu* if  $(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = \eta(E)\phi(G, D) + 2\eta(G)\phi(D, E)$  and  $d\eta = 0$ ;
- (41) *quasi-K-Sasakian* if

$$(\nabla_D\varphi)E + (\nabla_{\varphi D}\varphi)\varphi E = 2g(D, E)\xi + \eta(E)(\nabla_{\varphi D}\xi) - 2\eta(E)D;$$

- (42)  $C_2$ -*manifold* if  $d\phi = 0$  and  $\nabla\eta = 0$ ;
- (43) *almost-K-contact* if  $\nabla_\xi\varphi = 0$ ;
- (44)  $C_5$ -*manifold* if

$$(\nabla_D\phi)(E, G) = \frac{1}{2m}\{\phi(D, G)\eta(E) - \eta(G)\phi(D, E)\}\delta\eta;$$

(45) *nearly-K-Sasakian* if

$$\begin{aligned} (\nabla_D\varphi)E + (\nabla_E\varphi)D &= 2g(D, E)\xi - \eta(E)D - \eta(D)E, \\ \text{and } \nabla_D\xi &= -\varphi D; \end{aligned}$$

(46) *almost trans Kenmotsu* if

$$\begin{aligned} \mathcal{G} \left\{ (\nabla_D\phi)(E, G) - \frac{1}{m}\phi(D, E)\delta\phi(\varphi G) - 2\eta(D)\phi(E, G) \right\} &= 0, \\ \text{and } d\eta &= 0. \end{aligned}$$

Looking through the defining relations of all various classes of almost contact metric structures, it appears that the differential  $d\phi$  and the covariant derivative  $\nabla\phi$  of the fundamental form  $\phi$  can be expressed in formulae such as the following in which  $b$  is a real number.

$$d\phi(D, E, G) = \frac{b}{3}\mathcal{G} \{ \eta(D)\mathcal{C} \}, \quad (1.18)$$

$$d\phi(D, E, G) = \frac{b}{3}\mathcal{G} \{ \phi(D, E)\mathcal{C} \}, \quad (1.19)$$

$$d\phi = b.\phi \wedge \omega, \quad (1.20)$$

$$(\nabla_D\phi)(D, E) - (\nabla_{\varphi D}\phi)(\varphi D, E) + b.\eta(D)\mathcal{C} = 0, \quad (1.21)$$

$$\mathcal{G} \{ (\nabla_D\phi)(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) + b.\eta(D)\mathcal{C} \} = 0. \quad (1.22)$$

Note that in the above formulae,  $\mathcal{C}$  is a factor which is determined by the class of the manifold. For instance, if we take  $b = 2$  and  $\mathcal{C} = \phi(E, G)$  in (1.18), we obtain

$$d\phi(D, E, G) = \frac{2}{3}\mathcal{G} \{ \eta(D)\phi(E, G) \},$$

which is one of the defining relations of an almost Kenmotsu manifold [28].

Taking  $b = 1$  and  $\mathcal{C} = (\nabla_G\eta)\varphi E + (\nabla_{\varphi G}\eta)E$  in the same formula, we get

$$d\phi(D, E, G) = \frac{1}{3}\mathcal{G} \{ \eta(D) [(\nabla_G\eta)\varphi E + (\nabla_{\varphi G}\eta)E] \},$$

which leads to define a  $C_7$ -manifold as in [15]. However, if  $b = 0$  in the same formula, we get one of the defining relations of almost cosymplectic, quasi Sasakian or a  $C_2$ -manifold. Taking  $b = -1$  and  $\mathcal{C} = (\nabla_{\varphi D}\eta)E$ , in (1.21), we obtain the defining relations of  $G_1$ -Sasakian structures; in the same formula, if  $b = -1$  and  $\mathcal{C} = \phi(E, D)$  we then obtain the defining relations of a  $G_1$ -Kenmotsu manifold.

Clearly, (1.22) can be illustrated by  $G_2$ -almost contact metric structures such as:  $G_2$ -Sasakian,  $G_2$ -semi-cosymplectic or  $G_2$ -Kenmotsu.

Frequently, we will use the above formulas which generalize the defining relations of some structures.

**Proposition 1.2.1.** *Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  be an almost contact metric manifold. Then, we have,*

$$2g((\nabla_D \varphi)E, G) = 3d\phi(D, \varphi E, \varphi G) - 3d\phi(D, E, G) + g(N^{(1)}(E, G), \varphi D) \\ + N^{(2)}(E, G)\eta(D) + 2d\eta(\varphi E, D)\eta(G) - 2d\eta(\varphi G, D)\eta(E).$$

*Proof.* See Blair [6, p.52] or [7, Lemma 6.1, p.65]. □

The above proposition leads to express the defining relations of some structures in the function of the covariant or the exterior derivative of the tensors. For instance

**Proposition 1.2.2.** *Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  be an almost contact metric manifold. If it is*

- (1) *quasi-Sasakian, then  $g((\nabla_D \varphi)E, G) = d\eta(\varphi E, D)\eta(G) - d\eta(\varphi G, D)\eta(E)$ ;*
- (2) *Sasakian, then  $(\nabla_D \varphi)E = g(D, E)\xi - \eta(E)D$ ;*
- (3) *almost cosymplectic, then  $2g((\nabla_D \varphi)E, G) = g(N_\varphi(E, G), \varphi D)$ ;*
- (4) *cosymplectic, then  $\nabla_D \varphi = 0$ ;*
- (5) *Kenmotsu, then  $(\nabla_D \varphi)E = g(\varphi D, E)\xi - \eta(E)\varphi D$ .*

*Proof.* (1) Recall that a quasi-Sasakian manifold is defined by  $d\phi = 0$  and  $N^{(1)} = 0$ . Using Proposition 1.2.1, we have,

$$2g((\nabla_D \varphi)E, G) = N^{(2)}(E, G)\eta(D) + 2d\eta(\varphi E, D)\eta(G) - 2d\eta(\varphi E, D)\eta(E).$$

On the other hand, it is known that  $N^{(1)} = 0$  implies that  $N^{(2)} = 0$  from which, the preceding relation reduces to

$$g((\nabla_D \varphi)E, G) = d\eta(\varphi E, D)\eta(G) - d\eta(\varphi G, D)\eta(E)$$

which is the proof of (1). Concerning the statement (2), we claim that

$$2g((\nabla_D \varphi)E, G) = 3d\phi(D, \varphi E, \varphi G) - 3d\phi(D, E, G) \\ + 2d\eta(\varphi E, D)\eta(G) - 2d\eta(\varphi G, D)\eta(E),$$

because a Sasakian manifold is normal. Since  $\phi = d\eta$ , we have  $d\phi = 0$  so that the above relation reduces to

$$2g((\nabla_D \varphi)E, G) = 2d\eta(\varphi E, D)\eta(G) - 2d\eta(\varphi G, D)\eta(E).$$

Thus,  $g((\nabla_D \varphi)E, G) = d\eta(\varphi E, D)\eta(G) - d\eta(\varphi G, D)\eta(E)$ , which becomes

$$g((\nabla_D \varphi)E, G) = \phi(\varphi E, D)\eta(G) - \phi(\varphi G, D)\eta(E).$$

On the other hand,  $\phi(\varphi E, D) = g(\varphi E, \varphi D) = g(E, D) - \eta(E)\eta(D)$  and  $\phi(\varphi G, D) = g(\varphi G, \varphi D) = g(G, D) - \eta(G)\eta(D)$ . These lead to  $g((\nabla_D \varphi)E, G) = g(D, E)\eta(G) - g(G, D)\eta(E)$ , which is

$$g((\nabla_D \varphi)E, G) = g(D, E)g(G, \xi) - g(G, D)g(E, \xi),$$

from which  $(\nabla_D \varphi)E = g(D, E)\xi - \eta(E)D$ , follows. This is the defining relation currently used in the definition of a Sasakian structure. Let us consider the case of statement (3) concerning the almost cosymplectic manifolds. From the relation  $d\phi = 0$  and  $d\eta = 0$ , we get  $2g((\nabla_D \varphi)E, G) = g(N^{(1)}(E, G), \varphi D)$  by Proposition 1.2.1. But

$$N^{(1)}(E, G) = N_\varphi(E, G) + 2d\eta(E, G)\xi.$$

Since  $d\eta = 0$ , the above relation becomes  $N^{(1)}(E, G) = N_\varphi(E, G)$  and then

$$2g((\nabla_D \varphi)E, G) = g(N_\varphi(E, G), \varphi D),$$

as claimed in statement (3). Considering (4), it is known that a cosymplectic manifold is normal and then  $N^{(1)}(E, G) = 0$  which leads to  $2g((\nabla_D \varphi)E, G) = 0$  from which  $g((\nabla_D \varphi)E, G) = 0$ . Using the non-degeneracy of  $g$ , the last relation implies that  $(\nabla_D \varphi)E = 0$  which is the proof of (4). In the literature, this is the defining relation currently used to define a cosymplectic manifold. Let us consider the case of Kenmotsu manifold. Since a Kenmotsu manifold is normal and  $d\eta = 0$ , we then have

$$2g((\nabla_D \varphi)E, G) = 3d\phi(D, \varphi E, \varphi G) - 3d\phi(D, E, G).$$

Considering  $d\phi(D, E, G)$  in a Kenmotsu manifold we have

$$3d\phi(D, E, G) = 2\{\eta(D)g(E, \varphi G) + \eta(E)g(G, \varphi D) + \eta(G)g(D, \varphi E)\} \quad (1.23)$$

Similarly,

$$3d\phi(D, \varphi E, \varphi G) = 2\{\eta(D)g(E, \varphi G)\}, \quad (1.24)$$

because  $\eta(\varphi E) = 0 = \eta(\varphi G)$ . Making (1.24)- (1.23), leads to  $3d\phi(D, \varphi E, \varphi G) - 3d\phi(D, E, G) = -2\{\eta(E)g(G, \varphi D) + \eta(G)g(D, \varphi E)\}$  and with this, we get

$$2g((\nabla_D \varphi)E, G) = -2\{\eta(E)g(G, \varphi D) + \eta(G)g(D, \varphi E)\},$$

which is equivalent to

$$g((\nabla_D \varphi)E, G) = -g(D, \varphi E)g(G, \xi) - g(G, \varphi D)g(E, \xi)$$

from which we deduce  $(\nabla_D \varphi)E = g(\varphi D, E)\xi - \eta(E)\varphi D$ . This is the defining relation usually used for a Kenmotsu manifold.  $\square$

The inclusion relationships of certain almost contact metric structures are given through the following sketch that consists of three diagrams.

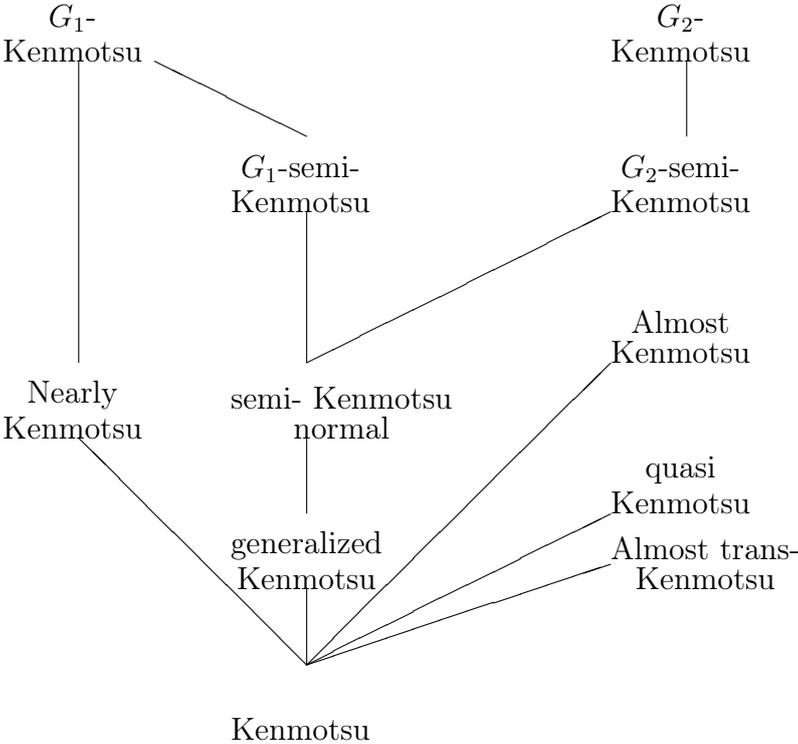


Figure 1. Diagram in the Kenmotsu case.

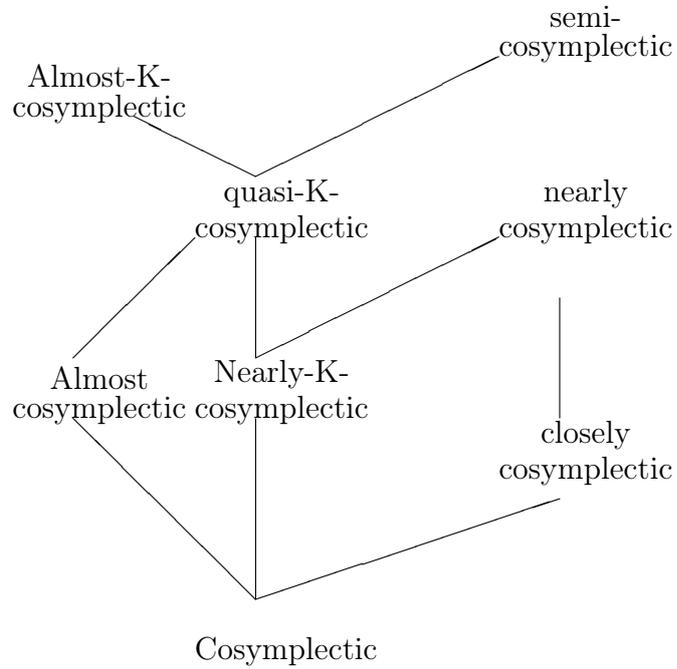


Figure 2. Diagram in the Cosymplectic case.

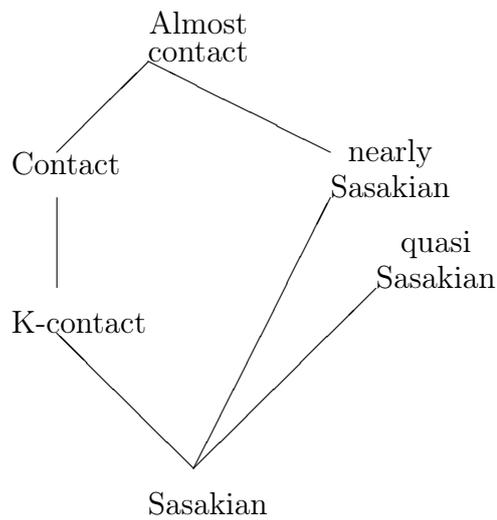


Figure 3. Diagram in the Sasakian case.

More details on almost contact metric manifolds can be found in [6], [7], [40] and [41].

## 1.3 Some Examples of Almost Contact Metric Manifolds

### 1.3.1 Cosymplectic structure on $\mathbb{R}^3$

Let  $\mathbb{R}^3$  with its cartesian coordinates  $(x^1, x^2, x^3)$ . Set  $\xi = \frac{\partial}{\partial x^2}$ ,  $\eta = dx^2$  and the tensor field of type  $(1, 1)$ ,

$$\varphi \frac{\partial}{\partial x^1} = -\frac{\partial}{\partial x^3}, \quad \varphi \frac{\partial}{\partial x^2} = 0, \quad \varphi \frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^1}.$$

It is clear that  $d\eta = 0$ ,  $\eta(\xi) = 1$ ,  $\varphi\xi = 0$ . Using the linearity of  $\varphi$ , the relation (1.10) is satisfied. Therefore,  $(\varphi, \xi, \eta)$  is an almost contact structure. Now, we want to show that the metric  $g$  is compatible with the almost contact structure  $(\varphi, \xi, \eta)$ . For this, let us express the natural formulations of  $\varphi X$  and  $\varphi Y$  where  $X = X^i \frac{\partial}{\partial x^i}$  and  $Y = Y^i \frac{\partial}{\partial x^i}$  are two vector fields of  $\mathbb{R}^3$ . We have,

$$\varphi X = X^i \varphi \frac{\partial}{\partial x^i} = X^3 \frac{\partial}{\partial x^1} - X^1 \frac{\partial}{\partial x^3} \quad \text{and} \quad \varphi Y = Y^i \varphi \frac{\partial}{\partial x^i} = Y^3 \frac{\partial}{\partial x^1} - Y^1 \frac{\partial}{\partial x^3}.$$

We then get,  $g(\varphi X, \varphi Y) = X^3 Y^3 + X^1 Y^1$ . But,  $g(X, Y) = X^1 Y^1 + X^2 Y^2 + X^3 Y^3$  and  $\eta(X)\eta(Y) = X^2 Y^2$  from which  $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$  which shows that  $(\varphi, \eta, \xi, g)$  is an almost contact metric structure and then  $(\mathbb{R}^3, \varphi, \eta, \xi, g)$  is an almost contact metric manifold. Referring to the basis  $(dx^1, dx^2, dx^3)$ , we have,  $\phi(X, Y) = 2dx^1 \wedge dx^3$  from which  $d\phi = 0$  is clear.

We then get  $d\phi = 0$  and  $d\eta = 0$  allowing to say that  $(\mathbb{R}^3, \varphi, \eta, \xi, g)$  is an almost cosymplectic manifold. To confirm or dis-confirm this, we must calculate  $N_\varphi(X, Y)$ . Recall that

$$N_\varphi(X, Y) = \varphi^2 [X, Y] + [\varphi X, \varphi Y] - \varphi [X, \varphi Y] - \varphi [\varphi X, Y].$$

A straightforward calculation gives  $N_\varphi(X, Y) = 0$ . Therefore  $(\mathbb{R}^3, \varphi, \eta, \xi, g)$  is cosymplectic.

### 1.3.2 Kenmotsu structure on $M^3$

As in the case of cosymplectic structure, let  $M^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ , where  $(x, y, z)$  are the standard coordinates in  $\mathbb{R}^3$ . The vector fields

$$E_1 = z \frac{\partial}{\partial x}, \quad E_2 = z \frac{\partial}{\partial y}, \quad E_3 = -z \frac{\partial}{\partial z}$$

are linearly independent at each point of  $M^3$ . Let  $g$  be the Riemannian metric on  $M^3$  defined by,  $g(E_i, E_i) = 1$  and  $g(E_i, E_j) = 0$ ,  $\forall i \neq j$ ,  $i, j = 1, 2, 3$ . Let  $\eta$  be the 1-form defined by  $\eta(\cdot) = g(E_3, \cdot)$ .

Let  $\varphi$  be the  $(1, 1)$  tensor field defined by

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0.$$

Then using the linearity of  $\varphi$  and  $g$ , we have,  $\eta(E_3) = 1$ ,  $\varphi^2 = -\mathbb{I} + \eta \otimes E_3$ ,  $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ , for any  $X, Y \in \Gamma(TM^3)$ . Thus, for  $E_3 = \xi$ ,  $(\varphi, \xi, \eta, g)$  is an almost contact metric structure. Using the Koszul formula, we have, for any  $X, Y \in \Gamma(TM^3)$ ,

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

which characterizes the Kenmotsu structure.

### 1.3.3 Sasakian structure on $\mathcal{S}^3$

Let  $(\mathbb{R}^4, g, J)$  where  $\mathbb{R}^4 = \{(x^i, y^i)\}$  with  $x, y \in \mathbb{R}$  and  $i = 1, 2$ .  $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$ ,  $J(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}$ , and  $g = \sum_{i=1}^2 (dx^i)^2 + (dy^i)^2$ . It is clear that  $(g, J)$  is an almost Hermitian structure.

Consider the unit sphere  $\mathcal{S}^3$  immersed in  $\mathbb{R}^4$  and a unit vector field  $\mu$  normal to  $\mathcal{S}^3$  defined by

$$\mu = \sum_{i=1}^2 (x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}).$$

Set

$$\xi = J\mu,$$

$$\eta = \sum_{i=1}^2 (x^i dy^i - y^i dx^i),$$

and

$$\varphi = J + \eta \otimes \mu.$$

Let us show that  $(\varphi, \xi, \eta)$  is an almost contact structure.

Indeed,

$$\begin{aligned} \varphi^2 D &= J(JD + \eta(D)\mu) + \eta(JD + \eta(D)\mu)\mu \\ &\quad - D + \eta(D)J\mu + \eta(JD)\mu + \eta(D)\eta(\mu)\mu. \end{aligned}$$

It is clear that  $\eta(\mu) = 0$  because  $\eta(\mu) = g(\xi, \mu) = 0$  since  $\mu$  is orthogonal to  $\xi$ . In this way,

$$\varphi^2 D = -D + \eta(D)\xi + \eta(JD)\mu.$$

Recall that  $\eta(JD) = g(JD, \xi) = g(JD, J\mu)$ . Since  $(g, J)$  is an almost Hermitian structure, then

$$g(JD, J\mu) = g(D, \mu).$$

But  $g(D, \mu) = 0$  because  $\mu$  is normal. Thus,  $\varphi^2 = -D + \eta(D)\xi$ .

Let us verify that  $\eta(\xi) = 1$ . Since  $\xi = J\mu$ , then  $\eta(J\mu) = \eta(\xi)$ . But,  $J\mu = \sum_{i=1}^2 (x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i})$ . So, we have,

$$\eta(\xi) = \eta(J\mu) = \sum_{i=1}^2 (x^i)^2 + (y^i)^2 = 1$$

We have to show that  $\eta \circ \varphi = 0$ . In fact,

$$(\eta \circ \varphi)D = \eta(\varphi D) = \eta(JD + \eta(D)\mu) = \eta(JD) + \eta(D)\eta(\mu).$$

On the other hand,  $\eta(\mu) = 0$  and  $\eta(JD) = 0$ . Therefore,  $(\eta \circ \varphi)(D) = 0$ . To show that  $\varphi\xi = 0$ , one must calculate  $\varphi\xi = J\xi + \eta(\xi)\mu$ . Since  $J\xi = -\mu$  and  $\eta(\xi) = 1$ , then  $\varphi\xi = -\mu + \mu = 0$ .

### 1.3.4 Product $M^{2m'} \times M^{2m+1}$

The product of manifolds plays an important role in the construction of some examples of submersions. Some results in this paragraph have been published in [47].

Let  $(M', g', J')$  be a  $2m'$ -dimensional almost Hermitian manifold and  $(M, g, \varphi, \xi, \eta)$  be an almost contact metric manifold of dimension  $2m + 1$ . It is known that the product  $\widetilde{M} = M' \times M$  is a differentiable manifold of dimension  $2(m' + m) + 1$ . One can put  $n = m' + m$  so that the dimension of  $\widetilde{M}$  is  $2n + 1$ .

On the product  $\widetilde{M} = M' \times M$ , one defines an almost contact metric structure  $(\tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$  by setting

$$\tilde{\varphi}(D', D) = (J'D', \varphi D), \quad (1.25)$$

$$\tilde{\eta}(D', D) = \frac{m}{n}\eta(D), \quad (1.26)$$

$$\tilde{g}((D', D), (E', E)) = g'(D', E') + \frac{n^2}{m^2}g(D, E), \quad (1.27)$$

$$\tilde{\xi} = \frac{n}{m}(0, \xi). \quad (1.28)$$

**Proposition 1.3.1.** *Let  $(M', g', J')$  be an almost Hermitian manifold and  $(M, g, \varphi, \xi, \eta)$  an almost contact metric manifold. If  $(M' \times M, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$  is an almost contact metric manifold obtained as above, then it is:*

- (1) *semi-Sasakian if, and only if,  $M'$  is semi Kähler and  $M$  is semi-Sasakian;*
- (2)  *$G_1$ -Sasakian if, and only if,  $M'$  is a  $G_1$ -manifold and  $M$  is  $G_1$ -Sasakian;*
- (3) *semi-cosymplectic normal if, and only if,  $M'$  is a  $W_3$ -manifold and  $M$  is semi-cosymplectic normal;*
- (4)  *$G_1$ -semi-Sasakian if, and only if,  $M'$  is a  $W_1 \oplus W_3$ -manifold and  $M$  is  $G_1$ -semi-Sasakian.*

*Proof.* First, note that since  $\tilde{M} = M'^{2m'} \times M^{2m+1}$ , we have  $\dim \tilde{M} = 2(m' + m) + 1$ . Suppose that  $\tilde{M}$  is semi-Sasakian, we then have

$$\tilde{\eta} = \frac{1}{2(m' + m)} \delta \tilde{\phi}, \quad (1.29)$$

$$\tilde{\eta}(D', D) = \frac{m}{m' + m} \eta(D), \quad (1.30)$$

$$\delta \tilde{\phi}(D', D) = \delta \Omega'(D') + \delta \phi(D). \quad (1.31)$$

Thus, combining (1.30) with (1.29) and (1.31) gives

$$\frac{m}{m' + m} \eta(D) = \frac{1}{2(m' + m)} (\delta \Omega'(D') + \delta \phi(D)),$$

which leads to

$$\frac{m}{m' + m} \eta(D) = \frac{1}{2(m' + m)} \delta \Omega'(D') + \frac{1}{2(m' + m)} \delta \phi(D).$$

Therefore

$$\eta(D) = \frac{m' + m}{2m(m' + m)} \delta \Omega'(D') + \frac{m' + m}{2(m' + m)} \delta \phi(D),$$

$$\eta(D) = \frac{1}{2m} \delta \Omega'(D') + \frac{1}{2m} \delta \phi(D),$$

and we deduce that  $\eta = \frac{1}{2m} \delta \phi$  if and only if  $\delta \Omega' = 0$ . This means that  $\tilde{M}$  is semi-Sasakian if and only if  $M'$  is semi-Kähler and  $M$  is semi-Sasakian. Other statements are proved in the same way.  $\square$

Some illustrations can be pointed out from [15] as follows.

$\mathcal{S}^6 \times \mathbb{R}^{2m+1}$  is nearly- $K$ -cosymplectic,

$\mathcal{S}^2 \times \mathbb{R}^{2m+1}$  is quasi- $K$ -cosymplectic,

$\mathcal{S}^{2m+1} \times \mathbb{R}^{2p}$  is quasi Sasakian.

Looking through these examples, it is known that:

$\mathcal{S}^6$  is nearly Kählerian and  $\mathbb{R}^{2m+1}$  is cosymplectic,

$\mathcal{S}^2$  is quasi Kählerian and  $\mathbb{R}^{2m+1}$  is cosymplectic,

$\mathcal{S}^{2m+1}$  is Sasakian and  $\mathbb{R}^{2p}$  is Kählerian.

It is known that there are 4,096 classes of almost contact metric structures; thus the above proposition should take many pages; we then generalize it in the following

**Theorem 1.3.2.** *Let  $(M', g', J')$  be an almost Hermitian manifold and  $(M, g, \varphi, \xi, \eta)$  an almost contact metric manifold. If  $(\tilde{M}, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$  is an almost contact metric manifold obtained as above, then, it is so that:*

(1)

$$d\tilde{\phi}((D', D), (E', E), (G', G)) = \frac{b}{3}\mathcal{G}\{\tilde{\eta}(D', D)\tilde{\mathcal{C}}\}$$

if and only if,

$$d\Omega'(D', E', G') = 0$$

and

$$d\phi(D, E, G) = \frac{b}{3}\mathcal{G}\{\eta(D)\mathcal{C}\};$$

(2)

$$d\tilde{\phi}((D', D), (E', E), (G', G)) = \frac{b}{3}\mathcal{G}\{\tilde{\phi}((D', D), (E', E))\tilde{\mathcal{C}}\}$$

if and only if,

$$d\Omega'(D', E', G') = \frac{b}{3}\mathcal{G}\{\Omega'(D', E')\mathcal{C}'\}$$

and

$$d\phi(D, E, G) = \frac{b}{3}\mathcal{G}\{\phi(D, E)\mathcal{C}\},$$

(3)

$$(\tilde{\nabla}_{(D', D)}\tilde{\phi})((D', D), (E', E)) = b.\tilde{\eta}(D', D)\tilde{\phi}((E', E), (D', D))$$

if and only if,

$$(\nabla'_{D'}\Omega')(D', E') = 0$$

and

$$(\nabla_D\phi)(D, E) = b.\eta(D)\phi(E, D);$$

(4)

$$(\tilde{\nabla}_{(D', D)}\tilde{\phi})((E', E), (G', G)) + (\tilde{\nabla}_{\tilde{\varphi}(D', D)}\tilde{\phi})(\tilde{\varphi}(E', E), (G', G)) = b.\tilde{\eta}(D', D)\tilde{\mathcal{C}}$$

if and only if,

$$(\nabla'_{D'}\Omega')(E', G') + (\nabla'_{J'D'}\Omega')(J'E', G') = 0$$

and

$$(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = b.\eta(D)\mathcal{C};$$

(5)

$$\mathcal{G}\{(\tilde{\nabla}_{(D', D)}\tilde{\phi})((E', E), (G', G))\} - \mathcal{G}\{(\tilde{\nabla}_{\tilde{\varphi}(D', D)}\tilde{\phi})(\tilde{\varphi}(E', E), (G', G))\} + b.\tilde{\eta}(D', D)\tilde{\mathcal{C}} = 0,$$

if and only if,

$$\mathcal{G}\{(\nabla'_{D'}\Omega')(E', G') - (\nabla'_{J'D'}\Omega')(J'E', G')\} = 0$$

and

$$\mathcal{G}\{(\nabla_D\phi)(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) + b.\eta(D)\mathcal{C}\} = 0;$$

(6)

$$\delta\tilde{\phi} = 0, \quad \delta\tilde{\eta} = 0, \quad d\tilde{\eta} = 0 \quad \text{or} \quad N^{(1)} = 0$$

if and only if,

$$\delta\tilde{\phi} = 0 = \delta\Omega' = 0 = N^{(1)}.$$

*Proof.* (1) If

$$d\tilde{\phi}((D', D), (E', E), (G', G)) = \frac{b}{3}\mathcal{G}\{\tilde{\eta}(D', D)\tilde{\mathcal{C}}\},$$

then, by the fact that, in this product,

$$d\tilde{\phi}((D', D), (E', E), (G', G)) = d\Omega'(D', E', G') + \frac{m^2}{n^2}d\phi(D, E, G),$$

we have

$$d\Omega'(D', E', G') + \frac{m^2}{n^2}d\phi(D, E, G) = \frac{b}{3}\left\{\frac{m}{n}\eta(D)\mathcal{C}\right\},$$

and this implies

$$d\Omega'(D', E', G') = 0 \quad \text{and} \quad \frac{m}{n}d\phi(D, E, G) = \frac{b}{3}\mathcal{G}\{\eta(D)\mathcal{C}\}.$$

Putting  $a = \frac{b.n}{m}$ , one gets  $d\phi(D, E, G) = \frac{a}{3}\mathcal{G}\{\eta(D)\mathcal{C}\}$ . Conversely, if

$$d\Omega'(D', E', G') = 0 \quad \text{and} \quad d\phi(D, E, G) = \frac{a}{3}\mathcal{G}\{\eta(D)\mathcal{C}\},$$

then,

$$d\Omega'(D', E', G') + d\phi(D, E, G) = \frac{a}{3}\mathcal{G}\{\eta(D)\mathcal{C}\}.$$

Since  $d\phi(D, E, G) = \frac{a}{3}\mathcal{G}\{\eta(D)\mathcal{C}\}$ , then,

$$\frac{m}{n}d\phi(D, E, G) = \frac{a.m}{3n}\mathcal{G}\{\eta(D)\mathcal{C}\}.$$

On the other hand, taking  $\frac{a.m}{n} = b$ , one gets

$$\frac{m}{n}d\phi(D, E, G) = \frac{b}{3}\mathcal{G}\{\eta(D)\mathcal{C}\},$$

from which we have

$$\frac{m^2}{n^2}d\phi(D, E, G) = \frac{b}{3}\mathcal{G}\left\{\frac{m}{n}\eta(D)\mathcal{C}\right\}.$$

Since  $\frac{m}{n}\eta(D) = \tilde{\eta}(D', D)$ , then

$$d\Omega'(D', E', G') + \frac{m^2}{n^2}d\phi(D, E, G) = \frac{b}{3}\mathcal{G}\{\tilde{\eta}(D', D)\tilde{\mathcal{C}}\},$$

which shows that

$$d\tilde{\phi}((D', D), (E', E), (G', G)) = \frac{b}{3}\mathcal{G}\{\tilde{\eta}(D', D)\tilde{\mathcal{C}}\}.$$

(2) If  $d\tilde{\phi}((D', D), (E', E), (G', G)) = \frac{b}{3}\mathcal{G}\{\tilde{\phi}((D', D), (E', E))\tilde{\mathcal{C}}\}$ , then by the fact that

$$d\tilde{\phi}((D', D), (E', E), (G', G)) = d\Omega'(D', E', G') + \frac{m^2}{n^2}d\phi(D, E, G),$$

and

$$\tilde{\phi}((D', D), (E', E)) = \Omega'(D', E') + \frac{m^2}{n^2}\phi(D, E),$$

in this product, we get

$$d\Omega'(D', E', G') = \frac{b}{3}\mathcal{G}\{\Omega'(D', E', G')\}$$

and

$$d\phi(D, E, G) = \frac{b}{3}\mathcal{G}\{\phi(D, E)\mathcal{C}\}.$$

The converse is established as in the above assertion (1). Other statements are proved in similar way.  $\square$

This theorem is important according to the following proposition, due to Oubiña [37, Prop.2.1].

**Proposition 1.3.3.** *The manifold  $(\tilde{M}, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$  defined as above cannot be quasi-K-Sasakian.*

*Proof.* Recall that a quasi-K-Sasakian manifold is defined by the relation

$$\begin{aligned} (\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = \\ 2g(D, E)\eta(G) - 2g(D, G)\eta(E) + g(\nabla_{\varphi D}\xi, G)\eta(E). \end{aligned}$$

Therefore, if  $(\tilde{M}, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$  is quasi-K-Sasakian, thus we get

$$\begin{aligned} (\nabla'_{D'}\Omega')(E', G') + (\nabla'_{J'D'}\Omega')(J'E', G') = 2g'(D', E') - 2g'(D', G'), \\ \frac{m^2}{n^2}(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = \frac{2n}{m}g(D, E)\eta(G) - g(D, G)\eta(E) \\ + g(\nabla_{\varphi D}\xi, G)\eta(E), \end{aligned}$$

which are absurd. Indeed, the first relation does not define a subclass in the classification of almost Hermitian structures from Gray and Hervella [22]. The second implies that  $m = n$  from which we deduce  $m' = 0$ .  $\square$

## 1.4 Examples of manifolds in Kenmotsu geometry

In Kenmotsu Geometry, we can refer to the warped product

$$\mathbb{R} \times_s M^{2m},$$

where  $s$  is the warping function defined by  $s(t) = ce^t$  with  $c \in \mathbb{R}^*$ ,  $M^{2m}$  being an almost Hermitian manifold.

With this warped product, we can construct some examples as follows.

### 1.4.1 Kenmotsu case

Let  $M_1$  and  $M_2$  be two cosymplectic manifolds. It can be shown that the direct product  $M_1 \times M_2$  is a Kähler manifold. Thus the warped product  $\mathbb{R} \times_s (M_1 \times M_2)$  is a Kenmotsu manifold.

In fact, it is known that  $\mathbb{R}^2$  is a Kähler manifold. So, the warped product  $\mathbb{R} \times_s \mathbb{R}^2$  is a Kenmotsu manifold. We have already seen that  $\mathbb{R}^3$  can be furnished with the Kenmotsu structure.

Considering that  $\mathbb{R}^3$  can be endowed with a cosymplectic structure as already established, it is true that  $\mathbb{R}^3 \times \mathbb{R}^3$  is Kähler and then  $\mathbb{R} \times_s (\mathbb{R}^3 \times \mathbb{R}^3)$  is a Kenmotsu manifold. This is to say that  $\mathbb{R}^7$  can be equipped with a Kenmotsu structure.

Following Massamba [32] and [33], we can show that  $M^7 = \{(x_1, x_2, \dots, x_7) \in \mathbb{R}^7 : x_7 > 0\}$  is a Kenmotsu manifold. In fact, the vector fields

$$\begin{aligned} E_1 &= x_7 \frac{\partial}{\partial x_1}, & E_2 &= x_7 \frac{\partial}{\partial x_2}, & E_3 &= x_7 \frac{\partial}{\partial x_3}, & E_4 &= x_7 \frac{\partial}{\partial x_4}, & E_5 &= -x_7 \frac{\partial}{\partial x_5}, \\ E_6 &= -x_7 \frac{\partial}{\partial x_6}, & E_7 &= -x_7 \frac{\partial}{\partial x_7}, \end{aligned}$$

are linearly independent at each point of  $M^7$ . Let  $g$  be the Riemannian metric on  $M^7$  defined by,  $g(E_i, E_i) = 1$  and  $g(E_i, E_j) = 0$ ,  $\forall i \neq j$ ,  $i, j = 1, 2, 3, \dots, 7$ . Let  $\eta$  be the 1-form defined by  $\eta(\cdot) = g(E_7, \cdot)$ .

Let  $\varphi$  be the  $(1, 1)$  tensor field defined by

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = -E_4, \quad \varphi E_4 = E_3, \quad \varphi E_5 = -E_6, \quad \varphi E_6 = E_5, \quad \varphi E_7 = 0.$$

Then using the linearity of  $\varphi$  and  $g$ , we have,  $\eta(E_3) = 1$ ,  $\varphi^2 = -\mathbb{I} + \eta \otimes E_3$ ,  $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ , for any  $X, Y \in \Gamma(TM^7)$ . Thus, for  $E_7 = \xi$ ,  $(\varphi, \xi, \eta, g)$  is an almost contact metric structure. Using the Koszul formula, we have, for any  $X, Y \in \Gamma(TM^7)$ ,

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

which proves that  $(M^7, \varphi, \xi, \eta, g)$  is a Kenmotsu manifold.

**Problem.** We have seen that  $\mathbb{R}^3$  and  $\mathbb{R}^7$  can be endowed with a Kenmotsu structure. *How can one characterize the Kenmotsu structure on  $\mathbb{R}^{2m+1}$ ?*

According to the product of manifolds, as already defined, we would like to construct the Kenmotsu structure on  $\mathbb{R}^{2m+1}$ . First, we have seen that  $\mathbb{R}^3$  can be endowed with the Kenmotsu structure. Since  $\mathbb{R}^2$  is a Kähler manifold, then  $\mathbb{R}^2 \times \mathbb{R}^3$  is also a Kenmotsu manifold. This is to say that  $\mathbb{R}^5$  is a Kenmotsu manifold.

Following this procedure, we claim that:  $\mathbb{R}^{2m+1}$  is Kenmotsu if  $\mathbb{R}^{2m-1}$  is Kenmotsu.

In fact, if  $\mathbb{R}^{2m-1}$  is Kenmotsu, since  $\mathbb{R}^2$  is Kähler, then

$$\mathbb{R}^{2m+1} = \mathbb{R}^2 \times \mathbb{R}^{2m-1}$$

is Kenmotsu. Let us indicate the construction.

Consider  $M^{2m+1} = \{(x_1, x_2, \dots, x_{2m+1}) \in \mathbb{R}^{2m+1} : x_{2m+1} > 0\}$ . Put  $\xi = \left(\frac{-1}{x_{2m+1}}\right) \frac{\partial}{\partial x_{2m+1}}$ ,

$\eta = -x_{2m+1} dx_{2m+1}$ ,  $E_i = x_{2m+1} \frac{\partial}{\partial x_i}$  for  $i \neq 2m+1$ . The vector fields  $E_i$  are linearly independent at each point of  $M^{2m+1}$ .

Note by  $\mathcal{M}_{2m+1}(\mathbb{R})$  the set of  $(2m+1)$  real matrices and  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Taking

$\varphi \in \mathcal{M}_{2m+1}(\mathbb{R})$  such that  $\varphi = \begin{pmatrix} J & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & J & 0 & . & . & . & . \\ 0 & 0 & J & 0 & . & . & . \\ 0 & 0 & 0 & J & 0 & . & . \\ 0 & 0 & 0 & 0 & J & 0 & . \\ . & . & . & . & . & J & . \\ 0 & . & . & . & . & . & 0 \end{pmatrix}$ , it is easy to verify that

$(\varphi, \xi, \eta)$  is an almost contact structure. With the canonical metric on  $\mathbb{R}^{2m+1}$ ,  $(g, \varphi, \xi, \eta)$  is an almost contact metric structure.

### 1.4.2 Almost Kenmotsu case

In the similar way as in the preceding example, if  $M_1$  and  $M_2$  are almost cosymplectic manifolds, then the direct product  $M_1 \times M_2$  is an almost Kähler manifold. Thus,  $M = \mathbb{R} \times_s (M_1 \times M_2)$  is almost Kenmotsu.

### 1.4.3 Nearly Kenmotsu case

Let  $M_1$  and  $M_2$  be closely cosymplectic. In [9], Capursi has shown that the product  $M_1 \times M_2$  is nearly Kählerian. So, the warped product  $M = \mathbb{R} \times_s (M_1 \times M_2)$  is nearly Kenmotsu. Since, from Gray-Hervella [22],  $\mathcal{S}^6$  is a nearly Kähler manifold, then  $M = \mathbb{R} \times_s \mathcal{S}^6$  is a nearly Kenmotsu manifold.

### 1.4.4 Quasi Kenmotsu case

It is known that  $\mathcal{S}^2 \times \mathbb{R}^4$  is a quasi Kähler manifold according to the almost complex structure defined by the Cayley numbers. Thus,  $M = \mathbb{R} \times_s (\mathcal{S}^2 \times \mathbb{R}^4)$  is a quasi Kenmotsu manifold.

### 1.4.5 Semi-Kenmotsu normal case

From A. Gray [20], it is known that any parallelizable complex manifold is a  $W_3$ -manifold. Since  $\mathcal{S}^2$  is parallelizable, then  $\mathbb{R} \times_s \mathcal{S}^2$  is a semi-Kenmotsu normal manifold.

Let  $Q$  be a minimal surface immersed in  $\mathbb{R}^3$ . It can be shown that  $Q \times \mathbb{R}^4$  is a  $W_3$ -manifold. So,  $M = \mathbb{R} \times_s (Q \times \mathbb{R}^4)$  is a semi-Kenmotsu normal manifold. Concretely, consider the following non planar surface  $Q$  defined by

$$\begin{cases} x = u \cos v \\ y = u \sin v \\ z = av, \end{cases}$$

where  $a > 0$ . This surface is minimal. Thus,  $Q \times \mathbb{R}^4$  is a  $W_3$ -manifold.

#### 1.4.6 $G_1$ -Kenmotsu case

L. M. Hervella and E. Vidal, have shown that the product of a nearly Kähler  $M'_1$  with a Hermitian manifold  $M'_2$  is a  $G_1$ -manifold. With this in mind, the warped product is a  $G_1$ -Kenmotsu manifold.

According to Capursi, if  $M_1$  and  $M_2$  are normal then  $M_1 \times M_2$  is Hermitian. On the other hand, since  $S^6$  is nearly Kählerian, then  $M = \mathbb{R} \times_s ((M_1 \times M_2) \times S^6)$  is a  $G_1$ -Kenmotsu manifold.

#### 1.4.7 $G_2$ -Kenmotsu case

In the similar way, let  $M'_1$  be Hermitian and  $M'_2$  almost Kähler. Then  $M'_1 \times M'_2$  is a  $G_2$ -manifold. Thus  $M = \mathbb{R} \times_s ((M_1 \times M_2))$  is a  $G_2$ -Kenmotsu manifold.

#### 1.4.8 Generalized Kenmotsu case

It is known that the product of two spheres  $S^{2p+1}$  and  $S^{2p'+1}$  is Hermitian. Thus  $M = \mathbb{R} \times_s (S^{2p+1} \times S^{2p'+1})$  is generalized Kenmotsu. More on this class can be found in [50].

### 1.5 The common properties of some classes of almost contact metric manifolds

Some classes of almost contact metric manifolds have a common property. Here, we present those which have in common the main defining relations.

- (1) *Cosymplectic, quasi-Sasakian and Kenmotsu manifolds*

$$d\phi(D, E, G) = \frac{b}{3} \mathcal{G} \{ \eta(D)\phi(E, G) \}$$

and  $N^{(1)} = 0$ .

(2) *Almost cosymplectic,  $C_2$  and almost Kenmotsu manifolds*

$$d\phi(D, E, G) = \frac{b}{3}\mathcal{G} \{\eta(D)\phi(E, G)\}.$$

(3) *Nearly cosymplectic, nearly-K-cosymplectic, closely cosymplectic and nearly Kenmotsu manifolds*

$$(\nabla_D\phi)(D, E) = b.\eta(D)\mathcal{C}.$$

(4) *Quasi-K-cosymplectic and quasi Kenmotsu manifolds*

$$(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = b.\eta(D)\mathcal{C}.$$

(5)  *$G_1$ -Sasakian and  $G_1$ -Kenmotsu manifolds*

$$(\nabla_D\phi)(D, E) - (\nabla_{\varphi D}\phi)(\varphi D, E) + b.\eta(D)\mathcal{C} = 0.$$

(6)  *$G_2$ -Sasakian and  $G_2$ -Kenmotsu manifolds*

$$\mathcal{G} \{(\nabla_D\phi)(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) + b.\eta(D)\mathcal{C}\} = 0.$$

(7) *Trans-Sasakian, locally conformal cosymplectic and  $C_4$ -manifolds*

$$d\phi(D, E, G) = \frac{b}{3}\mathcal{G} \{\phi(D, E)\mathcal{C}\}.$$

(8) *Almost trans-Sasakian and locally conformal almost cosymplectic*

$$d\phi = b.\phi \wedge \omega.$$

# Chapter 2

## On the structure of the fibres

In this chapter, we determine the structure of the fibres according to that of the total space.

Using the product of manifolds, we construct examples of almost contact metric submersions before examining the structure of the fibres. This chapter can be viewed as a classification of submersions whose fibres lie in a fixed class of almost Hermitian structures for type I submersions. Many common properties of almost contact metric manifolds are exploited.

### 2.1 Background on almost contact metric submersions

We begin by recalling, from O'Neill [36] the concept of Riemannian submersion before attacking that of almost contact metric one.

Let  $(M, g)$  and  $(M', g')$  be two smooth, connected and complete Riemannian manifolds. By a Riemannian submersion, one understands a smooth surjective mapping

$$\pi : M \longrightarrow M'$$

such that:

- (i)  $\pi$  has maximal rank, and
- (ii)  $\pi_*|_{(Ker\pi_*)^\perp}$  is a linear isometry.

Here,  $\pi_*$  denotes the differential of  $\pi$  whose kernel is denoted by  $\ker \pi_*$  and  $(\ker \pi_*)^\perp$  is orthogonal to the kernel,  $\ker \pi_*$ , of  $\pi_*$ .

Vectors in  $\ker \pi_*$  are vertical while those in  $(\ker \pi_*)^\perp$  are horizontal. For each  $x' \in M'$ ,  $\pi^{-1}(x')$  is a closed, embedded submanifold of  $M$ , called the fibre of  $\pi$  over  $x'$ . Noting

by  $F_{x'} = \pi^{-1}(x')$  it is known that  $\dim F_{x'} = \dim M - \dim M'$ . The tangent bundle  $TM$  of the total space  $M$  has an orthogonal decomposition

$$T(M) = V(M) \oplus H(M),$$

where  $V(M)$  is the vertical distribution while  $H(M)$  designates the horizontal one.

A vector field  $X$  of the horizontal distribution is called a *basic* vector field if it is  $\pi$ -related to a vector field  $X_*$  which means  $\pi_*X = X_*$ .

In this work, we will denote horizontal vector fields by  $X, Y$  and  $Z$ , while those of the vertical distribution will be denoted by  $U, V$  and  $W$ . On the base space, tensors and other operators will be specified by a prime ( $'$ ), while those of the fibres will be denoted by a caret  $\hat{\cdot}$ . For instance,  $\nabla, \nabla'$  and  $\hat{\nabla}$  will designate the Levi-Civita connection of the total space, the base space and the fibres, respectively.

**Proposition 2.1.1 ([36]).** *Let  $\pi : (M, g) \longrightarrow (M', g')$  be a Riemannian submersion,  $X$  and  $Y$  two basic vector fields on  $M$ , then:*

- (1)  $g(X, Y) = g'(\pi_*X, \pi_*Y)$ ;
- (2)  $\mathcal{H}[X, Y]$  is the basic vector field associated to  $[X_*, Y_*]$ ;
- (3)  $\mathcal{H}(\nabla_X Y)$  is the basic vector field associated to  $(\nabla'_{X_*} Y_*)$ .

*Proof.* See O'Neill [36]. □

Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  and  $(M'^{2m'+1}, g', \varphi', \xi', \eta')$  be two almost contact metric manifolds. By an almost contact metric submersion of type  $I$ , in the sense of Watson [52], one understands a Riemannian submersion

$$\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$$

satisfying

- (i)  $\pi_*\varphi = \varphi'\pi_*$ ,
- (ii)  $\pi_*\xi = \xi'$ .

Next, we overview some of the fundamental properties of this type of submersions, established by Watson [52].

**Proposition 2.1.2.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type  $I$ . Then,*

- (1)  $\pi^*\eta' = \eta$ ,
- (2)  $\pi^*\phi' = \phi$ ;

- (3)  $U \in V(M)$  implies that  $\varphi U \in V(M)$ ;
- (4)  $X \in H(M)$  implies that  $\varphi X \in H(M)$ ;
- (5)  $\xi \in H(M)$ ;
- (6)  $\eta(U) = 0$  for all  $U \in V(M)$ ;
- (7)  $\hat{N}_J(U, V) = N^{(1)}(U, V)$ ;
- (8)  $\pi_* N^{(1)} = N'^{(1)}$ ;
- (9)  $\varphi X$  is basic associated to  $\varphi' X_*$  when  $X$  is basic;
- (10)  $\mathcal{H}(\nabla_X \varphi)Y$  is basic associated to  $(\nabla'_{X_*} \varphi')Y_*$  when  $X$  and  $Y$  are basic.

*Proof.* See Watson [52]. □

As consequences of assertions (1) and (2) in the above proposition, we have  $\pi^* d\eta' = d\eta$  and  $\pi^* d\phi' = d\phi$  respectively. Statements (3) and (4) mean that the vertical and horizontal distributions are invariant by  $\varphi$ .

Now, let us deal with another type of almost contact metric submersions introduced again by Watson.

Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  be an almost contact metric manifold and  $(M'^{2m'}, g', J')$  an almost Hermitian one. A Riemannian submersion

$$\pi : M^{2m+1} \longrightarrow M'^{2m'}$$

is called an almost contact metric submersion of type *II* if it satisfies  $\pi_* \varphi = J' \pi_*$ . This means that  $\pi$  is  $(\varphi, J')$ -holomorphic.

**Proposition 2.1.3.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type *II*. Then,*

- (1)  $\pi^* \Omega' = \phi$ ;
- (2)  $U \in V(M)$  implies that  $\varphi U \in V(M)$ ;
- (3)  $X \in H(M)$  implies that  $\varphi X \in H(M)$ ;
- (4)  $\xi \in \ker \pi_*$ ;
- (5)  $X \in H(M)$  implies that  $\eta(X) = 0$ ;
- (6)  $\pi_* N^{(1)} = N_{J'}$ ;
- (7)  $\mathcal{H}(\nabla_X \varphi)Y$  is basic associated to  $(\nabla'_{X_*} J')Y_*$  when  $X$  and  $Y$  are basic.

*Proof.* See again Watson [52]. □

## 2.2 Examples of almost contact metric submersions

**Theorem 2.2.1.** *Let  $(M'^{2m'}, g', J')$  be an almost Hermitian manifold and  $(M^{2m+1}, g, \varphi, \xi, \eta)$  an almost contact metric one. Consider the almost contact metric manifold product  $\tilde{M} = M' \times M$  defined as in the product manifold, then:*

- (1) *the projection  $\pi_1 : M' \times M \longrightarrow M$  is an almost contact metric submersion of type I,*
- (2) *the projection  $\pi_2 : M' \times M \longrightarrow M'$  is an almost contact metric submersion of type II.*

*Proof.* It is known that these two projections are Riemannian submersions. We have to show that they are  $(\tilde{\varphi}, \varphi)$ -holomorphic for the first type and  $(\tilde{\varphi}, J')$ -holomorphic for the second type. Since  $\tilde{\varphi}(D', D) = (J'D', \varphi D)$ , then,

$$\pi_{1*}\tilde{\varphi}(D', D) = \pi_{1*}(J'D', \varphi D) = \varphi D = \varphi\pi_{1*}(D', D),$$

from which  $\pi_{1*}\tilde{\varphi} = \varphi\pi_{1*}$ . On the other hand,  $\pi_{1*}\tilde{\xi} = \pi_{1*}(0, \xi) = \xi$  which achieves the proof of (1). In the similar way, we have,

$$\pi_{2*}\tilde{\varphi}(D', D) = \pi_{2*}(J'D', \varphi D) = J'D' = J'\pi_{2*}(D', D),$$

which shows that  $\pi_{2*}\tilde{\varphi} = J'\pi_{2*}$  and establishes (2). We note that, in the first case, the fibres are constituted by  $M'$  and in the second case by  $M$ . Since  $\tilde{\varphi}(D', 0) = (J'D', 0)$  and  $\tilde{\varphi}(0, D) = (0, \varphi D)$ , then  $M'$  and  $M$  are invariant submanifolds of  $\tilde{M} = M \times M'$ .  $\square$

More concretely, we have seen that  $M^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$  can be endowed with a Kenmotsu structure. Using the Kählerian structure of  $\mathbb{R}^2$ , it is clear that the product  $\mathbb{R}^2 \times M^3$  is a Kenmotsu manifold as proved in Proposition 1.3.1.

According to the preceding Theorem 2.2.1, we have:

(i) *The projection  $\pi_1 : \mathbb{R}^2 \times M^3 \longrightarrow M^3$  is an almost contact metric submersion of type I whose fibres are the Kählerian manifold  $\mathbb{R}^2$  where the total and the base space are Kenmotsu manifolds.*

(ii) *The projection  $\pi_2 : \mathbb{R}^2 \times M^3 \longrightarrow \mathbb{R}^2$  is an almost contact metric submersion of type II whose total space is Kenmotsu, the base space is the Kählerian manifold  $\mathbb{R}^2$  while the fibres are Kenmotsu manifold  $M^3$ .*

## 2.3 On the fibres of a type I submersion

**Proposition 2.3.1.** *The fibres of an almost contact metric submersion of type I are almost Hermitian manifolds.*

*Proof.* Noting by  $(2m + 1)$  the dimension of the total space and by  $(2m' + 1)$  that of the base space, it is known that the fibres have  $(2m + 1) - (2m' + 1) = 2(m - m')$  as dimension. Since the dimension of the fibres are even, it is possible to endow them with an almost complex structure. Note by  $\hat{\varphi}$ , the restriction of  $\varphi$  to the fibres, we have

$$\hat{\varphi}^2(U) = -U + \eta(U)\xi.$$

The vanishing of  $\eta$  on vertical vector fields gives rise to  $\hat{\varphi}^2(U) = -U$ . On the other hand,

$$\hat{g}(\hat{\varphi}U, \hat{\varphi}V) = \hat{g}(U, V) - \eta(U)\eta(V),$$

from which

$$\hat{g}(\hat{\varphi}U, \hat{\varphi}V) = \hat{g}(U, V)$$

is clear and shows the compatibility of  $\hat{g}$  with  $\hat{\varphi}$ . We then conclude that  $(\hat{g}, \hat{\varphi})$  is an almost Hermitian structure. Note that  $\hat{\varphi} = J$ .  $\square$

Now we give a remarkable result on the fact that some manifolds are not total space of almost contact metric submersions of type  $I$ . This is why we have to look at any class of almost contact metric manifolds.

**Proposition 2.3.2.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type  $I$ . Then the total space cannot be a nearly Sasakian, nearly- $K$ -Sasakian or a quasi- $K$ -Sasakian manifold.*

*Proof.* Recall that the structures under consideration are defined as follows:

- \* *nearly Sasakian* if  $(\nabla_D\varphi)E + (\nabla_E\varphi)D = 2g(D, E)\xi - \eta(D)E - \eta(E)D$ ;
- \* *nearly- $K$ -Sasakian* if  $(\nabla_D\varphi)E + (\nabla_E\varphi)D = 2g(D, E)\xi - \eta(E)D - \eta(D)E$  and  $\nabla_D\xi = -\varphi D$ ;
- \* *quasi- $K$ -Sasakian* if  $(\nabla_D\varphi)E + (\nabla_{\varphi D}\varphi)E = 2g(D, E)\xi + \eta(E)(\nabla_{\varphi D}\xi) - 2\eta(E)D$ .

Consider the case where the total space is nearly Sasakian. Since  $\eta$  vanishes on vertical vector fields, we get

(a)

$$(\nabla_U\varphi)V + (\nabla_V\varphi)U = 2g(U, V)\xi$$

for all  $U, V$  vertical vector fields. Now, from (a), using the formula of Gauss we obtain

$$(\hat{\nabla}_U\varphi)V + (\hat{\nabla}_V\varphi)U = 0$$

and (b)

$$\alpha(U, JV) + \alpha(V, JU) - 2\varphi(\alpha(U, V)) = 2g(U, V)\xi$$

where  $\alpha$  denotes the second fundamental form of  $M$ . Now, if we consider  $JU, JV$  we have

(c)

$$-\alpha(JU, V) - \alpha(JV, U) - 2\varphi(\alpha(JU, JV)) = 2g(JU, JV)\xi$$

Thus if we add (b) and (c) we obtain

$$-\varphi(\alpha(U, , V) + \alpha(JU, JV)) = 2g(U, V)\xi$$

which is a contradiction. The same procedure applies to the case of nearly- $K$ -Sasakian and quasi- $K$ -Sasakian manifolds.  $\square$

**Proposition 2.3.3.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is cosymplectic, quasi Sasakian, Kenmotsu,  $C_9$ ,  $C_{10}$ ,  $C_{11}$ , or  $C_{12}$ -manifold, then the fibres are Kählerian.*

*Proof.* Let  $U$ ,  $V$  and  $W$  be three vector fields tangent to the fibres. The first three manifolds have in common the following relations  $d\phi(U, V, W) = \frac{b}{3}\mathcal{G}\{\eta(U)\phi(V, W)\}$  and  $N^{(1)} = 0$ .

Since  $\eta$  vanishes on vertical vector fields, we have  $d\phi(U, V, W) = 0$ . On the other hand,  $\hat{N}(U, V) = N^{(1)}(U, V) = 0$ . Therefore, the fibres are defined by  $d\hat{\phi} = 0 = \hat{N}$ , which are the defining relations of the Kähler structure. For the rest of manifolds, their defining relations become  $(\nabla_U\phi)(V, W) = 0$ , because of the vanishing of  $\eta$  on the vertical distribution. Then  $g(W, (\nabla_U\phi)V) = 0$ . Now, using the formula of Gauss we have

$$(\nabla_U\phi)V = (\hat{\nabla}_U\phi)V + \alpha(U, \phi V) - \varphi(\alpha(U, V)).$$

Then

$$g(W, (\nabla_U\phi)V) = g(W, (\hat{\nabla}_U\phi)V) = 0$$

and thus  $(\hat{\nabla}_U\phi)V = (\hat{\nabla}_U J)V = 0$ .  $\square$

**Proposition 2.3.4.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is almost cosymplectic, a  $C_2$ -manifold or an almost Kenmotsu manifold, then the fibres are almost Kählerian.*

*Proof.* As in the preceding proposition, all these manifolds have in common the following defining relation

$$d\phi(U, V, W) = \frac{b}{3}\mathcal{G}\{\eta(U)\phi(V, W)\},$$

which becomes  $d\phi(U, V, W) = 0$  because of the vanishing of  $\eta$  on vertical vector fields. Thus, on the fibres, we have  $d\hat{\phi}(U, V, W) = 0$  which defines the almost Kähler structure.  $\square$

**Proposition 2.3.5.** *Assume that  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  is an almost contact metric submersion of type I. If the total space is nearly cosymplectic, nearly Kenmotsu, nearly- $K$ -cosymplectic or closely cosymplectic, then the fibres are nearly Kählerian.*

*Proof.* The common defining relation for all these manifolds is

$$(\nabla_U \phi)(U, V) = b \cdot \eta(U) \mathcal{C}$$

where  $\mathcal{C}$  is the factor determined by the class of the manifold. For instance, taking  $\mathcal{C} = \phi(U, V)$  and  $b = 1$ , we have nearly Kenmotsu. If  $b = 0$ , we get the nearly cosymplectic, nearly- $K$ -cosymplectic or closely cosymplectic structure. As in the preceding proposition, we have  $(\hat{\nabla}_U \hat{\phi})(U, V) = 0$  because  $\hat{\phi}(U, V) = \phi(U, V)$  and  $\eta(U) = 0 = \eta(V)$ . On the other hand, in the classification of almost Hermitian structures [22], the nearly Kähler structure is the only structure which is defined by  $(\hat{\nabla}_U \hat{\phi})(U, V) = 0$ .  $\square$

**Proposition 2.3.6.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is quasi- $K$ -cosymplectic, or a quasi Kenmotsu manifold, then the fibres are quasi Kählerian.*

*Proof.* As in the preceding proposition, all these manifolds have in common the following defining relation

$$(\nabla_U \phi)(V, W) + (\nabla_{\varphi U} \phi)(\varphi V, W) = \eta(V) \mathcal{C}_1 + b \eta(W) \mathcal{C}_2.$$

The proof follows by the use of the procedure of Proposition 2.3.5.  $\square$

**Proposition 2.3.7.** *Suppose that the total space of an almost contact metric submersion of type I is  $G_i$ -Sasakian or  $G_i$ -Kenmotsu, for  $i \in \{1, 2\}$ , then the fibres are  $G_i$ -manifolds.*

*Proof.* Note that for  $i = 1$ , the manifolds under consideration have in common the defining relation

$$(\nabla_U \phi)(U, W) - (\nabla_{\varphi U} \phi)(\varphi U, W) + b \eta(U) \mathcal{C} = 0.$$

If  $i = 2$ , the common defining relation is

$$\mathcal{G} \{ (\nabla_U \phi)(V, W) - (\nabla_{\varphi U} \phi)(\varphi V, W) + b \eta(V) \mathcal{C} \} = 0.$$

We can proceed as in Proposition 2.3.6 to obtain the required statement.  $\square$

Now, we shall be concerned with the manifolds defined by the codifferential of the fundamental 2-form.

Recall that in [36], O'Neill has defined two configuration tensors  $T$  and  $A$ , of the total space of a Riemannian submersion by setting

$$T_D E = \mathcal{H} \nabla_{\mathcal{V}D} \mathcal{V}E + \mathcal{V} \nabla_{\mathcal{V}D} \mathcal{H}E;$$

$$A_D E = \mathcal{V} \nabla_{\mathcal{H}D} \mathcal{H}E + \mathcal{H} \nabla_{\mathcal{H}D} \mathcal{V}E.$$

Among their fundamental properties, we have settled the following

$$T_U V = T_V U, \quad (2.1)$$

$$T_E = T_{\mathcal{V}E}, \quad (2.2)$$

$$T_X E = 0, \quad (2.3)$$

$$\mathcal{H}\nabla_U V = T_U V, \quad (2.4)$$

$$A_X Y = -A_Y X, \quad (2.5)$$

$$A_E = A_{\mathcal{H}E}, \quad (2.6)$$

$$A_V E = 0. \quad (2.7)$$

If  $X$  is basic, then

$$\mathcal{H}\nabla_U X = A_X U \quad (2.8)$$

and  $[U, X]$  is vertical.

Using the tensor  $A$ , China [13] has defined an associated tensor  $A^*$  on horizontal vector fields by setting

$$A^*(X, Y) = A_X \varphi Y - A_{\varphi X} Y,$$

and has established the following structure equations

$$\delta\phi(U) = \delta\hat{\phi}(U) + \frac{1}{2}g(\text{tr}A^*, U), \quad (2.9)$$

$$\delta\phi(X) = \delta\phi'(X_*) + g(H, \varphi X), \quad (2.10)$$

$$\delta\eta = \delta\eta' \circ \pi - g(H, \xi), \quad (2.11)$$

where,  $\text{tr}A^*$  is the trace of  $A^*$ .

$\omega$  and  $\theta$ , we have the following.

**Lemma 2.3.8.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I, then  $\omega(U) = \frac{m-m'-1}{m}\theta(U)$  if and only if  $\text{tr}A^* = 0$ .*

*Proof.* Let us recall that on the total space  $M^{2m+1}$ , the Lee form is

$$\omega(D) = \frac{1}{m} \{ \delta(\varphi D) - \eta(D)\delta\eta \}$$

and the Lee form on the fibres is given by

$$\theta(U) = \frac{1}{m - m' - 1} \delta\hat{\phi}(\varphi U).$$

Now, using equation (2.9)

$$\begin{aligned} \omega(U) &= \frac{1}{m} \delta\phi(\varphi U) = \frac{1}{m} (\delta\hat{\phi}(\varphi U) + \frac{1}{2}g(\text{tr}A^*, U)) \\ &= \frac{m - m' - 1}{m} \theta(U) + \frac{1}{2m} g(\text{tr}A^*, U) \end{aligned}$$

Thus  $\omega(U) = \frac{m-m'-1}{m}\theta(U)$  if and only if  $\text{tr}A^* = 0$ .

□

**Proposition 2.3.9.** *If the total space of an almost contact metric submersion of type I is trans-Sasakian or locally conformal cosymplectic, then the fibres are Kähler.*

*Proof.* The case of trans-Sasakian structure on the total space is treated by China [12, Thm.2.1]. The case of locally conformal cosymplectic is established in [18, Thm.4.4 page 112]. □

**Proposition 2.3.10.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is almost trans-Sasakian or a locally conformal almost cosymplectic, then the fibres are respectively  $W_2 \oplus W_4$  or almost Kähler.*

*Proof.* The case of almost trans-Sasakian is established by China [12, Thm.2.1].

Now, let us consider the case of locally conformal almost cosymplectic. In this case, the proof follows from relationships between  $d\phi$  and  $d\Omega$  on the one hand, and on the other hand, between  $\omega$  and  $\theta$ . □

**Proposition 2.3.11.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is nearly trans-Sasakian or quasi trans-Sasakian, then the fibres are nearly Kähler or  $W_1 \oplus W_2 \oplus W_4$  respectively.*

*Proof.* See China [12, Thm.2.1]. □

**Proposition 2.3.12.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is semi-cosymplectic or semi-Sasakian, then the fibres are semi-Kählerian if and only if  $\text{tr}A^* = 0$ .*

*Proof.* See China [13]. □

**Proposition 2.3.13.** *If the total space of an almost contact metric submersion of type I is a  $C_7$ -manifold, semi-Sasakian normal, semi-cosymplectic normal or semi-Kenmotsu normal, then the fibres are  $W_3$ -manifolds if and only if  $\text{tr}A^* = 0$ .*

*Proof.* If  $M$  is a  $C_7$ -manifold, for all  $U, V, W$  vertical vector fields we have that  $(\nabla_U \phi)(V, W) = 0$ , and thus  $(\hat{\nabla}_U \hat{\phi})(V, W) = 0$  and the fibres are Kähler. Remember that a  $C_7$  manifold enjoys the codifferential  $\delta = 0$ ; applying equation (2.9), we get the proof.

The semi-Sasakian normal and semi-cosymplectic normal cases are treated as follows. Since  $\eta$  vanishes on vertical vector fields, it is clear that  $\delta\phi = 0$  on a semi-Sasakian normal manifold. Thus, we have  $\delta\phi = 0$  and  $N^{(1)} = 0$  from which  $\delta\hat{\Omega} = 0$  and  $\hat{N}_J = 0$  follow respectively; showing that the fibres are  $W_3$ -manifolds. Now, the case of semi-cosymplectic normal and semi-Kenmotsu normal follow in the same procedure as in the case of semi-Sasakian normal. □

**Proposition 2.3.14.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is  $G_i$ -semi-cosymplectic  $G_i$ -semi-Sasakian or  $G_i$ -semi-Kenmotsu for  $i \in \{1, 2\}$ , then the fibres are  $W_i \oplus W_3$ -manifolds if and only if  $\text{tr}A^* = 0$ .*

*Proof.* We proceed as in the proof of Propositions 2.3.7 and 2.3.9 by adding  $\delta\phi = 0$ .  $\square$

China [11, Thm 2.2] and Watson [52, Thm 3.1] have shown, in another manner, that the total space of an almost contact metric submersion of type *I* cannot be contact, *K*-contact, Sasakian, nearly Sasakian or a quasi-*K*-Sasakian manifold. Indeed, for Sasakian, contact and *K*-contact manifolds which are defined by  $\phi = d\eta$ , the vanishing of  $\eta$  on the vertical vector fields gives  $\hat{\phi} = d\eta(U, V) = 0$ . This implies that, on the fibres, the Kähler form is identically null.

## 2.4 On the fibres of a Submersion of type *II*

**Proposition 2.4.1 ([48]).** *The fibres of an almost contact metric submersions of type *II* are almost contact metric manifolds.*

*Proof.* It is clear that the dimension of the fibres is  $2(m - m') + 1$  which is odd. Let  $(\hat{g}, \hat{\varphi}, \hat{\xi}, \hat{\eta})$  be the restriction of the almost contact metric structure  $(g, \varphi, \xi, \eta)$  of the total space to the fibres. We have to show that  $(\hat{g}, \hat{\varphi}, \hat{\xi}, \hat{\eta})$  is an almost contact metric structure. Indeed,

- (i)  $\hat{\varphi}^2 U = -U + \hat{\eta}(U)\hat{\xi}$ ;
- (ii)  $\hat{\eta}(\hat{\xi}) = \hat{g}(\hat{\xi}, \hat{\xi}) = g(\xi, \xi) = 1$ ;
- (iii)  $\hat{g}(\hat{\varphi}U, \hat{\varphi}V) = -\hat{g}(U, \hat{\varphi}^2 V) = \hat{g}(U, V) - \hat{g}(U, \hat{\eta}(V)\hat{\xi})$ ,

but  $\hat{g}(U, \hat{\eta}(V)\hat{\xi}) = \hat{g}(U, \hat{\xi})\hat{\eta}(V) = \hat{\eta}(U)\hat{\eta}(V)$ . Thus,  $\hat{g}(\hat{\varphi}U, \hat{\varphi}V) = \hat{g}(U, V) - \hat{\eta}(U)\hat{\eta}(V)$ .  $\square$

**Proposition 2.4.2.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type *II*. Then  $\omega(U) = \frac{m-m'}{m}\hat{\omega}(U)$  if and only if  $\text{tr}A^* = 0$ .*

*Proof.* Since it is a submersion of type *II*, we know that

$$\delta\phi(U) = \delta\hat{\phi}(U) + \frac{1}{2}g(\text{tr}A^*, U) \text{ and}$$

$$\delta\eta = \delta\hat{\eta} + \sum_{i=1}^m g(A_{X_i}X_i + A_{\varphi X_i}\varphi X_i, \xi)$$

where  $\{X_1, \dots, X_{m'}, \varphi X_1, \varphi X_{m'}\}$  is a local horizontal  $\varphi$ -basis.

Thus, if  $\text{tr}A^* = 0$  and  $\sum_{i=1}^m (A_{X_i}X_i + A_{\varphi X_i}\varphi X_i) = 0$ , then

$$\omega(U) = \frac{m - m'}{m}\hat{\omega}(U)$$

$\square$

**Proposition 2.4.3.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space verifies  $\delta\phi = 0$ , then the fibres verify  $\delta\hat{\phi} = 0$  if and only if  $\text{tr}A^* = 0$ .*

*Proof.* In [46, Thm 2], it is shown that

$$\delta\phi(E) = g(H, \varphi\mathcal{H}E) + \delta'\phi'(\mathcal{H}E) + \delta\hat{\phi}(\mathcal{V}E) + \frac{1}{2}g(\text{tr}A^*, \mathcal{V}E).$$

Since, from equation (2.9) which is valid also in the case of a type II submersion, we have

$$\delta\phi(U) = \delta\hat{\phi}(U) + \frac{1}{2}g(\text{tr}A^*, U)$$

so that if  $\delta\phi = 0$ , then  $\delta\hat{\phi} = 0$  if and only if  $\text{tr}A^* = 0$ .  $\square$

As a consequence of this proposition we have the following.

**Corollary 2.4.4.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is a  $C_3$  or a  $C_7$ -manifold, then the fibres inherit the structure of the total space if and only if  $\text{tr}A^* = 0$ .*

**Corollary 2.4.5.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is semi-cosymplectic, semi-Sasakian, almost trans-Sasakian or locally conformal almost cosymplectic, then the fibres inherit the structure of the total space if and only if  $\text{tr}A^* = 0$ .*

**Theorem 2.4.6.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is  $G_i$ -semi-cosymplectic,  $G_i$ -semi-Sasakian or  $G_i$ -semi-Kenmotsu, then the fibres inherit the structure of the total space if and only if  $\text{tr}A^* = 0$ .*

*Proof.* The manifolds under consideration have in common the following defining relations, apart from that of the codifferential of the fundamental 2-form  $\phi$ . We have

$$(\nabla_D\varphi)D - (\nabla_{\varphi D}\varphi)\varphi D + \eta(D)\mathcal{C} = 0,$$

for  $i = 1$ , and  $\mathcal{G}\{(\nabla_D\phi)(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) + S\} = 0$ , for  $i = 2$ . In fact, considering the first case where  $i = 1$ , if  $\mathcal{C} = (\nabla_{\varphi D}\xi)$  we get one of the defining relations of  $G_1$ -semi-cosymplectic or  $G_1$ -semi-Sasakian structures. Taking  $\mathcal{C} = \phi(D, E)$ , we go to the  $G_1$ -semi-Kenmotsu structure. Now, consider the case where  $i = 2$ . Taking  $S = \eta(E)(\nabla_{\varphi D}\eta)G$ , we get one of the defining relations of  $G_2$ -semi-Sasakian or  $G_2$ -semi-cosymplectic structure. If  $S = \eta(D)\phi(E, G)$ , we find the case of  $G_2$ -semi-Kenmotsu manifolds.

Let us consider three vector fields  $U, V$  and  $W$  tangent to the fibres. It is not difficult to show that, on the fibres, we have  $(\hat{\nabla}_U\hat{\phi})U - (\hat{\nabla}_{\varphi U}\hat{\phi})\varphi U + \hat{\eta}(U)\hat{\mathcal{C}} = 0$  for  $i = 1$  and  $\mathcal{G}\{(\hat{\nabla}_U\hat{\phi})(V, W) - (\hat{\nabla}_{\varphi U}\hat{\phi})(\varphi V, W) + \hat{S}\} = 0$  for  $i = 2$ . With the use of Proposition 2.4.1 and Corollary 2.4.4 the proof follows.  $\square$

**Proposition 2.4.7.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is semi-cosymplectic normal, semi-Sasakian normal or semi-Kenmotsu normal, then the fibres inherit the structure of the total space if and only if  $\text{tr}A^* = 0$ .*

*Proof.* In the light of the preceding theorem and the fact that  $\hat{N}^{(1)}(U, V) = N^{(1)}(U, V)$  from which  $N^{(1)} = 0$  implies  $\hat{N}^{(1)} = 0$ , we get the proof.  $\square$

According to Blair [7], it is known that almost Kähler manifolds enjoy symplectic structures. But in this study, one of the interesting problems is to know how can symplectic structures be interrelated with almost contact metric ones via the theory of almost contact metric submersions.

For this, we have already obtained the following proposition [2].

**Proposition 2.4.8.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is quasi-K-cosymplectic or quasi Kenmotsu, then the base space is a  $(1, 2)$ -symplectic manifold.*

*Proof.* Note that all these manifolds have in common the following relation

$$(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) = \alpha.\eta(D)\mathcal{C},$$

where  $\mathcal{C}$  is a factor determined by the class of the manifold. For instance, if  $\alpha = 1$  and  $\mathcal{C} = \eta(E)(\nabla_{\varphi D}\xi)$ , we get the defining relation of a quasi-K-cosymplectic structure. If  $\alpha = 1$  and  $\mathcal{C} = \eta(E)\phi(G, D) + 2\eta(G)\phi(D, E)$ , we obtain the principal defining relation of a quasi Kenmotsu structure.

Let  $X, Y$  and  $Z$  be three basic vector fields. Since  $\eta$  vanishes on horizontal vector fields, the common relation becomes

$$(\nabla_X\phi)(Y, Z) + (\nabla_{\varphi X}\phi)(\varphi Y, Z) = 0.$$

As  $\pi^*\Omega' = \phi$ , we get

$$(\nabla'_{X_*}\Omega')(Y_*, Z_*) + (\nabla'_{J'X_*}\Omega')(J'Y_*, Z_*) = 0.$$

This last relation is the defining relation of a quasi Kählerian structure on the base space.

Recalling that

$$(\nabla'_{X_*}\Omega')(Y_*, Z_*) = g'((\nabla'_{X_*}J')Y_*, Z_*)$$

and

$$(\nabla'_{J'X_*}\Omega')(J'Y_*, Z_*) = g'((\nabla'_{J'X_*}J')J'Y_*, Z_*),$$

we then get

$$g'((\nabla'_{X_*}J')Y_*, Z_*) + g'((\nabla'_{J'X_*}J')J'Y_*, Z_*) = 0$$

which is equivalent to

$$g'((\nabla'_{X_*} J')Y_*) + (\nabla'_{J'X_*} J')J'Y_*, Z_* = 0$$

from which

$$(\nabla'_{X_*} J')Y_* + (\nabla'_{J'X_*} J')J'Y_* = 0$$

follows. This is the defining relation of a  $(1, 2)$ -symplectic manifold as noted in [8].  $\square$

When studying submersions of contact  $CR$ -submanifolds, [34], we obtained the following result which is analogous to the preceding .

**Proposition 2.4.9.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be a submersion of type II of contact  $CR$ -submanifold of a quasi- $K$ -cosymplectic manifold  $\overline{M}$  onto an almost contact metric manifold. Then, the base space  $M'$  is a  $(1, 2)$ -symplectic manifold.*

**Proposition 2.4.10.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the base space is a  $(1, 2)$ -symplectic manifold, then the horizontal distribution of the total space looks like quasi Kähler manifold.*

*Proof.* Let  $X, Y$  and  $Z$  be basic vector fields. It is known that on the base space  $\pi_*X = X_*$ ,  $\pi_*Y = Y_*$  and  $\pi_*Z = Z_*$ . Consider that the base space is defined by

$$(\nabla'_{X_*} J')Y_* + (\nabla'_{J'X_*} J')J'Y_* = 0.$$

This is to say that

$$(\nabla'_{X_*} \Omega')(Y_*, Z_*) + (\nabla'_{J'X_*} \Omega')(J'Y_*, Z_*) = 0.$$

Since  $\pi^*\Omega' = \phi$ , we have,

$$\pi^*(\nabla'_{X_*} \Omega')(Y_*, Z_*) = (\nabla_X \phi)(Y, Z)$$

and

$$\pi^*(\nabla'_{J'X_*} \Omega')(J'Y_*, Z_*) = (\nabla_{\varphi X} \phi)(\varphi Y, Z),$$

which lead to

$$(\nabla_X \phi)(Y, Z) + (\nabla_{\varphi X} \phi)(\varphi Y, Z) = 0.$$

Taking into account that the horizontal distribution of an almost contact metric submersion of type II is even dimensional, the proof follows.  $\square$

# Chapter 3

## Curvature relation and the structure of the fibres

In this Chapter, we show how  $C(\alpha)$ -curvature tensors on the total space are interrelated with  $K_i$ -curvature identities on the fibres. We begin with the properties of the O'Neill tensors of configuration. These properties are useful in the study of Riemannian curvature properties and the holomorphic sectional curvature.

### 3.1 On the $\varphi$ -linearity of the O'Neill's tensors

Let us recall that  $T$  is used in the geometry of the fibres and  $A$  is the integrability tensor of the horizontal distribution.

Following Watson and Vanhecke [55], the  $\varphi$ -symmetry of a smooth tensor field  $L$  of type  $(1, 2)$  on almost contact metric manifold can be defined by

- (1)  $L$  is  $\varphi$ -linear in the first variable if  $L_{\varphi D}E = \varphi L_D E$ ;
- (2)  $L$  is  $\varphi$ -linear in the second variable if  $L_D \varphi E = \varphi L_D E$ ;
- (3)  $L$  is  $\varphi$ -symmetric if  $L_{\varphi D}E = L_D \varphi E$ .

Since  $T$  and  $A$  are smooth tensors fields of type  $(1, 2)$ , we can examine their  $\varphi$ -linearity properties.

**Proposition 3.1.1.** *Let  $\pi : M^{2m+1} \longrightarrow B$  be an almost contact metric submersion of type I or type II. If the configuration tensor  $T$  (resp.  $A$ ) is  $\varphi$ -linear in one of the variables on the vertical (resp. horizontal) distribution, then it is  $\varphi$ -linear in the other one.*

In this proposition,  $B$  means that the base space can be endowed with an almost contact or almost Hermitian structure.

*Proof.* Suppose that  $T_{\varphi U}V = \varphi T_U V$ ; we have to show that  $T_U \varphi V = \varphi T_U V$ . Indeed, since  $U$  and  $V$  are vertical, it is known that  $\varphi U$  and  $\varphi V$  are also vertical by virtue of Propositions 2.1.2 and 2.1.3. On the other hand,  $T$  is symmetric on the vertical distribution according to equation (2.1). Thus  $T_U \varphi V = T_{\varphi V} U = \varphi T_V U = \varphi T_U V$ . In the similar manner, we can show that if  $T_{\varphi V} U = \varphi T_U V$ , then  $T_{\varphi U} V = \varphi T_U V$ . Consider the configuration tensor  $A$ . Since  $X$  and  $Y$  are horizontal vector fields, then  $\varphi X$  and  $\varphi Y$  are also horizontal by virtue of Propositions 2.1.2 and 2.1.3; the fact that  $A$  is skew-symmetric on horizontal vector fields, according to equation (2.5), gives rise to  $A_{\varphi X} Y = -A_Y \varphi X = -\varphi A_Y X = -\varphi(-A_X Y) = \varphi A_X Y$ .  $\square$

From the above proposition it turns out that, on a given distribution the  $\varphi$ -linearity implies the  $\varphi$ -symmetries of these tensors.

**Corollary 3.1.2.** *Let  $\pi : M^{2m+1} \longrightarrow B$  be an almost contact metric submersion of type I or type II. If the configuration tensor  $A$  is  $\varphi$ -symmetric on the horizontal distribution, then,  $A_X \varphi X = 0$ .*

*Proof.* Since  $A_{\varphi X} Y = A_X \varphi Y = -A_{\varphi Y} X$ , taking  $Y = X$  one gets  $A_{\varphi X} X = -A_{\varphi X} X$  and the proof clearly follows.  $\square$

**Proposition 3.1.3.** *Let  $\pi : M^{2m+1} \longrightarrow B$  be an almost contact metric submersion of type I or type II. Suppose that the total space is cosymplectic, then  $A = 0$ .*

*Proof.* See Watson [52, Thms 4.4 and 4.17].  $\square$

**Corollary 3.1.4.** *Let  $\pi : M^{2m+1} \longrightarrow B$  be an almost contact metric submersion of type I or type II. If the total space is endowed with the nearly cosymplectic or closely cosymplectic structure, then:*

- (1)  $T_U \varphi U = \varphi T_U U$ ;
- (2)  $A_X \varphi X = \varphi A_X X$ .

*Proof.* Recall that the structures under consideration verify the relation  $(\nabla_D \varphi)D = 0$ . Taking a vertical vector field  $U$ , we have  $(\nabla_U \varphi)U = 0$  from which the proof of (1) is obtained. Concerning assertion (2), we consider,  $X$ , horizontal and then  $(\nabla_X \varphi)X = 0$ , gives the proof.  $\square$

**Proposition 3.1.5.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is  $\alpha$ -Kenmotsu, then:*

- (1)  $T_U \varphi X = \varphi T_U X - \alpha \cdot \eta(X) \varphi U$ ;
- (2)  $T_U V = \varphi T_U V + \alpha \cdot g(\varphi U, V) \xi$ ;
- (3)  $A_X \varphi U = \varphi A_X U$ ;

$$(4) A_X\varphi Y = \varphi A_X Y;$$

$$(5) A_\xi \xi = 0.$$

*Proof.* (1) On the total space, the condition under consideration becomes  $(\nabla_U\varphi)X = \alpha \{g(\varphi U, X)\xi - \eta(X)\varphi U\}$ . Since  $\varphi U$  is vertical and  $X$  is horizontal, then  $g(\varphi U, X) = 0$  and this implies that  $(\nabla_U\varphi)X = -\alpha\eta(X)\varphi U$ . The vertical part of this last equation gives (1). The vanishing of  $\eta$  on the vertical vector fields, as shown in Proposition 2.1.2 leads to  $(\nabla_U\varphi)V = \alpha.g(\varphi U, V)\xi$ , which gives the proof of (2) by taking its horizontal projection. Concerning assertion (3), the condition on the total space becomes

$$(\nabla_X\varphi)U = \alpha \{g(\varphi X, U)\xi - \eta(U)\varphi X\}.$$

Since, by Proposition 2.1.2,  $\eta(U) = 0$ , the condition reduces to  $(\nabla_X\varphi)U = 0$ . Taking the horizontal projection of this equation, we obtain the proof of (3). To establish (4), we have  $(\nabla_X\varphi)Y = \alpha. \{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}$ . The vertical projection of this relation gives  $\mathcal{V}(\nabla_X\varphi)Y = 0$  because  $\varphi X$  and  $\xi$  are horizontal. Therefore,  $A_X\varphi Y = \varphi A_X Y$ .

(5) Since  $\varphi\xi = 0$ , we have  $\varphi A_\xi \xi = 0$  from which  $A_\xi \xi = 0$  follows.  $\square$

**Proposition 3.1.6.** *Let  $\pi : M^{2m+1} \longrightarrow B$  be almost contact metric submersion of type I or type II. If the total space is endowed with the nearly Kenmotsu structure, then:*

$$(1) T_U\varphi U = \varphi T_U U;$$

$$(2) T_\xi \xi = 0;$$

$$(3) A_X\varphi X = 0;$$

$$(4) A_\xi \xi = 0.$$

*Proof.* Remember that a nearly Kenmotsu manifold is defined by  $(\nabla_D\varphi)D = -\eta(D)\varphi D$ . In the case of a submersion of type I, the vanishing of  $\eta$  on vertical vector fields gives  $\eta(U) = 0$  so that the defining relation becomes  $(\nabla_U\varphi)U = 0$  from which the horizontal projection gives  $T_U\varphi U = \varphi T_U U$ . Considering the case of a submersion of type II, the horizontal projection gives also  $(\nabla_U\varphi)U = 0$  because  $\varphi U$  is vertical. We then get the proof of (1). Now, let us examine assertion (2). If we have a type I submersion,  $\xi$  is horizontal (basic). In this case,  $T_\xi \xi = 0$  according to the fact that  $T_E = T_{\mathcal{V}E}$ . If we have a submersion of type II, one has  $(\nabla_\xi\varphi)\xi = 0$  from which  $\nabla_\xi \xi = 0$  follows. Since  $\xi$  is vertical, the horizontal projection of  $\nabla_\xi \xi = 0$  gives  $T_\xi \xi = 0$ . Concerning assertion (3), it is clear that the relation becomes  $(\nabla_X\varphi)X = -\eta(X)\varphi X$ . If we have a submersion of type I, since  $\varphi X$  is horizontal, the vertical projection gives  $\mathcal{V}(\nabla_X\varphi)X = 0$  from which  $A_X\varphi X = \varphi A_X X$  follows. Using equation (2.5), one gets  $A_X\varphi X = 0$ . If we have a submersion of type II, the relation gives  $(\nabla_X\varphi)X = 0$  because  $\eta(X) = 0$  and thus,  $A_X\varphi X = 0$  is deduced. Considering assertion (4). In the case of a submersion of type I, we have  $(\nabla_\xi\varphi)\xi = 0$  as above from which  $\nabla_\xi \xi = 0$  follows. Since  $\xi$  is horizontal, the vertical projection of  $\nabla_\xi \xi = 0$  gives  $A_\xi \xi = 0$ . If we have a submersion of type II, it is known that  $\xi$  is vertical so that by virtue of equation (2.7) the proof follows.  $\square$

**Proposition 3.1.7.** *Let  $\pi : M^{2m+1} \longrightarrow B$  be an almost contact metric submersion of type I or type II. If the total space is endowed with the nearly  $\alpha$ -Kenmotsu structure, then:*

- (1)  $T_U\varphi U = \varphi T_U U$ ;
- (2)  $T_\xi \xi = 0$ ;
- (3)  $A_X\varphi X = 0$ ;
- (4)  $A_\xi \xi = 0$ .

*Proof.* Recall that a nearly  $\alpha$ -Kenmotsu manifold is defined by

$$(\nabla_D\varphi)E + (\nabla_E\varphi)D = \alpha\{-\eta(E)\varphi D - \eta(D)\varphi E\}.$$

Setting  $D = E = U$  in the above relation, one gets  $(\nabla_U\varphi)U = -\alpha\eta(U)\varphi U$ , which is the relation in Proposition 3.1.6.  $\square$

**Proposition 3.1.8.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is furnished with the quasi Sasakian structure, then:*

- (1)  $T_U\varphi V = \varphi T_U V$ ;
- (2)  $T_U\xi = 0$ ;
- (3)  $A_X\varphi Y = \varphi A_X Y$ ;
- (4)  $A_X\xi = 0$ .

*Proof.* We refer to Watson [52].  $\square$

Now, let us turn to the case of almost contact metric submersions of type II.

**Proposition 3.1.9.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is  $\alpha$ -Kenmotsu then:*

- (1)  $T_U\varphi V = \varphi T_U V$ ;
- (2)  $A_X\varphi Y = \varphi A_X Y + \alpha.g(\varphi X, Y)\xi$ .

*Proof.* (1) On vertical vector fields, the condition on the total space is

$$(\nabla_U\varphi)V = \alpha.\{g(\varphi U, V)\xi - \eta(V)\varphi U\}.$$

Since, according to Proposition 2.1.3 (1) and (3), respectively,  $\varphi U$  and  $\xi$  are vertical,  $\mathcal{H}(\varphi U) = 0 = \mathcal{H}(\xi)$  which imply that  $\mathcal{H}(\nabla_U\varphi)V = 0$ . Therefore,  $T_U\varphi V = \varphi T_U V$ .

Concerning assertion(2), it is clear that  $(\nabla_X\varphi)Y = \alpha.g(\varphi X, Y)\xi$  because  $\eta(Y) = 0$ . Thus,  $\mathcal{V}(\nabla_X\varphi)Y = \mathcal{V}\alpha.g(\varphi X, Y)\mathcal{V}(\xi)$  which implies that  $A_X\varphi Y = \varphi A_X Y + \alpha.g(\varphi X, Y)\xi$ .  $\square$

**Proposition 3.1.10.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is  $\alpha$ -Sasakian, then:*

- (1)  $T_U\varphi V = \varphi T_U V$ ;
- (2)  $A_X\varphi Y = \varphi A_X Y + \alpha.g(X, Y)\xi$ .

*Proof.* (1) Using vertical vector fields  $U$  and  $V$ , the condition becomes

$$(\nabla_U\varphi)V = \alpha.\{g(U, V)\xi - \eta(V)U\}.$$

Since  $\xi$  and  $U$  are vertical, we have  $\mathcal{H}(\xi) = 0 = \mathcal{H}(U)$  so that  $\mathcal{H}(\nabla_U\varphi)V = 0$  which yields  $T_U\varphi V = \varphi T_U V$ .

For assertion (2), since  $\eta(Y) = 0$ , the condition becomes

$$(\nabla_X\varphi)Y = \alpha.g(X, Y)\xi$$

from which the vertical projection gives rise to  $A_X\varphi Y = \varphi A_X Y + \alpha.g(X, Y)\xi$ . □

**Proposition 3.1.11.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is Sasakian, then:*

- (1)  $T_U\varphi V = \varphi T_U V$ ;
- (2)  $T_U\xi = 0$ .

*Proof.* We refer to Watson [52]. □

**Proposition 3.1.12.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is almost cosymplectic, then:*

- (1)  $A_X\varphi Y = \varphi A_X Y$ ;
- (2)  $A_X\xi = 0$ .

*Proof.* We refer again to Watson [52]. □

## 3.2 Riemannian curvature properties

Recall that the Riemannian curvature tensor  $\mathcal{R}$  of a Kählerian manifold satisfies the  $K_1$ -identity, named the Kähler identity, defined by

$$\mathcal{R}(D, E, F, G) = \mathcal{R}(D, E, JF, JG). \quad (3.1)$$

Other  $K_i$ -identities ( $i = 1, 2, 3$ ) have been studied by A. Gray in [21], but their interrelations with the theory of almost Hermitian submersions can be found in [54] and [55].

Let  $(M^{2m}, g, J)$  be an almost Hermitian manifold. The  $K_i$ -curvature properties are defined in the following way.

$$\begin{aligned} K_1 : \mathcal{R}(D, E, F, G) &= \mathcal{R}(D, E, JF, JG), \\ K_2 : \mathcal{R}(D, E, F, G) &= \mathcal{R}(JD, E, JF, G) + \mathcal{R}(JD, JE, F, G) \\ &\quad + \mathcal{R}(JD, E, F, JG), \\ K_3 : \mathcal{R}(D, E, F, G) &= \mathcal{R}(JD, JE, JF, JG). \end{aligned}$$

In their study of curvature tensors of almost contact metric manifolds, D. Janssens and L. Vanhecke [28], have obtained the following properties of the Riemannian curvature tensor.

- (1) the *cosymplectic curvature property*, defined by

$$\mathcal{R}(D, E, F, G) = \mathcal{R}(D, E, \varphi F, \varphi G);$$

- (2) the *Kenmotsu curvature property*, defined by

$$\begin{aligned} \mathcal{R}(D, E, F, G) &= \mathcal{R}(D, E, \varphi F, \varphi G) + g(D, F)g(E, G) - g(D, G)g(E, F) \\ &\quad - g(D, \varphi F)g(E, \varphi G) + g(D, \varphi G)g(E, \varphi F); \end{aligned}$$

- (3) the *Sasakian curvature property*, defined by

$$\begin{aligned} \mathcal{R}(D, E, F, G) &= \mathcal{R}(D, E, \varphi F, \varphi G) - g(D, F)g(E, G) + g(D, G)g(E, F) \\ &\quad + g(D, \varphi F)g(E, \varphi G) - g(D, \varphi G)g(E, \varphi F). \end{aligned}$$

The curvature tensors of an almost contact metric manifold are called  $C(\alpha)$ -curvature tensors where  $\alpha$  is a real number. For instance, the cosymplectic curvature tensor is a  $C(0)$ -curvature tensor, the Kenmotsu curvature tensor is a  $C(-1)$ -curvature tensor and the Sasakian curvature tensor is a  $C(1)$ -curvature tensor. For more details, we refer to [28]. It is clear that the cosymplectic curvature tensor resembles to the Kähler identity.

Now, we want to determine the classes of almost contact metric manifolds satisfying the cosymplectic curvature property.

**Theorem 3.2.1 ([28]).** *Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  be an almost contact metric manifold. If  $M$  satisfies the condition*

$$(\nabla_D \varphi)E = 0,$$

*then it has the cosymplectic curvature property.*

*Proof.* For an almost contact metric manifold, the Ricci identity is given by

$$\mathcal{R}(D, E)\varphi - \varphi\mathcal{R}(D, E) = [\nabla_D, \nabla_E]\varphi - \nabla_{[D, E]}\varphi.$$

The condition on  $M$  being equivalent to  $\nabla\varphi = 0$ , the right hand side of the above relation vanishes. We get  $\mathcal{R}(D, E)\varphi F - \varphi\mathcal{R}(D, E)F = 0$  which gives

$$g(\mathcal{R}(D, E)\varphi F, \varphi G) = g(\varphi\mathcal{R}(D, E)F, \varphi G) = -g(\mathcal{R}(D, E)F, \varphi^2 G)$$

from which we get

$$g(\mathcal{R}(D, E)\varphi F, \varphi G) = -g(\mathcal{R}(D, E)F, -G) - g(\mathcal{R}(D, E)F, \eta(G)\xi).$$

It remains to show that  $g(\mathcal{R}(D, E)F, \eta(G)\xi) = 0$ . Indeed,  $g(\mathcal{R}(D, E)F, \eta(G)\xi) = g(\mathcal{R}(D, E)F, \xi)\eta(G)$ , but

$$g(\mathcal{R}(D, E)F, \xi)\eta(G) = \mathcal{R}(D, E, F, \xi) = -\mathcal{R}(D, E, \xi, F) = -g(\mathcal{R}(D, E)\xi, F).$$

Since,  $\nabla_D \xi = 0$ , we get  $\mathcal{R}(D, E)\xi = 0$  from which we deduce  $g(\mathcal{R}(D, E)F, \xi) = 0$  so that  $g(\mathcal{R}(D, E)\varphi F, \varphi G) = g(\mathcal{R}(D, E)F, G)$ , hence  $\mathcal{R}(D, E, \varphi F, \varphi G) = \mathcal{R}(D, E, F, G)$  follows immediately.  $\square$

**Theorem 3.2.2.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. Suppose that the total space satisfies the condition*

$$(\nabla_D \varphi)E = 0,$$

*then the base space verifies the cosymplectic curvature property and, on the fibres, this property corresponds to the Kähler identity.*

*Proof.* Let  $X$  and  $Y$  be basic vector fields, it is known that  $\mathcal{H}(\nabla_X \varphi)Y$  is basic associated to  $(\nabla'_{X_*} \varphi')Y_*$ . Thus, since  $(\nabla_X \varphi)Y = 0$ , one deduces that  $(\nabla'_{X_*} \varphi')Y_* = 0$ . Therefore, according to the preceding Theorem 3.2.1, the base space verifies the cosymplectic curvature property. Now, consider the vector fields  $U, V, W$  and  $S$  tangent to the fibres. For a Riemannian submersion, the Gauss equation is given by

$$\mathcal{R}(U, V, W, S) = \hat{\mathcal{R}}(U, V, W, S) - g(T_U W, T_V S) + g(T_V W, T_U S) \quad (3.2)$$

This equation can be transformed in

$$\mathcal{R}(U, V, \varphi W, \varphi S) = \hat{\mathcal{R}}(U, V, \hat{\varphi} W, \hat{\varphi} S) - g(T_U \varphi W, T_V \varphi S) + g(T_V \varphi W, T_U \varphi S). \quad (3.3)$$

Since  $T$  is  $\varphi$ -linear in the second variable, as shown in Proposition 3.1.3, we have

$$\begin{aligned} g(T_U\varphi W, T_V\varphi S) &= g(\varphi T_U W, \varphi T_V S) = -g(T_U W, \varphi^2 T_V S) \\ &= g(T_U W, T_V S) - g(T_U W, \eta(T_V S)\xi); \end{aligned}$$

Also  $\eta(T_V S) = g(\xi, T_V S) = -g(S, T_V \xi) = 0$ , since  $T_V \xi = 0$ . Thus,

$$g(T_U\varphi W, T_V\varphi S) = g(T_U W, T_V S),$$

$g(T_V\varphi W, T_U\varphi S) = g(T_V W, T_U S)$ . In this case, (3.3) leads to

$$\mathcal{R}(U, V, \varphi W, \varphi S) = \hat{\mathcal{R}}(U, V, \hat{\varphi} W, \hat{\varphi} S) - g(T_U W, T_V S) + g(T_V W, T_U S). \quad (3.4)$$

Subtracting (3.3) from (3.2), we get,

$$R(U, V, W, S) - R(U, V, \varphi W, \varphi S) = \hat{\mathcal{R}}(U, V, W, S) - \hat{\mathcal{R}}(U, V, \hat{\varphi} W, \hat{\varphi} S).$$

Since  $R(U, V, W, S) = R(U, V, \varphi W, \varphi S)$ , then  $\hat{\mathcal{R}}(U, V, W, S) = \hat{\mathcal{R}}(U, V, \hat{\varphi} W, \hat{\varphi} S)$  which shows that the fibres have the  $K_1$ -curvature identity.  $\square$

The above theorem can be viewed as a way to establish the following.

**Corollary 3.2.3.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. Suppose the following conditions satisfied*

- (1) *the total space satisfies the cosymplectic curvature property,*
- (2) *the configuration tensor  $T$  is  $\varphi$ -linear on the vertical distribution,*
- (3)  *$T_U \xi = 0$  for all vertical vector fields  $U$ .*

*Then the fibres have the Kähler identity.*

**Theorem 3.2.4.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I satisfying the following conditions*

- (1) *the total space satisfies the Kenmotsu curvature property,*
- (2) *the configuration tensor  $T$  is  $\varphi$ -linear on the vertical distribution,*
- (3)  *$T_U \xi = 0$  for all vertical vector fields  $U$ .*

*Then the fibres have the  $K_2$ -curvature identity.*

*Proof.* Since  $T$  is  $\varphi$ -linear and  $T_U\xi = 0$ , by calculation we get  $g(T_U\varphi W, T_V\varphi S) = g(T_UW, T_VS)$  and  $g(T_{\varphi U}\varphi W, T_VS) = -g(T_UW, T_VS)$ . By virtue of the Kenmotsu curvature property, we have

$$\mathcal{R}(U, V, W, S) = \mathcal{R}(U, V, \varphi W, \varphi S) + \mathcal{R}(\varphi U, V, \varphi W, S) + \mathcal{R}(\varphi U, V, W, \varphi S). \quad (3.5)$$

So, the Gauss equation gives

- (i)  $\mathcal{R}(U, V, \varphi W, \varphi S) = \hat{\mathcal{R}}(U, V, \hat{\varphi}W, \hat{\varphi}S) - g(T_UW, T_VS) + g(T_VW, T_US)$ ,
- (ii)  $\mathcal{R}(\varphi U, V, \varphi W, S) = \hat{\mathcal{R}}(\hat{\varphi}U, V, \hat{\varphi}W, S) + g(T_UW, T_VS) + g(T_VW, T_US)$ ,
- (iii)  $\mathcal{R}(\varphi U, V, W, \varphi S) = \hat{\mathcal{R}}(\hat{\varphi}U, V, W, \hat{\varphi}S) - g(T_UW, T_VS) - g(T_VW, T_US)$ .

Therefore, summing (i), (ii) and (iii), we obtain a relation yielding to

$$\hat{\mathcal{R}}(U, V, W, S) = \hat{\mathcal{R}}(U, V, \hat{\varphi}W, \hat{\varphi}S) + \hat{\mathcal{R}}(\hat{\varphi}U, V, \hat{\varphi}W, S) + \hat{\mathcal{R}}(\hat{\varphi}U, V, W, \hat{\varphi}S),$$

which shows that the fibres verify the  $K_2$ -curvature identity.  $\square$

**Proposition 3.2.5.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. Suppose that the total space satisfies the conditions  $(\nabla_D\varphi)D = 0$ , and  $\nabla_D\xi = 0$ , then the fibres possess the Kähler identity.*

*Proof.* Let us recall that the conditions under consideration are the defining relations of a nearly- $K$ -cosymplectic structure. In [11], Chinea has proved that if the total space of an almost contact metric submersion of type I is nearly- $K$ -cosymplectic, then the configuration tensor,  $T$ , satisfies  $T_U\varphi V = T_{\varphi U}V = \varphi T_UV$  and  $T_U\xi = 0$ . On the other hand, it is known that for a submersion of this class, the fibres are nearly Kähler [12]. Since the total space has the cosymplectic curvature property, then the fibres have the Kähler identity and then are Kähler.  $\square$

**Proposition 3.2.6.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. Suppose that the total space satisfies the condition*

$$(\nabla_D\phi)(D, E) = \alpha.\eta(D)\phi(E, D),$$

*then, the fibres possess the Kähler identity.*

*Proof.* If  $\alpha = 0$ , then the given condition reduces to  $(\nabla_D\phi)(D, E) = 0$  which leads to the case that the fibres are nearly Kähler. According to the preceding Proposition 3.2.5, the fibres possess the  $NK_1$ -identity which is the Kähler one. Suppose that  $\alpha \neq 0$ , the vanishing of  $\eta$  on vertical vector fields gives rise to  $(\nabla_U\phi)(U, V) = 0$  from which the proof follows as in the case where  $\alpha = 0$ .  $\square$

Considering the fact that, a nearly  $\alpha$ -Kenmotsu manifold is also defined by the relation

$$(\nabla_D\phi)(D, E) = \alpha.\eta(D)\phi(E, D);$$

the above Proposition 3.2.6 can be replaced by the following.

**Theorem 3.2.7.** *The fibres of a nearly  $\alpha$ -Kenmotsu submersion of type I verify the Kähler identity.*

Now, we are going to examine the analogous properties in the case of almost contact metric submersions of type II.

**Theorem 3.2.8.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. Suppose that the condition*

$$(\nabla_D \varphi)E = 0,$$

*is verified on the total space, then the fibres have the cosymplectic curvature property, which corresponds to the  $K_1$ -curvature identity on the base space.*

*Proof.* By Theorem 3.2.1, the total space has the cosymplectic curvature property. As in Theorem 3.2.2, setting  $D = U$  and  $E = V$  in the given condition on the total space, we obtain  $(\hat{\nabla}_U \hat{\varphi})V = 0$ , hence the fibres have the cosymplectic curvature property. To see that the base space verifies the Kähler identity, let  $X, Y, Z$  and  $P$  be basic vector fields. As in Theorem 3.2.2, the Gauss equation is

$$\mathcal{R}(X, Y, Z, P) = \mathcal{R}'(X_*, Y_*, Z_*, P_*) - 2g(A_X Y, A_Z P) + g(A_Y Z, A_X P) + g(A_X \varphi Z, A_Y \varphi P).$$

This equation gives

$$\begin{aligned} \mathcal{R}(X, Y, \varphi Z, \varphi P) &= \mathcal{R}'(X_*, Y_*, \varphi' Z_*, \varphi' P_*) - 2g(A_X Y, A_{\varphi Z} \varphi P) \\ &\quad + g(A_Y \varphi Z, A_X \varphi P) + g(A_X \varphi Z, A_Y \varphi P). \end{aligned}$$

Taking account into the fact that, in the context of Proposition 3.1.3, the configuration tensor  $A$  is  $\varphi$ -linear on the horizontal distribution and  $A_X \xi = 0$ , then this last equation can be rewritten in the following way

$$\begin{aligned} \mathcal{R}(X, Y, \varphi Z, \varphi P) &= \mathcal{R}'(X_*, Y_*, \varphi' Z_*, \varphi' P_*) - 2g(A_X Y, A_Z P) \\ &\quad + g(A_Y Z, A_X P) + g(A_X Z, A_Y P). \end{aligned}$$

Thus, subtracting this last equation from the Gauss equation, we get

$$\mathcal{R}(X, Y, Z, P) - \mathcal{R}(X, Y, \varphi Z, \varphi P) = \mathcal{R}'(X_*, Y_*, Z_*, P_*) - \mathcal{R}'(X_*, Y_*, \varphi' Z_*, \varphi' P_*) \quad (3.6)$$

Since the total space satisfies the cosymplectic curvature property, then  $\mathcal{R}(X, Y, Z, P) = \mathcal{R}(X, Y, \varphi Z, \varphi P)$ , which implies that  $\mathcal{R}'(X_*, Y_*, Z_*, P_*) = \mathcal{R}'(X_*, Y_*, \varphi' Z_*, \varphi' P_*)$ . The base space being an almost Hermitian manifold, it follows that it has the  $K_1$ -curvature identity for which  $\varphi' = J$ .  $\square$

**Theorem 3.2.9.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II which satisfies the following conditions:*

- (1) *the total space verifies the cosymplectic curvature property,*
- (2) *the configuration tensor  $A$  is  $\varphi$ -linear on the horizontal distribution, and*
- (3)  *$A_X \xi = 0$ , for all horizontal vector fields.*

*Then, the fibres have the cosymplectic curvature property and, on the base space, this property corresponds to the Kähler identity.*

*Proof.* Similar to the preceding.  $\square$

**Theorem 3.2.10.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II such that*

- (1) *the total space has the Sasakian curvature property,*
- (2) *the configuration tensor  $A$  is  $\varphi$ -linear on the horizontal distribution, and*
- (3)  *$A_X \xi = 0$  for all horizontal vector fields  $X$ .*

*Then, the fibres have the Sasakian curvature property which corresponds to the  $K_3$ -curvature identity on the base space.*

*Proof.* Similar to that of Theorem 3.2.4.  $\square$

### 3.3 Holomorphic sectional curvature

Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  be an almost contact metric manifold,  $D$  and  $E$  two vector fields orthogonal to  $\xi$ . The holomorphic bisectional curvature tensor is defined in [11] and [52] by setting

$$B_\varphi(D, E) = \|D\|^{-2} \|E\|^{-2} g(\mathcal{R}(D, \varphi D)E, \varphi E).$$

Letting  $D = E$  in the above formula, one gets the definition of the  $\varphi$ -holomorphic sectional curvature tensor which is

$$H_\varphi(E) = \|E\|^{-4} g(\mathcal{R}(E, \varphi E)E, \varphi E).$$

**Theorem 3.3.1 ([52]).** *Let  $\pi : M^{2m+1} \longrightarrow B$  be an almost contact metric submersion of type I or type II. Then the  $\varphi$ -holomorphic sectional curvature tensor is given by:*

- (1)  $H_\varphi(U) = H_{\hat{\varphi}}(U) + \|U\|^{-4} \{ \|T_U \varphi U\|^2 - g(T_U U, T_{\varphi U} \varphi U) \},$
- (2)  $H_\varphi(X) = H_{\varphi'}(X_*) - 3 \|X\|^{-4} \|A_X \varphi X\|^2.$

*Proof.* See Watson [52].  $\square$

**Lemma 3.3.2.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type I. If the configuration tensors  $T$  and  $A$  are  $\varphi$ -linear in the second variable, then:*

- (1)  $B_\varphi(U, V) = B_{\hat{\varphi}}(U, V) + 2 \|U\|^{-2} \|V\|^{-2} \|T_U V\|^2;$
- (2)  $H_\varphi(U) = H_{\hat{\varphi}}(U) + 2 \|U\|^{-4} \|T_U U\|^2;$
- (3)  $B_\varphi(X, Y) = B_{\varphi'}(X_*, Y_*);$
- (4)  $H_\varphi(X) = H_{\varphi'}(X_*).$

*Proof.* We know that if  $T_U\varphi V = \varphi T_U V$  then  $T_{\varphi U}\varphi V = -T_U V$ .  
Therefore,

$$g(T_U V, T_{\varphi U}\varphi V) = -g(T_U V, T_U V).$$

On the other hand

$$\begin{aligned} g(T_U\varphi V, T_{\varphi U}V) &= g(\varphi T_U V, \varphi T_U V) \\ &= -g(T_U V, \varphi^2 T_U V) \\ &= g(T_U V, T_U V). \end{aligned}$$

Thus,

$$\begin{aligned} g(T_U\varphi V, T_{\varphi U}V) - g(T_U V, T_{\varphi U}\varphi V) &= 2g(T_U V, T_U V) \\ &= 2\|T_U V\|^2, \end{aligned}$$

which is the proof of (1) from which (2) is a consequence.

Since  $A_X\varphi Y = \varphi A_X Y$ , then from Corollary 3.1.2, one has  $A_X\varphi X = 0$ . On the other hand,

$$\begin{aligned} g(\varphi A_X Y, \varphi A_X Y) &= -g(A_X Y, \varphi^2 A_X Y) \\ &= g(A_{\varphi X} Y, A_{\varphi X} Y) \end{aligned}$$

But  $\varphi^2 A_X Y = -A_X Y + \eta(A_X Y)\xi$ , since  $A_X\xi = 0$  then  $\eta(A_X Y)\xi = 0$ . Therefore

$$g(A_{\varphi X} Y, A_{\varphi X} Y) = \|A_X Y\|^2$$

and then

$$-g(A_{\varphi X} Y, A_{\varphi X} Y) = -\|A_X Y\|^2.$$

In the same way,

$$-g(A_X Y, A_{\varphi X}\varphi Y) = -\|A_X Y\|^2$$

which leads to the proof of (3) from which (4) is a consequence.  $\square$

# Chapter 4

## Minimality, superminimality and umbilicity of the fibres

The main objective of this chapter is to investigate specific properties which impact the geometry of the base space.

### 4.1 Submersions with minimal fibres

It is known that an invariant submanifold of a Riemannian manifold is minimal if its mean curvature vector field,  $H$ , is null. On the other hand, the mean curvature vector field is the trace of the second fundamental form; since the O'Neill configuration tensor  $T$ , is the second fundamental form of the fibres of a Riemannian submersion, it is clear that the minimality of the fibres is related to the properties of  $T$ .

In [20], A. Gray had shown that the minimality of a submanifold of an almost Hermitian manifold can be established by using the O'Neill configuration tensor  $T$ .

On the other hand, Watson and Vanhecke [55] have shown that if  $T$  is  $J$ -symmetric on vertical vector fields, then the fibres are minimal. In a similar way, it can be shown that, for an almost contact metric submersion, if  $T$  is  $\varphi$ -symmetric on the vertical distribution, then the fibres are minimal submanifolds. Furthermore, the property of being  $\varphi$ -symmetric derives from the  $\varphi$ -linearity.

In this subsection, we refer to the  $\varphi$ -linearity of the O'Neill's tensors and determine the classes of submersions with minimal fibres.

**Proposition 4.1.1.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is defined by*

- (1)  $(\nabla_D \varphi)E = 0$ , or
- (2)  $d\phi = 0$  and  $(N^{(1)} = 0$  or  $d\eta = 0)$ .

Then, the fibres are minimal.

*Proof.* Combine Proposition 3.1.3 and Corollary 3.1.4.  $\square$

The above proposition is a generalization of the following.

**Proposition 4.1.2.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is cosymplectic, closely cosymplectic, nearly cosymplectic, nearly-K-cosymplectic, nearly Kenmotsu, quasi-Sasakian or almost cosymplectic, then the fibres are minimal.*

In the case of almost contact metric submersions of type II, we have

**Theorem 4.1.3.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the total space is defined by*

- (1)  $(\nabla_D \varphi)E = \alpha \{g(\varphi D, E)\xi - \eta(E)\varphi D\}$ , or
- (2)  $N^{(1)} = 0$  and  $(d\phi = 0$  or  $\phi = d\eta)$ .

Then, the fibres are minimally embedded.

*Proof.* It follows from Propositions 3.1.9, 3.1.10 and 3.1.11.  $\square$

The analogous of Proposition 4.1.2 is the following.

**Proposition 4.1.4.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the total space is cosymplectic, closely cosymplectic, nearly cosymplectic, nearly-K-cosymplectic, nearly Kenmotsu, nearly  $\alpha$ -Kenmotsu, Sasakian, quasi-Sasakian or nearly Sasakian, then the fibres are minimal.*

## 4.2 Implications of minimality

**Lemma 4.2.1.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. Then  $\pi^*\omega' = \frac{m}{m'}\omega$  if and only if the fibres are minimal.*

*Proof.* Let  $X$  be a basic vector field. We have

$$(\pi^*\omega')(X) = \frac{1}{m'}[\delta\phi(X) - g(H, \varphi X) - (\delta\eta + g(H, \xi))(\eta(X))].$$

Thus  $(\pi^*\omega') = \frac{m}{m'}\omega$  if and only if the fibres are minimal.  $\square$

**Lemma 4.2.2.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. Then  $\theta' = \frac{m}{m'-1}\omega$  if and only if the fibres are minimal.*

*Proof.* As in the proof of the preceding lemma since  $\eta$  vanishes on horizontal vector fields.  $\square$

**Proposition 4.2.3.** *If the total space of an almost contact metric submersion of type I is trans-Sasakian or locally conformal cosymplectic, then the base space is respectively cosymplectic or locally conformal cosymplectic if, and only, if the fibres are minimal.*

*Proof.* The case of trans-Sasakian is treated in China [12, Thm.2.1].

Concerning the case of locally conformal cosymplectic, we refer to (China, Marrero and Rocha in Annales Fac. Sc. Toulouse, 4, 473-517 (1995)).  $\square$

**Proposition 4.2.4.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1'}$  be an almost contact metric submersion of type I. If the total space is almost trans-Sasakian or nearly trans-Sasakian, then the base space is respectively almost cosymplectic or nearly-K-cosymplectic if, and only, if the fibres are minimal.*

*Proof.* China [12, Thm.2.1].  $\square$

**Proposition 4.2.5.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1'}$  be an almost contact metric submersion of type I. If the total space is quasi trans-Sasakian, then the base space is quasi-K-cosymplectic if, and only if the fibres are minimal.*

*Proof.* See again [12, Thm.2.1].  $\square$

**Proposition 4.2.6.** *If the total space of a type I almost contact metric submersion is semi-cosymplectic or semi-Sasakian, then the base space inherits the structure of the total space if and only if the fibres are minimal.*

*Proof.* See China [13].  $\square$

**Proposition 4.2.7.** *If the total space of a type I almost contact metric submersion is semi-cosymplectic normal, semi-Sasakian normal or semi-Kenmotsu normal, then the base space inherits the structure of the total space if and only if the fibres are minimal.*

*Proof.* Note that the manifolds under consideration have in common the relation  $N^{(1)} = 0$  which corresponds to  $\hat{N}_J = 0$  on the fibres. As in the proof of Proposition 4.2.6, the proof follows.  $\square$

**Proposition 4.2.8.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1'}$  be an almost contact metric submersion of type I. If the total space is  $G_i$ -semi-cosymplectic,  $G_i$ -semi-Sasakian or  $G_i$ -semi-Kenmotsu, then the base space inherits the structure of the total space if and only if the fibres are minimal.*

*Proof.* As in the previous propositions.  $\square$

Looking through all the preceding results related to the transference of the structure from the total to the base space, we can summarize them in the following.

**Theorem 4.2.9 ([49]).** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m+1}$  be an almost contact metric submersion of type I. If among the defining relations of the total space there is the codifferential  $\delta\phi$  or  $\delta\eta$ , then the base space inherits the structure of the total space if and only if the fibres are minimal.*

Now, let us look at the submersions of type II. The implication of minimality of the fibres concerns the structure of the base space.

**Proposition 4.2.10.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the total space is semi-cosymplectic or semi-Sasakian, then the base space is semi-Kählerian if and only if the fibres are minimal.*

*Proof.* Let  $E$  be an arbitrary vector field. In [13], it is shown that

$$\delta\phi(E) = g(H, \varphi\mathcal{H}E) + \delta'\phi'(\mathcal{H}E_*) + \delta\hat{\phi}(\mathcal{V}E) + \frac{1}{2}(\text{tr}A^*, \mathcal{V}E)$$

Let  $X$  be a horizontal basic vector field, then the above equation becomes

$\delta\phi(X) = g(H, \varphi\mathcal{H}X) + \delta'\phi'(\mathcal{H}X_*) + \delta\hat{\phi}(\mathcal{V}X) + \frac{1}{2}(\text{tr}A^*, \mathcal{V}X)$  Since the vertical projection  $\mathcal{V}(X) = 0$ , the formula then reduces to

$\delta\phi(X) = g(H, \varphi\mathcal{H}X) + \delta'\phi'(\mathcal{H}X_*)$ . With Proposition 2.1.3 in mind,  $\pi^*\Omega = \phi$ . If the total space is semi-cosymplectic then  $\delta\phi = 0$  and  $\delta\eta = 0$ . In this case,  $0 = g(H, \varphi\mathcal{H}X) + \delta'\phi'(\mathcal{H}X_*)$  which shows that  $\delta'\phi' = 0$  if and only if  $H = 0$ ; but from Proposition 2.1.3(1), on the base space which is almost Hermitian,  $\delta'\phi' = \delta'\Omega'$ . If the total space is semi-Sasakian then  $\eta = \frac{1}{2m}\delta\phi$ . The vanishing of  $\eta$  on horizontal vector fields leads to  $\delta\phi(X) = 0$  which means that on the base space, we have  $\delta\Omega = 0$ .  $\square$

**Proposition 4.2.11.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the total space is almost trans-Sasakian or locally conformal almost cosymplectic, then the base space is locally conformal almost Kählerian if and only if the fibres are minimal.*

*Proof.* The under consideration manifolds have in common the following property  $d\phi = b\phi \wedge \theta$  as shown at page 27. Remember that a locally conformal almost Kähler structure is defined by  $d\Omega = \Omega \wedge \theta$ . In this case, the proof follows from relationships between  $d\phi$  and  $d\Omega$  on the one hand, and on the other hand, between  $\omega$  and  $\theta$ .  $\square$

**Proposition 4.2.12.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the total space is  $G_2$ -semi-cosymplectic,  $G_2$ -semi-Sasakian or  $G_2$ -semi-Kenmotsu, then the base space is a  $W_2 \oplus W_3$ -manifold if and only if the fibres are minimal.*

*Proof.* Let us indicate that all these manifolds have in common the following relations  $\mathcal{G} \{(\nabla_D)\phi(E, G) - (\nabla_{\varphi D}\phi)(\varphi E, G) + \mathcal{C}\} = 0$  and  $\delta\phi = 0$ , where  $\mathcal{C}$  is determined by the class of the manifold. For instance, if  $\mathcal{C} = \eta(D)\phi(E, G)$ , we get one of the defining relations of a  $G_2$ -semi-Kenmotsu structure. Taking  $\mathcal{C} = \eta(E)(\nabla_{\varphi D}\eta)G$ , one gets one of the defining relations of  $G_2$ -semi-Sasakian or  $G_2$ -semi-cosymplectic structure. Let  $X, Y$  and  $Z$  be basic vector fields. It is known that  $\varphi X, \varphi Y$  and  $\varphi Z$  are basic associated to  $J'X_*, J'Y_*$  and  $J'Z_*$  respectively. Since  $\pi^*\Omega' = \phi$  and the vanishing of  $\eta$  on horizontal vector fields, then on the base space we have  $\mathcal{G} \{(\nabla'_{X_*}\Omega')(Y_*, Z_*) - (\nabla'_{J'X_*}\Omega')(J'Y_*, Z_*)\} = 0$ . On the other hand, since  $\delta\phi = 0$ , equation (2.10) implies that  $\delta\Omega'(X_*) = 0$  if and only if  $g(H, \varphi X) = 0$  which implies that  $H = 0$ . Therefore, the base space is defined by  $\mathcal{G} \{(\nabla'_{X_*}\Omega')(Y_*, Z_*) - (\nabla'_{J'X_*}\Omega')(J'Y_*, Z_*)\} = 0$  and  $\delta\Omega' = 0$  if and only if  $H = 0$ .  $\square$

**Proposition 4.2.13.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the total space is  $G_1$ -semi-cosymplectic,  $G_1$ -semi-Sasakian or  $G_1$ -semi-Kenmotsu, then the base space is  $W_1 \oplus W_3$ -manifold if and only if the fibres are minimal.*

*Proof.* We can proceed as in the preceding.  $\square$

**Proposition 4.2.14.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the total space is semi-cosymplectic normal, semi-Sasakian normal or semi-Kenmotsu normal, then the base space is  $W_3$ -manifold if and only if the fibres are minimal.*

*Proof.* It is known that  $\pi_*N^{(1)} = N'_J$ . Using the fact that  $N^{(1)} = 0$  for all these manifolds, we deduce  $N'_J = 0$  on the base space. From Proposition 4.2.10, where it is shown that  $\delta\Omega' = 0$ , we then get the proof.  $\square$

**Proposition 4.2.15.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the total space is nearly trans-Sasakian, then the base space is  $W_1 \oplus W_4$ -manifold if and only if the fibres are minimal.*

*Proof.* As in the case of Proposition 4.2.12, we have

$$(\nabla_X\phi)(X, Y) = \frac{-1}{2m} \{ \|X\|^2 \delta\phi(Y) - g(X, Y)\delta\phi(X) - g(\varphi X, Y)\delta\phi(\varphi X) \}.$$

On the base space, using equation (2.10) and the fact that  $\pi^*\Omega' = \phi$ , the above relation leads to the defining relation of a  $W_1 \oplus W_4$ -manifold.  $\square$

### 4.3 Submersions with superminimal fibres

Superminimal fibres of a Riemannian submersion have been introduced by M. Falcitelli and A. M. Pastore [19], who examined only the case of almost Kähler submersions. On the other hand, B. Watson [53], studied extensively superminimal fibres of an almost

Hermitian submersion. He used this property to derive the structure of the total space according to that of the base space.

In [45], we extended the definition of superminimal submanifolds to the  $\varphi$ -invariant fibres of almost contact metric manifolds, considering submersions whose total space is a nearly  $\alpha$ -Kenmotsu manifold. There, we showed that if the fibres of an almost contact metric submersion with total space a nearly  $\alpha$ -Kenmotsu manifold are superminimal, then the horizontal distribution is completely integrable. Furthermore, in [46], the study was extended to the case of almost contact metric submersions with a non specified total space.

In this subsection, we have examined the case where the total space is a  $G_i$ -almost contact metric manifold for  $i \in \{1, 2\}$ , a Chinea-Gonzalez manifold such as  $C_3, C_7, C_8, C_9$  and  $C_{10}$ ; generalized Kenmotsu, semi Kenmotsu normal and quasi Kenmotsu are also treated.

Now we want to examine the superminimality of the fibres. We would like to begin by investigating the classes of almost contact metric submersions whose fibres are, or are not, superminimal in a natural way.

Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  be an almost contact metric manifold and  $\bar{M}$  a  $\varphi$ -invariant submanifold of  $M$ . If,  $\nabla_V \varphi = 0$  for all  $V$  tangent to  $\bar{M}$ , then  $\bar{M}$  is said to be superminimal.

In order to verify the superminimality of the fibres of an almost contact metric submersion of type  $I$ , there are four components of  $g(\nabla_V \varphi)D, E$  to be considered on the total space  $M$ . From [45] and [46], we recall that

$$\text{SM-1) } g((\nabla_V \varphi)U, W) = g(\hat{\nabla}_V(\hat{J}U) - \hat{J}\hat{\nabla}_V U, W),$$

$$\text{SM-2) } g((\nabla_V \varphi)U, X) = g(T_V(\varphi U) - \varphi(T_V U), X),$$

$$\text{SM-3) } g((\nabla_V \varphi)X, U) = -g((\nabla_V \varphi)U, X),$$

$$\text{SM-4) } g((\nabla_V \varphi)X, Y) = -g(A_{\varphi X} Y + A_X(\varphi Y), V).$$

In the case of an almost contact metric submersion of type  $II$ , we easily find

$$\text{SM-5) } g((\nabla_V \varphi)U, W) = g(\hat{\nabla}_V(\hat{\varphi}U) - \hat{\varphi}\hat{\nabla}_V U, W),$$

$$\text{SM-6) } g((\nabla_V \varphi)U, X) = g(T_V(\varphi U) - \varphi(T_V U), X),$$

$$\text{SM-7) } g((\nabla_V \varphi)X, U) = -g((\nabla_V \varphi)U, X),$$

$$\text{SM-8) } g((\nabla_V \varphi)X, Y) = -g(A_{\varphi X} Y + A_X(\varphi Y), V).$$

It is clear that  $SM-1)$  implies that if the fibres are superminimal, then they are Kähler.

**Proposition 4.3.1.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type  $I$ . If the total space is cosymplectic, a  $C_{11}$  or a  $C_{12}$ -manifold, then the fibres are superminimal.*

*Proof.* The case of a cosymplectic submersion is obvious. Let us consider the case of a  $C_{11}$ -submersion. Since the contact 1-form  $\eta$  vanishes on vertical vector fields, we have  $(\nabla_U\phi)(E, G) = 0$ . It is known that  $(\nabla_U\phi)(E, G) = g(E, (\nabla_U\varphi)G)$  which leads to  $g(E, (\nabla_U\varphi)G) = 0$ . According to the non-degeneracy of  $g$ , we deduce  $(\nabla_U\varphi)G = 0$  which shows that the fibres are superminimal. We apply the same procedure for a  $C_{12}$ -submersion.  $\square$

**Proposition 4.3.2.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II. If the total space is cosymplectic, then the fibres are superminimal.*

*Proof.* Since the total space is cosymplectic, obviously we have that, the four expressions  $SM - 5)$ ,  $SM - 6)$ ,  $SM - 7)$  and  $SM - 8)$ , vanish. Then the fibres are superminimal.  $\square$

**Proposition 4.3.3.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'}$  be an almost contact metric submersion of type II with  $M$ , either  $C_{11}$  or  $C_{12}$ , but not cosymplectic. Then the fibres cannot be superminimal.*

*Proof.* We first note that the base space of an almost contact metric submersion of type II with a  $C_{11}$  or  $C_{12}$  total space is Kähler. The fundamental 1-form,  $\eta$ , on  $M$  vanishes on the horizontal distribution, so the defining relations for  $C_{11}$  or  $C_{12}$ - manifolds imply that

$$(\nabla_X\phi)(E, G) = g(E, (\nabla_X\varphi)G).$$

Thus,  $(\nabla_X\varphi)G = 0$  for all horizontal vector fields  $X$ . Then, if the fibres are taken to be superminimal, we have  $\nabla_U\varphi = 0$ , contradicting the non-cosymplectic nature of the total space,  $M$ .  $\square$

**Proposition 4.3.4.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is a Kenmotsu manifold, then the fibres cannot be superminimal.*

*Proof.* Suppose that the fibres are superminimal. This means that  $\nabla_U\varphi = 0$  for all vector fields  $U$  tangent to the fibres. But on Kenmotsu manifold we have  $0 = g((\nabla_U\varphi)\varphi U, \xi) = g(\varphi U, \varphi U)g(\xi, \xi) = \|U\|^2$ . If  $\|U\|^2 = 0$  then  $U = 0$  which is not true. Thus, the fibres cannot be superminimal.  $\square$

## 4.4 Some implications of superminimality

In this subsection, we will present some implications of the superminimality of the fibres.

We begin by considering some minors remarks before going to the study of the integrability of the horizontal distribution. We end this subsection with the transference of the structure from the base to the total space when the fibres are superminimal.

If the fibres of an almost contact metric submersion of type  $I$  are superminimal, then the vanishing of expression  $SM - 4)$  yields  $A_{\varphi X}Y = -A_X\varphi Y$ . In this case,  $A^*(X, Y) = 2A_X\varphi Y$ .

**Proposition 4.4.1.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I such that on the total space,  $\delta\eta = 0$ . If the fibres are superminimal, then  $\delta\eta' = 0$  on the base space.*

*Proof.* Since the fibres are superminimal, the vanishing of  $SM - 1$ ) implies that they are Kähler. In this case, they are minimal. Using equation (2.11), it is clear that  $\delta\eta' = 0$  because  $\delta\eta = 0$ .  $\square$

Now, let us consider the integrability of the horizontal distribution.

Recall that the horizontal distribution of a Riemannian submersion is said to be integrable if the O'Neill tensor  $A$  vanishes identically (i.e.  $A \equiv 0$ ).

**Proposition 4.4.2.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I such that the total space is almost cosymplectic or quasi-Sasakian. If the fibres are superminimal, then the horizontal distribution is completely integrable.*

*Proof.* It is not difficult to show that  $A_X\varphi Y = \varphi A_X Y$  for the three mentioned almost contact metric submersions. If the fibres are superminimal, we have  $g((\nabla_U\varphi)X, Y) = -g(A_{\varphi X}Y + A_X\varphi Y, U)$ , which implies that  $A \equiv 0$ .  $\square$

**Proposition 4.4.3.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I whose total space  $M$ , is  $G_1$ -Kenmotsu. If the fibres are superminimal, then the horizontal distribution is completely integrable.*

*Proof.* Let  $X$  be a horizontal vector fields and  $U$  a vertical one. According to the defining relation of a  $G_1$ -Kenmotsu structure, we have

$$(\nabla_X\phi)(X, U) - (\nabla_{\varphi X}\phi)(\varphi X, U) = \eta(X)\phi(X, U).$$

Thus we obtain

$$g(A_X X, \varphi U) + g(A_X \varphi X, U) + g(A_{\varphi X} X, U) - g(A_{\varphi X} \varphi X, \varphi U) - \eta(X)g(A_{\varphi X} X, U) = 0,$$

yielding  $2g(A_{\varphi X} X, U) = 0$ , from which one gets  $A_{\varphi X} X = 0$ . Combining this result with the fact that expression  $SM - 4$ ) vanishes, we have  $A \equiv 0$ .  $\square$

As in [53], we are able to use the superminimality of the fibres to induce a specific almost contact metric structure onto the total space of an almost contact metric submersion, provided that certain necessary structures exist on the base space and the fibres.

Regarding the transference of structure from the base to the total space, three cases are examined.

- 1) Automatical transference when the fibres are superminimal.
- 2) Transference subjected to the condition  $(\nabla_X\varphi)U = 0$ , when the fibres are superminimal.

3) Not transferring even the above condition  $(\nabla_X \varphi)U = 0$ , holds when the fibres are superminimal.

We begin by proving a technical result.

**Lemma 4.4.4.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m+1}$  be an almost contact metric submersion of type I. Suppose that  $d\eta' = 0$  on the base space. If the fibres are superminimal, then  $d\eta = 0$  on the total space.*

*Proof.* In order to see that  $d\eta = 0$ , we begin by assuming that  $X$  and  $Y$  are basic vector fields on the total space. Then  $d\eta(X, Y) = d\eta'(X_*, Y_*) = 0$ . The vanishing of expression  $SM - 2)$  implies, along with  $A_{\varphi X} = 0$  that  $A \equiv 0$ . Now

$$\begin{aligned} 2d\eta(X, U) &= (\nabla_X \eta)U - (\nabla_U \eta)X = g(X, \nabla_U \xi) - g(U, \nabla_X \xi) \\ &= g(X, \nabla_U \xi) - g(U, A_X \xi) = g(X, \nabla_U \xi). \end{aligned}$$

The superminimality of the fibres implies that

$$0 = g((\nabla_U \varphi)\xi, X) = g(\nabla_U \varphi \xi, X) - g(\varphi \nabla_U \xi, X) = g(\nabla_U \xi, \varphi X).$$

Thus,  $\nabla_U \xi$  is  $g$ -orthogonal to all vector fields except, perhaps,  $\xi$ . Recall that  $\|\xi\|^2 = g(\xi, \xi)$  is constant 1, so that  $g(\nabla_U \xi, \xi) = 0$ . Hence  $d\eta(X, U) = 0$  and  $d\eta(U, X) = 0$ . Recall, too, that the Lie bracket  $[U, V]$  is vertical from the complete integrability of the vertical distribution. Then  $d\eta(U, V) = \frac{1}{2} \{U\eta(V) - V\eta(U) - \eta([U, V])\} = 0$ , because  $\eta$  vanishes on the vertical distribution.  $\square$

Lemma applies to the following almost contact metric structures among others:

- (1) closely cosymplectic,
- (2) almost cosymplectic,
- (3) cosymplectic,
- (4) nearly Kenmotsu,
- (5) quasi Kenmotsu,
- (6) generalized Kenmotsu.

**Theorem 4.4.5.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m+1}$  be an almost contact metric submersion of type I. Assume that the base space is nearly cosymplectic, nearly-K-cosymplectic, nearly Kenmotsu,  $G_1$ -Sasakian or  $G_1$ -Kenmotsu. If the fibres are superminimal, then the total space is respectively nearly cosymplectic, nearly-K-cosymplectic, nearly Kenmotsu,  $G_1$ -Sasakian or  $G_1$ -Kenmotsu.*

*Proof.* There are four expressions that must vanish in order to conclude that the total space is nearly cosymplectic:

$$NC-1) \ g((\nabla_U \varphi)U, V);$$

$$NC-2) \ g((\nabla_U \varphi)U, X);$$

$$NC-3) \ g((\nabla_X \varphi)X, U);$$

$$NC-4) \ g((\nabla_X \varphi)X, Y).$$

The superminimality of the fibres implies that the first two expressions are zero. We may assume that the horizontal vector fields  $X$  and  $Y$  are basic for expression  $NC-4$ ), in which case that expression vanishes because the base space is nearly cosymplectic. Finally,

$$g((\nabla_X \varphi)X, U) = g(\nabla_X \varphi X, U) - g(\varphi \nabla_X X, U) = g(\nabla_X \varphi X, U) = 0$$

yielding the vanishing of expression  $NC-3$ ). Concerning the case of nearly-K-cosymplectic structure on the base space, we need only establish that  $\nabla \eta = 0$  on the total space; that is, we must show that  $\nabla_E \xi = 0$  for all vector fields,  $E$ , on  $M$ . But  $\nabla_X \xi = 0$  by projection onto the base space. For  $\nabla_U \xi$ , we know that  $0 = (\nabla_U \varphi)\xi$  by the superminimality of the fibres. Thus  $0 = \nabla_U \varphi \xi - \varphi \nabla_U \xi = -\varphi \nabla_U \xi = \nabla_U \xi - \eta(\nabla_U \xi)\xi$ . But, during the proof of Lemma 4.4.4, we established that  $\eta(\nabla_U \xi) = g(\nabla_U \xi, \xi) = 0$ . Therefore,  $\nabla \eta = 0$  and  $M$  is nearly-K-cosymplectic.

Now, let us consider the case of the nearly Kenmotsu structure. Lemma 4.4.4 implies that  $d\eta = 0$  on the total space. Since  $\eta$  vanishes on the vertical distribution, we need only to show that  $(\nabla_U \varphi)U = 0$  and that  $0 = (\nabla_X \varphi)X + \eta(X)\varphi X$ . Let  $X$  be basic, then

$$(\nabla_X \varphi)X + \eta(X)\varphi X = (\nabla'_{X_*} \varphi')X_* + \eta'(X_*)\varphi'X_* = 0.$$

Clearly,  $(\nabla_U \varphi)U = 0$  because the fibres are superminimal. Therefore, the total space is nearly Kenmotsu.

There are four expressions which must vanish in order to prove that the total space,  $(M, g, \varphi, \xi, \eta)$ , is  $G_1$ -Sasakian.

$$G_1 - S-1) \ g((\nabla_U \varphi)U, V) - g((\nabla_{\varphi U} \varphi)\varphi U, V) + \eta(U)g(\nabla_{\varphi U} \xi, V);$$

$$G_1 - S-2) \ g((\nabla_U \varphi)U, X) - g((\nabla_{\varphi U} \varphi)\varphi U, X) + \eta(U)g(\nabla_{\varphi U} \xi, X);$$

$$G_1 - S-3) \ g((\nabla_X \varphi)X, U) - g((\nabla_{\varphi X} \varphi)\varphi X, U) + \eta(X)g(\nabla_{\varphi X} \xi, U);$$

$$G_1 - S-4) \ g((\nabla_X \varphi)X, Y) - g((\nabla_{\varphi X} \varphi)\varphi X, Y) + \eta(X)g(\nabla_{\varphi X} \xi, Y).$$

Since  $\eta$  vanishes on the vertical distribution, in the light of Proposition 2.1.2 (f), the first two expressions respectively become

$$g((\nabla_U \varphi)U, V) - g((\nabla_{\varphi U} \varphi)\varphi U, V),$$

and

$$g((\nabla_U \varphi)U, X) - g((\nabla_{\varphi U} \varphi)\varphi U, X),$$

which vanish identically because of the superminimality of the fibres.

Assuming that the horizontal vector fields  $X$  and  $Y$  are basic, expression  $G_1 - S - 4$  is zero because the base space is  $G_1$ -Sasakian. Now, let us consider expression  $G_1 - S - 3$ . We have to calculate

$$g((\nabla_X \varphi)X, U) - g((\nabla_{\varphi X} \varphi)\varphi X, U) + \eta(X)g(\nabla_{\varphi X} \xi, U)$$

and show that it is zero; this can be proved by a straightforward calculation. Therefore  $(M^{2m+1}, g, \varphi, \xi, \eta)$  is  $G_1$ -Sasakian. Let us consider the case of  $G_1$ -Kenmotsu structure. In [46], it is shown that if  $d\eta' = 0$  on the base space then the total space also verifies  $d\eta = 0$ . As in the case of  $G_1$ -Sasakian structure, there are four components which must vanish to verify the  $G_1$ -Kenmotsu structure on the total space. We have

$$\begin{aligned} G_1 - K - 1) & (\nabla_U \phi)(U, V) - (\nabla_{\varphi U} \phi)(\varphi U, V) - \eta(U)\phi(V, U); \\ G_1 - K - 2) & (\nabla_U \phi)(U, X) - (\nabla_{\varphi U} \phi)(\varphi U, X) - \eta(U)\phi(X, U); \\ G_1 - K - 3) & (\nabla_X \phi)(X, V) - (\nabla_{\varphi X} \phi)(\varphi X, V) - \eta(X)\phi(V, X); \\ G_1 - K - 4) & (\nabla_X \phi)(X, Y) - (\nabla_{\varphi X} \phi)(\varphi X, Y) - \eta(X)\phi(Y, X). \end{aligned}$$

For the first two components, we note that superminimality of the fibres mean that  $\nabla_U \varphi = 0$  because of the vanishing of  $\eta$  on the vertical vector fields. The fourth calculation vanishes on basic horizontal vector fields because the projected tensors by the submersion down to the base space vanish. Now, consider calculation  $G_1 - K - 3$ . Recall that  $(\nabla_X \phi)(X, V) = g(X, (\nabla_X \varphi)V) = g(X, \nabla_X \varphi V - \varphi(\nabla_X V)) = g(X, A_X \varphi V - \varphi A_X V) = -g(A_X X, \varphi V) - g(A_X \varphi X, V) = -g(A_X \varphi X, V)$ . Similarly,  $(\nabla_X \phi)(X, V) = g(A_{\varphi X} X, V)$ . Thus,  $(\nabla_X \phi)(\varphi X, V) - (\nabla_{\varphi X} \phi)(\varphi X, V) = -g(A_X \varphi X + A_{\varphi X} X, V) = 0$ . Therefore  $G_1 - K - 3$  vanishes and  $M$  is  $G_1$ -Kenmotsu.  $\square$

Considering  $G_1$ -Sasakian and  $G_1$ -Kenmotsu manifolds, they have in common the following relation

$$(\nabla_D \phi)(D, E) - (\nabla_{\varphi D} \phi)(\varphi D, E) + b \cdot \eta(D)\mathcal{C} = 0,$$

where  $\mathcal{C}$  is a factor determined by the class of the manifold and  $b$  is a real number. For instance, taking  $b = -1$  and  $\mathcal{C} = \phi(E, D)$ , we obtain one of the defining relations of a  $G_1$ -Kenmotsu structure; if  $b=1$  and  $\mathcal{C} = (\nabla_{\varphi D} \xi)$ , we get the defining relation of a  $G_1$ -Sasakian structure.

With this in mind, we can summarize the above results in the following way.

**Theorem 4.4.6 ([48]).** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I whose base space is a  $G_1$ -almost contact metric manifold. If the fibres are superminimal then the total space inherits the structure of the base space.*

**Proposition 4.4.7.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. Assume that the base space,  $M'$ , is  $G_1$ -semi-Sasakian,  $G_1$ -semi-cosymplectic or  $G_1$ -semi-Kenmotsu manifold and that the fibres are superminimal. Then the total space inherits the structure of the base space.*

*Proof.* In this proposition, it remains to consider the case where  $\delta\phi' = 0$ . It is known that, since the fibres are superminimal, they are Kähler and then minimal. With this, and the use of equation (2.10) of China, we then have  $\delta\phi = 0$ . Other properties of the defining relations are established as in the preceding Theorem 4.4.5.  $\square$

**Proposition 4.4.8.**  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the fibres are superminimal, then the total space is almost-K-contact.

*Proof.* Recall that  $\xi$  is a vertical vector field in the case of an almost contact metric submersion of type II. Since the fibres are superminimal, they satisfy  $\nabla_U\varphi = 0$ . Therefore,  $\nabla_\xi\varphi = 0$  which is the defining relation of an almost -K-contact structure.  $\square$

Let us look at the case where the total space enjoys with a  $G_2$ -structure as in [51].

**Proposition 4.4.9.** Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. Assume that the base space,  $M'$ , is a  $G_2$ -semi-Sasakian or a  $G_2$ -semi-cosymplectic manifold and that the fibres are superminimal. Then the total space inherits the structure of the base space.

*Proof.* Let us consider the case of  $G_2$ -semi-cosymplectic structure. It is known that this structure is  $G_2$ -Sasakian and  $\delta\phi = 0 = \delta\eta$ . We then have to consider the case where  $\delta\phi' = 0$ . It is known that since the fibres are superminimal, they are Kähler and then minimal. With this, and the use of equation (2.10), of China, we then have  $\delta\phi = 0$ ; equation (2.11) gives  $\delta\eta = 0$ . Considering the  $G_2$ -semi-Sasakian structure, it remains to show that  $\eta = \frac{1}{2m}\delta\phi$ , which follows from Proposition 2.1.2 (1) and 2.1.2 (2).  $\square$

Now we present some results on tranference subjected to the condition  $(\nabla_X\varphi)U = 0$ .

**Theorem 4.4.10.** Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. Assume that the base space  $M'$  is  $G_2$ -Sasakian and the fibres are superminimal. If  $(\nabla_X\varphi)U = 0$ , then the total space,  $M$ , is  $G_2$ -Sasakian.

*Proof.* There are six vanishing expressions to prove that the total space,  $(M, g, \varphi, \xi, \eta)$ , is  $G_2$ -Sasakian.

$$\begin{aligned}
G_2 - S-1) & \quad \mathcal{G} \{(\nabla_U\phi)(V, W) - (\nabla_{\varphi U}\phi)(\varphi V, W) - \eta(V)(\nabla_{\varphi U}\eta)W\}; \\
G_2 - S-2) & \quad \mathcal{G} \{(\nabla_U\phi)(V, X) - (\nabla_{\varphi U}\phi)(\varphi V, X) - \eta(V)(\nabla_{\varphi U}\eta)X\}; \\
G_2 - S-3) & \quad \mathcal{G} \{(\nabla_U\phi)(X, Y) - (\nabla_{\varphi U}\phi)(\varphi X, Y) - \eta(X)(\nabla_{\varphi U}\eta)Y\}; \\
G_2 - S-4) & \quad \mathcal{G} \{(\nabla_X\phi)(U, V) - (\nabla_{\varphi X}\phi)(\varphi U, V) - \eta(U)(\nabla_{\varphi X}\eta)V\}; \\
G_2 - S-5) & \quad \mathcal{G} \{(\nabla_X\phi)(Y, V) - (\nabla_{\varphi X}\phi)(\varphi Y, V) - \eta(Y)(\nabla_{\varphi X}\eta)V\}; \\
G_2 - S-6) & \quad \mathcal{G} \{(\nabla_X\phi)(Y, Z) - (\nabla_{\varphi X}\phi)(\varphi Y, Z) - \eta(Y)(\nabla_{\varphi X}\eta)Z\}.
\end{aligned}$$

Obviously, the first two calculations vanish because the fibres are superminimal and  $\eta$  vanishes on the vertical distribution. Recall that  $(\nabla_U\phi)(X, Y) = g(X, (\nabla_U\varphi)Y)$  and  $(\nabla_U\eta)Y = g(Y, \nabla_U\xi) = (\nabla_U\phi)(\xi, \varphi Y)$ . But,  $(\nabla_U\phi)(\xi, \varphi Y) = g(\xi, (\nabla_U\varphi)\varphi Y)$ ; since

the fibres are superminimal,  $(\nabla_U\varphi)\varphi Y = 0$  from which  $(\nabla_U\eta)Y = 0$  and similarly  $(\nabla_{\varphi U}\eta)Y = 0$ . With this, calculation  $G_2 - S-3$ ) vanishes. In  $G_2 - S-4$ ), we have to show that  $(\nabla_X\phi)(U, V) = 0$ . In fact,  $(\nabla_X\phi)(U, V) = g(U, (\nabla_X\varphi)V)$ ; using the fact that  $(\nabla_X\varphi)V = 0$ , we get  $(\nabla_X\phi)(U, V) = 0 = (\nabla_{\varphi X}\phi)(\varphi U, V)$ . Thus,  $G_2 - S-4$ ) vanishes. Now, taking  $G_2 - S-5$ ), it is known that  $(\nabla_{\varphi X}\eta)V = (\nabla_{\varphi X}\phi)(\xi, \varphi V) = g(\xi, (\nabla_{\varphi X}\varphi)\varphi V)$ . On the other hand,  $\varphi X$  is horizontal and  $\varphi V$  is vertical according to Proposition 2.1.2. Since  $(\nabla_{\varphi X}\varphi)\varphi V = 0$ , we get  $-\eta(Y)(\nabla_{\varphi X}\eta)V = 0$ . Let us look at  $(\nabla_X\phi)(Y, V)$  and  $(\nabla_{\varphi X}\phi)(\varphi Y, V)$ . We have  $(\nabla_{\varphi X}\phi)(\varphi Y, V) = (Y, (\nabla_X\varphi)V)$ , and  $(\nabla_{\varphi X}\phi)(\varphi Y, V) = g(\varphi Y, (\nabla_{\varphi X}\varphi)V)$ . Since  $(\nabla_X\varphi)V = 0$  and  $(\nabla_{\varphi X}\varphi)V = 0$ , we see that  $G_2 - S-5$ ) vanishes. The last calculation vanishes on basic horizontal vector fields because the projected tensors by the submersion down to the base space vanish. Therefore  $(M^{2m+1}, g, \varphi, \xi, \eta)$  is  $G_2$ -Sasakian.  $\square$

**Proposition 4.4.11.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. Assume that the base space is generalized Kenmotsu, semi-Kenmotsu normal or a quasi Kenmotsu manifold and the fibres are superminimal. If  $(\nabla_X\varphi)U = 0$ , then the total space inherits the structure of the base space.*

*Proof.* As in the case of the preceding proposition, there are also six calculations which must vanish to prove that the total space inherits the structure of the base space. We have

- GeK-1)  $(\nabla_U\phi)(V, W) - (\nabla_{\varphi U}\phi)(\varphi V, W) - \eta(V)\phi(W, U)$ ;
- GeK-2)  $(\nabla_U\phi)(V, X) - (\nabla_{\varphi U}\phi)(\varphi V, X) - \eta(V)\phi(X, U)$ ;
- GeK-3)  $(\nabla_U\phi)(X, Y) - (\nabla_{\varphi U}\phi)(\varphi X, Y) - \eta(X)\phi(Y, U)$ ;
- GeK-4)  $(\nabla_X\phi)(U, V) - (\nabla_{\varphi X}\phi)(\varphi U, V) - \eta(U)\phi(V, X)$ ;
- GeK-5)  $(\nabla_X\phi)(Y, V) - (\nabla_{\varphi X}\phi)(\varphi Y, V) - \eta(Y)\phi(V, X)$ ;
- GeK-6)  $(\nabla_X\phi)(Y, Z) - (\nabla_{\varphi X}\phi)(\varphi Y, Z) - \eta(Y)\phi(Z, X)$ .

We first calculate GeK-4). It is clear that the condition  $(\nabla_X\varphi)U = 0$ , applies to  $(\nabla_{\varphi X}\phi)(\varphi U, V)$ , and  $(\nabla_X\phi)(U, V)$ . Since  $\eta(U)\phi(V, X) = 0$ , expression GeK-4) vanishes. In GeK-5), we have to treat  $(\nabla_X\phi)(Y, V)$  and  $(\nabla_{\varphi X}\phi)(\varphi Y, V)$ . Recall that  $(\nabla_X\phi)(Y, V) = g(Y, (\nabla_X\varphi)V)$  and  $(\nabla_{\varphi X}\phi)(\varphi Y, V) = g(\varphi Y, (\nabla_{\varphi X}\varphi)V)$ , we can apply the condition  $(\nabla_X\varphi)U = 0$ . The last calculation vanishes on basic horizontal vector fields because the projected tensors by the submersion down to the base space vanish. We have then proved the case of generalized Kenmotsu manifold. Considering the case of semi-Kenmotsu normal manifold, it remains to consider the case of  $\delta\phi' = 0$  and  $d\eta = 0$ . Since  $\delta\phi' = 0$ , we can use equation (2.10) of Chinea to get  $\delta\phi = 0$ . In the light of Lemma 4.4.4, since  $d\eta' = 0$  then  $d\eta = 0$ . The case of quasi Kenmotsu is treated in a way similar to that of generalized Kenmotsu.  $\square$

**Proposition 4.4.12 ([49]).** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. Assume that the base space is a  $C_7$ ,  $C_8$ ,  $C_9$ , or a  $C_{10}$ -manifold and the fibres are superminimal. If  $(\nabla_X\varphi)U = 0$ , then the total space inherits the structure of the base space.*

*Proof.* Let us consider the case where the base space is a  $C_7$ -manifold. In order to prove that the total space is a  $C_7$ -manifold, there are six calculations which must vanish.

$$\begin{aligned}
C_7-1) \quad & (\nabla_U \phi)(V, W) - \eta(W)(\nabla_V \eta)\varphi U - \eta(V)(\nabla_{\varphi U} \eta)W; \\
C_7-2) \quad & (\nabla_U \phi)(V, X) - \eta(X)(\nabla_V \eta)\varphi U - \eta(V)(\nabla_{\varphi U} \eta)X; \\
C_7-3) \quad & (\nabla_U \phi)(Y, X) - \eta(X)(\nabla_Y \eta)\varphi U - \eta(Y)(\nabla_{\varphi U} \eta)X; \\
C_7-4) \quad & (\nabla_X \phi)(U, V) - \eta(V)(\nabla_U \eta)\varphi X - \eta(U)(\nabla_{\varphi X} \eta)V; \\
C_7-5) \quad & (\nabla_X \phi)(Y, V) - \eta(V)(\nabla_Y \eta)\varphi X - \eta(Y)(\nabla_{\varphi X} \eta)V; \\
C_7-6) \quad & (\nabla_X \phi)(Y, Z) - \eta(Z)(\nabla_Y \eta)\varphi X - \eta(Y)(\nabla_{\varphi X} \eta)Z;
\end{aligned}$$

Since the fibres are superminimal, the two first calculations vanish. Regarding  $C_7-3$ ), it is known that  $(\nabla_U \phi)(Y, X) = g(Y, (\nabla_U \varphi)X)$  from which  $(\nabla_U \varphi)X = 0$  because of the superminimality of the fibres. Note that  $(\nabla_Y \eta)\varphi U = (\nabla_Y \phi)(\xi, \varphi^2 U) = g(\xi, (\nabla_Y \varphi)\varphi^2 U)$ . Applying the condition  $(\nabla_X \varphi)U = 0$  we deduce that  $(\nabla_Y \eta)\varphi U = 0$ . In  $C_7-4$ ), we have to examine only  $(\nabla_X \phi)(U, V)$ , others terms vanish because  $\eta$  vanishes on vertical vector fields. But  $(\nabla_X \phi)(U, V) = g(U, (\nabla_X \varphi)V) = 0$  by the use of the condition  $(\nabla_X \varphi)U = 0$ . Concerning  $C_7-5$ ), we have to examine  $(\nabla_X \phi)(Y, V)$  and  $\eta(Y)(\nabla_{\varphi X} \eta)V$ . Recall that  $(\nabla_X \phi)(Y, V) = g(Y, (\nabla_X \varphi)V)$ . Using the condition  $(\nabla_X \varphi)V = 0$ , we have then  $(\nabla_X \phi)(Y, V) = 0$ . Considering  $(\nabla_{\varphi X} \eta)V$ , we have  $(\nabla_{\varphi X} \eta)V = (\nabla_{\varphi X} \phi)(\xi, V) = g(\xi, (\nabla_{\varphi X} \varphi)V)$ . Using the condition  $(\nabla_X \varphi)U = 0$ , we conclude that  $C_7-5$ ) vanishes. The last calculation  $C_7-6$ ) vanishes on basic horizontal vector fields because the projected tensors by the submersion down to the base space vanish. Therefore  $(M, g, \varphi, \xi, \eta)$  is a  $C_7$ -manifold. Consider the case of  $C_8$ -manifold. As in the case of  $C_7$ -manifold, we have to examine the following six calculations

$$\begin{aligned}
C_8-1) \quad & (\nabla_U \phi)(V, W) + \eta(W)(\nabla_V \eta)\varphi U - \eta(V)(\nabla_{\varphi U} \eta)W; \\
C_8-2) \quad & (\nabla_U \phi)(V, X) + \eta(X)(\nabla_V \eta)\varphi U - \eta(V)(\nabla_{\varphi U} \eta)X; \\
C_8-3) \quad & (\nabla_U \phi)(Y, X) + \eta(X)(\nabla_Y \eta)\varphi U - \eta(Y)(\nabla_{\varphi U} \eta)X; \\
C_8-4) \quad & (\nabla_X \phi)(U, V) + \eta(V)(\nabla_U \eta)\varphi X - \eta(U)(\nabla_{\varphi X} \eta)V; \\
C_8-5) \quad & (\nabla_X \phi)(Y, V) + \eta(V)(\nabla_Y \eta)\varphi X - \eta(Y)(\nabla_{\varphi X} \eta)V; \\
C_8-6) \quad & (\nabla_X \phi)(Y, Z) + \eta(Z)(\nabla_Y \eta)\varphi X - \eta(Y)(\nabla_{\varphi X} \eta)Z;
\end{aligned}$$

As in the case of  $C_7$ -manifold, since  $\delta\eta' = 0$  on the base space, equation (2.11) gives  $\delta\eta = 0$ . Other calculations are treated as in the case of  $C_7$ -manifold.  $\square$

**Proposition 4.4.13.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. Assume that the base space is a  $C_3$ -manifold and the fibres are superminimal. If the transference criterion  $(\nabla_X \varphi)U = 0$ , then the total space inherits the structure of the base space.*

*Proof.* As in the preceding situation, we have six calculations which must vanish.

$$C_3-1) \quad (\nabla_U \phi)(V, W) - (\nabla_{\varphi U} \phi)(\varphi V, W);$$

- $C_3-2)$   $(\nabla_U\phi)(V, X) - (\nabla_{\varphi U}\phi)(\varphi V, X);$
- $C_3-3)$   $(\nabla_U\phi)(X, Y) - (\nabla_{\varphi U}\phi)(\varphi X, Y);$
- $C_3-4)$   $(\nabla_X\phi)(U, V) - (\nabla_{\varphi X}\phi)(\varphi U, V);$
- $C_3-5)$   $(\nabla_X\phi)(Y, V) - (\nabla_{\varphi X}\phi)(\varphi Y, V);$
- $C_3-6)$   $(\nabla_X\phi)(Y, Z) - (\nabla_{\varphi X}\phi)(\varphi Y, Z).$

The first two calculations vanish because of the superminimality of the fibres. Consider  $C_3 - 3)$ , we have,

$$(\nabla_U\phi)(X, Y) = g(Y, (\nabla_U\varphi)X) \text{ and } (\nabla_{\varphi U}\phi)(\varphi X, Y) = g(Y, (\nabla_{\varphi U}\varphi)\varphi U),$$

which must vanish because of the superminimality of the fibres since  $U$  and  $\varphi U$  are vertical. In  $C_3 - 4)$ , we have  $(\nabla_X\phi)(U, V) = g(V, (\nabla_X\varphi)U)$  and  $(\nabla_{\varphi X}\phi)(\varphi U, V) = g(V, (\nabla_{\varphi X}\varphi)\varphi U)$ , which must vanish in case one used the criterion  $(\nabla_X\varphi)U = 0$ . Considering  $C_3 - 5)$ , we have  $(\nabla_X\phi)(Y, V) = g(Y, (\nabla_X\varphi)V)$ , and  $(\nabla_{\varphi X}\phi)(\varphi Y, V) = g(\varphi Y, (\nabla_{\varphi X}\varphi)V)$ , which must also vanish as in  $C_3 - 4)$ . The calculation  $C_3 - 6)$  vanishes on basic horizontal vector fields because the projected tensors by the submersion down to the base space vanish. Concerning the codifferential, since  $\delta\phi' = 0$  on the base space, we can use equation (2.10) to get  $\delta\phi = 0$ .  $\square$

The significance of the above criterion,  $(\nabla_X\varphi)U = 0$ , is that, when the fibres are superminimal, it ensures the transference of the structure from the base to the total space. For instance, by Proposition 4.3.1, it is proven that submersions whose total space is a  $C_{11}$  or a  $C_{12}$ -manifold have superminimal fibres; but it can be shown that, in this case, the structure of the base space does not transfer to the total space unless this criterion is fulfilled.

As in [49], we can state the following

**Proposition 4.4.14.** *Let  $\pi : M^{2m+1} \longrightarrow M^{2m'+1}$  be an almost contact metric submersion of type I with superminimal fibres. If the base space is a  $C_{11}$  or a  $C_{12}$ -manifold then these structures do not transfer to the total space unless  $(\nabla_X\varphi)U = 0$ .*

*Proof.* Let us consider the case where the base space is a  $C_{11}$ -manifold. As in the case of the preceding Proposition 4.4.13, the following six calculations must vanish

- $C_{11}-1)$   $(\nabla_U\phi)(V, W) + \eta(U)(\nabla_\xi\phi)(\varphi V, \varphi W);$
- $C_{11}-2)$   $(\nabla_U\phi)(V, X) + \eta(U)(\nabla_\xi\phi)(\varphi V, \varphi X);$
- $C_{11}-3)$   $(\nabla_U\phi)(Y, X) + \eta(U)(\nabla_\xi\phi)(\varphi Y, \varphi X);$
- $C_{11}-4)$   $(\nabla_X\phi)(U, V) + \eta(X)(\nabla_\xi\phi)(\varphi U, \varphi V);$
- $C_{11}-5)$   $(\nabla_X\phi)(Y, V) + \eta(X)(\nabla_\xi\phi)(\varphi Y, \varphi V);$
- $C_{11}-6)$   $(\nabla_X\phi)(Y, Z) + \eta(X)(\nabla_\xi\phi)(\varphi Y, \varphi Z).$

Considering expressions  $C_{11}-4)$  and  $C_{11}-5)$ , we encounter  $(\nabla_X\varphi)V$  and  $(\nabla_\xi\varphi)\varphi V$  which must vanish in order to conclude that the total space is a  $C_{11}$ -manifold.  $\square$

**Proposition 4.4.15.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. Assume that the base space,  $M'$ , is a  $G_2$ -Kenmotsu or almost trans-Kenmotsu manifold and the fibres are superminimal. Even  $(\nabla_X \varphi)V = 0$ , the total space does not inherit the structure of the base space.*

*Proof.* From Lemma 4.4.4, it is established that if  $d\eta' = 0$  on the base space, then the total space also verifies  $d\eta = 0$ . As in the case of  $G_2$ -Sasakian structure, there are six components which must vanish to verify the  $G_2$ -Kenmotsu structure on the total space. We have

$$\begin{aligned}
G_2 - K-1) & \quad \mathcal{G} \{(\nabla_U \phi)(V, W) - (\nabla_{\varphi U} \phi)(\varphi V, W) - \eta(U)\phi(V, W)\}; \\
G_2 - K-2) & \quad \mathcal{G} \{(\nabla_U \phi)(V, X) - (\nabla_{\varphi U} \phi)(\varphi V, X) - \eta(U)\phi(V, X)\}; \\
G_2 - K-3) & \quad \mathcal{G} \{(\nabla_U \phi)(X, Y) - (\nabla_{\varphi U} \phi)(\varphi X, Y) - \eta(U)\phi(X, Y)\}; \\
G_2 - K-4) & \quad \mathcal{G} \{(\nabla_X \phi)(U, V) - (\nabla_{\varphi X} \phi)(\varphi U, V) - \eta(X)\phi(U, V)\}; \\
G_2 - K-5) & \quad \mathcal{G} \{(\nabla_X \phi)(Y, V) - (\nabla_{\varphi X} \phi)(\varphi Y, V) - \eta(X)\phi(Y, V)\}; \\
G_2 - K-6) & \quad \mathcal{G} \{(\nabla_X \phi)(Y, Z) - (\nabla_{\varphi X} \phi)(\varphi Y, Z) - \eta(X)\phi(Y, Z)\}.
\end{aligned}$$

Since  $\eta$  vanishes on vertical vector fields, and the fibres are superminimal, the three first calculations vanish.  $G_2 - K-5)$  vanishes because  $\phi(Y, V) = 0$  since  $Y$  is horizontal and  $V$  is vertical;  $(\nabla_X \phi)(Y, V) = 0$  by the use of  $(\nabla_X \varphi)V$  as in the calculation of  $G_2 - S-4)$ , in the same way, we get  $(\nabla_{\varphi X} \phi)(\varphi Y, V) = 0$ . The obstruction to the transfer of the structure to the total space is the calculation  $G_2 - K-4)$ . Indeed, in this calculation,  $\eta(X)\phi(U, V) \neq 0$  because  $\phi(U, V) = g(U, \varphi V)$  and  $\eta(X) \neq 0$ .

In order to prove that the total space,  $(M^{2m+1}, g, \varphi, \xi, \eta)$ , is almost trans-Kenmotsu, the following six calculations must vanish.

$$\begin{aligned}
\text{ATK-1)} & \quad \mathcal{G} \{(\nabla_U \phi)(V, W) - \frac{1}{m}\phi(U, V)\delta\phi(\varphi W) - 2\eta(U)\phi(V, W)\}; \\
\text{ATK-2)} & \quad \mathcal{G} \{(\nabla_U \phi)(Y, X) - \frac{1}{m}\phi(U, Y)\delta\phi(\varphi X) - 2\eta(U)\phi(Y, X)\}; \\
\text{ATK-3)} & \quad \mathcal{G} \{(\nabla_U \phi)(Y, X) - \frac{1}{m}\phi(U, Y)\delta\phi(\varphi X) - 2\eta(U)\phi(Y, X)\}; \\
\text{ATK-4)} & \quad \mathcal{G} \{(\nabla_X \phi)(U, V) - \frac{1}{m}\phi(X, U)\delta\phi(\varphi V) - 2\eta(X)\phi(U, V)\}; \\
\text{ATK-5)} & \quad \mathcal{G} \{(\nabla_X \phi)(Y, V) - \frac{1}{m}\phi(X, Y)\delta\phi(\varphi V) - 2\eta(X)\phi(Y, V)\}; \\
\text{ATK-6)} & \quad \mathcal{G} \{(\nabla_X \phi)(Y, Z) - \frac{1}{m}\phi(X, Y)\delta\phi(\varphi Z) - 2\eta(X)\phi(Y, Z)\}.
\end{aligned}$$

Since  $\phi(U, V) \neq 0$ , it is clear that ATK-1) and ATK-4) cannot vanish. So, the total space does not inherit the structure of the base space. Considering ATK-5), we have  $\phi(X, Y)\delta\phi(\varphi V) \neq 0$  which obstructs this calculation to vanish.  $\square$

## 4.5 Umbilicity of the fibres

We begin by recalling some basic concepts before applying them to the case of Riemannian submersions.

Let  $\bar{M}$  be a submanifold of a Riemannian manifold  $(M^m, g)$  and  $\sigma$  the second fundamental form of  $\bar{M}$ . It is well known that the mean curvature vector field,  $H$ , of  $\bar{M}$  is given by  $H = \frac{1}{m}tr(\sigma)$  where  $tr(\sigma)$  denotes the trace of  $\sigma$ .

If  $H = 0$ , then  $\bar{M}$  is said to be minimal;

If  $\sigma = 0$ , then  $\bar{M}$  is said to be totally geodesic, and

If  $\sigma(D, E) = g(D, E)H$ , then  $\bar{M}$  is totally umbilical.

**Example.** Let  $\pi : B \times_s F \rightarrow B$  be a Riemannian submersion whose total space is a warped product. In this case, we have a submersion with totally umbilic fibres as noted in [18]. In [29], it is established that a Kenmotsu manifold is a warped product; therefore any almost contact metric submersion with total space a Kenmotsu manifold is of totally umbilic fibres.

Now, let us regard some implications of the total umbilicity.

**Proposition 4.5.1.** *Let  $\pi : (M, g) \rightarrow (M', g')$  be a Riemannian submersion with totally umbilic fibres. If the mean curvature vector field,  $H$ , of the fibres is parallel, then:*

- (1)  $\mathcal{R}(U, V, U, V) = \hat{\mathcal{R}}(U, V, U, V) + [g(U, V)^2 - g(U, U)g(V, V)]g(H, H)$ ;
- (2)  $\mathcal{R}(X, U, X, U) = g(A_X U, A_X U)$ ;
- (3)  $\mathcal{R}(X, Y, X, Y) = \mathcal{R}'(X_*, Y_*, X_*, Y_*) - 3g(A_X Y, A_X Y)$ .

*Proof.* Assertions (1) and (3) follow from O'Neill's equations [36]. Let us consider (2). Recall from Baditoiu and Ianus [1], that

$$\mathcal{R}(X, U, X, U) = g(U, U)[g(\nabla_X H, X) - g(X, H) - g(H, H)^2] + g(A_X U, A_X U).$$

Since,  $H$  is vertical and  $X$  is horizontal, we have  $g(X, H) = 0$ ; on the other hand, the parallelism of  $H$  implies that  $\nabla_X H = 0$  so that  $g(\nabla_X H, X) = 0$ ; therefore  $\mathcal{R}(X, U, X, U) = g(A_X U, A_X U)$ .  $\square$

In the case of contact geometry, Tripathi and Shukla, [42], have defined the concepts of totally contact umbilic and totally contact geodesic in the following way.

Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  be an almost contact metric manifold. For a distribution  $\mathcal{D}$  on  $\bar{M}$ ,  $\bar{M}$  is said to be  $\mathcal{D}$ -totally geodesic if, for all  $D, E \in \mathcal{D}$ , we have  $\sigma(D, E) = 0$ .

If for all  $D, E \in \mathcal{D}$ , we have  $\sigma(D, E) = g(D, E)N$  for some normal vector field  $N$ , then  $\bar{M}$  is  $\mathcal{D}$ -totally umbilical.

Suppose that  $\bar{M}$  is tangent to the structure vector field  $\xi$ . Consider the distribution  $\{\xi\}$  generated by this vector field and  $\{\xi\}^\perp$  its complement. In this case, Tripathi and Shukla have defined the following two concepts. The submanifold  $\bar{M}$  is said to be :

- (1) **totally contact umbilic** if it is  $\{\xi\}^\perp$ -totally umbilic;
- (2) **totally contact geodesic** if it is  $\{\xi\}^\perp$ -totally geodesic.

**Definition 4.5.1.** Let  $\bar{M}$  be a submanifold of an almost contact metric manifold  $(M^{2m+1}, g, \varphi, \xi, \eta)$ . Then  $\bar{M}$  is:

- (1) totally contact umbilic if,  $\sigma(\varphi^2 D, \varphi^2 E) = g(\varphi^2 D, \varphi^2 E)N$ , for all  $D, E \in \Gamma(\bar{M})$ ,
- (2) totally contact geodesic if,  $\sigma(\varphi^2 D, \varphi^2 E) = 0$ , for all  $D, E \in \Gamma(\bar{M})$ .

In his study of lightlike hypersurfaces of indefinite Sasakian manifolds, F. Massamba [31] used also these two concepts.

**Proposition 4.5.2.** Let  $\bar{M}$  be a submanifold of an almost contact metric manifold  $(M^{2m+1}, g, \varphi, \xi, \eta)$ . If  $\bar{M}$  is :

- (1) totally contact umbilic, then,

$$\sigma(D, E) = g(\varphi D, \varphi E)N + \eta(D)\sigma(E, \xi) + \eta(E)\sigma(D, \xi) - \eta(D)\eta(E)\sigma(\xi, \xi),$$

- (2) totally contact geodesic, then,

$$\sigma(D, E) = \eta(D)\sigma(E, \xi) + \eta(E)\sigma(D, \xi) - \eta(D)\eta(E)\sigma(\xi, \xi).$$

*Proof.* (1) Let us consider the relation

$$\sigma(\varphi^2 D, \varphi^2 E) = g(\varphi^2 D, \varphi^2 E)N.$$

It is known that

$$g(\varphi^2 D, \varphi^2 E) = g(D, E) - \eta(D)\eta(E),$$

from which  $g(\varphi^2 D, \varphi^2 E) = (g(D, E) - \eta(D)\eta(E))N$ . On the other hand,

$$\begin{aligned} \sigma(\varphi^2 D, \varphi^2 E) &= \sigma(-D + \eta(D)\xi, -E + \eta(E)\xi) \\ &= \sigma(D, E) - \eta(E)\sigma(D, \xi) - \eta(D)\sigma(E, \xi) + \eta(D)\eta(E)\sigma(\xi, \xi). \end{aligned}$$

Remembering that

$$g(D, E) - \eta(D)\eta(E) = g(\varphi D, \varphi E),$$

we get the proof of assertion (1).

Concerning assertion (2), since  $\bar{M}$  is totally contact geodesic, we have  $\sigma(\varphi^2 D, \varphi^2 E) = 0$ , which implies that  $\sigma(-D + \eta(D)\xi, -E + \eta(E)\xi) = 0$ . But

$$\begin{aligned} \sigma(-D + \eta(D)\xi, -E + \eta(E)\xi) &= \sigma(D, E) - \sigma(D, \eta(E)\xi) \\ &\quad - \sigma(\eta(D)\xi, E) + \sigma(\eta(D)\xi, \eta(E)\xi) \\ &= \sigma(D, E) - \eta(E)\sigma(D, \xi) \\ &\quad - \eta(D)\sigma(E, \xi) + \eta(D)\eta(E)\sigma(\xi, \xi). \end{aligned}$$

which is  $\sigma(D, E) = \eta(D)\sigma(E, \xi) + \eta(E)\sigma(D, \xi) - \eta(D)\eta(E)\sigma(\xi, \xi)$ . □

**Proposition 4.5.3 ([42]).** *Let  $\bar{M}$  be a submanifold of an almost contact metric manifold tangent to  $\xi$ . If  $\bar{M}$  is totally contact umbilic or totally contact geodesic, then  $\xi$  is an asymptotic direction.*

*Proof.* See [42, Thm.7.2, p28]. □

Let us examine some applications in the case of almost contact metric submersions.

**Proposition 4.5.4.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. If the fibres are totally contact umbilic, then*

$$T_U V = g(U, V)H.$$

*Proof.* It is known that the O'Neill configuration tensor  $T$  is the second fundamental form of the fibres. Since  $\eta$  vanishes on vertical vector fields, then in Proposition 4.5.2, the defining equation (1) becomes  $T_U V = g(\varphi U, \varphi V)H$ , which is  $T_U V = g(U, V)H$  because  $g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V) = g(U, V)$ . □

**Corollary 4.5.5.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I with totally contact umbilic fibres. If the fibres are minimal, then they are totally geodesic.*

*Proof.* In such a case, we have  $T_U V = g(U, V)H$ , because of umbilicity of the fibres. If moreover the fibres are minimal, we have  $T_U V = 0$  which shows that  $T = 0$ . □

**Proposition 4.5.6.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. Suppose that the total space  $M$  is a  $K$ -contact manifold. If the fibres are totally contact umbilic, then they are totally geodesic.*

*Proof.* See [24]. □

**Proposition 4.5.7.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be an almost contact metric submersion of type II. If the fibres are totally contact umbilic, then the vector fields  $\xi$  defines an asymptotic direction..*

*Proof.* Since, for a type II submersion,  $\xi$  is tangent to the fibres, then we can apply Proposition 4.5.3. □

# Chapter 5

## Submersions of contact $CR$ -submanifolds

The Riemannian submersions between Riemannian manifolds were initiated by O'Neill [36]. The study of  $CR$ -submanifolds of an Hermitian manifold was initiated by Bejancu in [3]. He generalized both totally real and holomorphic immersions. Given an almost Hermitian manifold,  $(M, J, g)$ , a submanifold  $M$  is called  $CR$ -submanifold if there exists a differentiable distribution  $\mathcal{D}$  on  $M$  such that it is holomorphic, and its complementary orthogonal distribution  $\mathcal{D}^\perp$  is totally real  $J\mathcal{D}_x \subseteq \mathcal{D}_x$  and  $J(\mathcal{D}_x^\perp) \subseteq T_x M^\perp$ , for all  $x \in M$ . Since then, many authors have treated  $CR$ -submanifolds on different ambient manifolds and have amplified the definition to other decompositions of the tangent bundle (semi-slant and almost semi invariant submanifolds). In [39], Sahin considered horizontally conformal submersions and proved that every horizontally homothetic submersion is a Riemannian submersion.

The subject was considered later on for Riemannian manifolds with an almost contact structure. In this sense Benjancu and Papaghiuc studied semi-invariant submanifolds of a Sasakian manifold or a Sasakian space form (see [4], [5] and [38] and references therein).

In this chapter, submersions of contact  $CR$ -submanifolds of quasi- $K$ -cosymplectic and quasi-Kenmotsu manifolds are investigated. Remarkable distributions of the underlining submersions are examined and their implications on the total space are pointed out (for more details see [34] and [56]).

### 5.1 Contact $CR$ -submanifolds

In this section, we introduce the notion of contact  $CR$ -submanifold of a manifold (see [56], for details). Let  $M$  be an finite-dimensional isometrically immersed submanifold of a  $(2m + 1)$ -dimensional manifold  $\overline{M}$  and let  $g$  be the metric tensor on  $\overline{M}$  as well as the induced metric on  $M$ .

**Definition 5.1.1 ([56]).** A Riemannian submanifold  $M$  of a quasi- $K$ -cosymplectic (resp. quasi-Kenmotsu) manifold  $\overline{M}$  is called a contact CR-submanifold if  $\xi$  is tangent to  $M$  and there exists on  $M$  a differential distribution  $\mathcal{D} : x \mapsto \mathcal{D}_x \subset T_x M$  such that

- (i)  $\mathcal{D}_x$  is invariant under  $\varphi$  (i.e.  $\varphi\mathcal{D}_x \subset \mathcal{D}_x$ ), for each  $x \in M$ ,
- (ii) the orthogonal complementary distribution  $\mathcal{D}^\perp : x \mapsto \mathcal{D}_x^\perp \subset T_x M^\perp$  of the distribution  $\mathcal{D}$  on  $M$  is totally real, (i.e.  $\varphi\mathcal{D}^\perp \subset T_x M^\perp$ ),
- (iii)  $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\}$ , where  $T_x M$  and  $T_x M^\perp$  are the tangent space and the normal space of  $M$  at  $x$ , respectively, and  $\oplus$  denotes the orthogonal direct sum.

We call  $\mathcal{D}$  (resp.  $\mathcal{D}^\perp$ ) the horizontal (resp. vertical) distribution. We denote by  $g$  the metric tensor field of  $\overline{M}$  as well as that induced on  $M$ . Let  $\overline{\nabla}$  (resp.  $\nabla$ ) be the covariant differentiation with respect to the Levi-Civita connection on  $\overline{M}$  (resp.  $M$ ). The Gauss and Weingarten formulas for  $M$  are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (5.1)$$

$$\text{and } \overline{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (5.2)$$

for any  $X, Y \in \Gamma(TM)$ ,  $V \in \Gamma(TM^\perp)$ , where  $h : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM^\perp)$  is a normal bundle valued symmetric bilinear form on  $M$ , the linear operator  $A_V$  is the fundamental form tensor of Weingarten with respect to the normal section  $V$ , and the differential operator  $\nabla^\perp$  defines a linear connection on the normal bundle  $TM^\perp$  called the normal connection on  $M$ . Moreover, we have

$$g(h(X, Y), V) = g(A_V X, Y). \quad (5.3)$$

The submanifold  $M$  is said to be *totally geodesic* if  $h$  vanishes identically.

The projection of  $TM$  to  $\mathcal{D}$  and  $\mathcal{D}^\perp$  are denoted by  $h$  and  $v$ , respectively, i.e., for any  $X \in \Gamma(TM)$ , we have

$$X = hX + vX + \eta(X)\xi. \quad (5.4)$$

Applying  $\varphi$  to  $X$ , we have,

$$\varphi X = FX + NX, \quad \forall X \in \Gamma(TM), \quad (5.5)$$

where  $FX = \varphi hX$  and  $NX = \varphi vX$  are tangential and normal components of  $\varphi X$ , respectively.

The normal bundle to  $M$  has the decomposition

$$TM^\perp = \varphi\mathcal{D}^\perp \oplus \nu, \quad (5.6)$$

where  $\nu$  denotes the orthogonal complementary distribution of  $\varphi\mathcal{D}^\perp$ , and is an invariant normal subbundle of  $TM^\perp$  under  $\varphi$ . For any  $V \in TM^\perp$ , we put

$$V = pV + qV, \quad (5.7)$$

where  $pV \in \varphi\mathcal{D}^\perp$ ,  $qV \in \nu$ . From the above equation, we have,

$$\varphi V = fV + nV, \quad \forall V \in TM^\perp, \quad (5.8)$$

where  $fV = \varphi pV \in \mathcal{D}^\perp$  and  $nV = \varphi qV \in \nu$ .

Now, we study the distributions involved and we characterize the horizontally one. Let  $M$  be a contact  $CR$ -submanifold of a quasi- $K$ -cosymplectic (respectively, quasi-Kenmotsu) manifold  $\overline{M}$  and  $M'$  be an almost contact metric manifold with the almost contact metric structure  $(\varphi', \xi', \eta', g')$ .

Assume that there is a submersion  $\pi : M \longrightarrow M'$  such that:

- (1)  $\mathcal{D}^\perp = \ker(\pi_*)$ , where  $\pi_* : TM \longrightarrow TM'$  is the tangent mapping to  $\pi$ ,
- (2)  $\pi_* : \mathcal{D}_x \oplus \{\xi\} \longrightarrow T_{\pi(x)}M'$  is an isometry for each  $x$  which satisfies:  $\pi_* \circ \varphi = \varphi' \circ \pi_*$ ,  $\eta = \eta' \circ \pi_*$ ,  $\pi_*(\xi_x) = \xi'_{\pi(x)}$ , where  $T_{\pi(x)}M'$  denotes the tangent space of  $M'$  at  $\pi(x)$ .

Comparing tangential and normal components in (5.4) and (5.5), we obtain the next two Lemmas.

**Lemma 5.1.1.** *For a contact  $CR$ -submanifold  $M$  of a quasi- $K$ -cosymplectic (resp. quasi-Kenmotsu) manifold  $\overline{M}$ , the following equalities hold*

$$F^2 + fN = -I + \eta \otimes \xi, \quad (5.9)$$

$$NF + nN = 0, \quad (5.10)$$

$$Ff + fn = 0, \quad (5.11)$$

$$n^2 + Nf = -I. \quad (5.12)$$

**Lemma 5.1.2.** *Let  $M$  be a contact  $CR$ -submanifold  $M$  of an almost contact manifold  $(\overline{M}, \varphi, \xi, \eta, g)$ . Then,*

$$(\nabla_X F)Y - A_{NY}X - fh(X, Y) = F((\overline{\nabla}_X \varphi)Y), \quad (5.13)$$

$$(\nabla_X N)Y + h(X, FY) - nh(X, Y) = N((\overline{\nabla}_X \varphi)Y), \quad (5.14)$$

$$(\nabla_X f)V - A_{nV}X + FA_VX = f((\overline{\nabla}_X \varphi)V), \quad (5.15)$$

$$(\nabla_X^\perp n)V + h(X, fV) + NA_VX = n((\overline{\nabla}_X \varphi)V), \quad (5.16)$$

where  $F((\overline{\nabla}_X \varphi)Y)$ ,  $f((\overline{\nabla}_X \varphi)V)$ ,  $n((\overline{\nabla}_X \varphi)V)$  and  $N((\overline{\nabla}_X \varphi)Y)$  are, respectively, tangential and normal components of  $(\overline{\nabla}_X \varphi)Y$  and  $(\overline{\nabla}_X \varphi)V$ , for any  $X, Y \in \Gamma(TM)$  and  $V \in \Gamma(TM^\perp)$ .

**Proposition 5.1.3.** *For a contact  $CR$ -submanifold  $M$  of a quasi- $K$ -cosymplectic (resp. quasi-Kenmotsu) manifold  $\overline{M}$ , the following equalities hold*

$$(i) \ker(F) = \mathcal{D}^\perp \oplus \{\xi\},$$

$$(ii) \ker(N) = \mathcal{D} \oplus \{\xi\},$$

$$(iii) \ker(n) = N\mathcal{D}^\perp,$$

$$(iv) \ker(f) = \nu.$$

*Proof.* (i) and (ii) are directly deduced from the definition of contact  $CR$ -submanifold. For (iii), if  $X \in \mathcal{D}^\perp$ , then, by (5.10),  $nNX = -NFX = 0$ , i.e.  $n\mathcal{D}^\perp \subset \ker(n)$ . Conversely, let us consider  $U \in \ker(n)$ . From (5.11) and (5.12), it follows that,  $FfU = -fnU = 0$  and  $U = -n^2U - NfU = -NfU$ . From the first equality,  $fU \in \mathcal{D}^\perp$  and then the second one implies that  $U \in N\mathcal{D}^\perp$ . Now let us prove (iv). For  $V \in \ker(f)$ , we have  $fV = 0$  and, by (5.10) and (5.12),  $0 = FfV + fnV = fnV$  and  $n^2V + NfV = -V$  which implies  $\varphi nV = n^2V = -V$ , using (5.4). Applying  $\varphi$  and  $n$  to this equation, we have,  $n\varphi^2nV = -n\varphi V$ , i.e.,  $V = -n\varphi V \in \nu$ . Thus,  $\ker(f) \subset \nu$ . For the other inclusion, notice that, for any  $V$  normal to  $M$ ,  $fV \in \mathcal{D}^\perp$ . Then, using (5.10) and (5.12),  $fnV = FfV = 0$  and  $\varphi fV = NfV = -V - n^2V$ , i.e.  $n^2V = -V$ . Therefore,  $fV = 0$ .  $\square$

For a quasi-Kenmostu manifold, the defining relation is equivalent to, for any  $X, Y \in \Gamma(TM)$ ,

$$(\bar{\nabla}_X \varphi)Y - \varphi(\bar{\nabla}_{\varphi X} \varphi)Y = g(\varphi X, Y)\xi - 2\eta(Y)\varphi X. \quad (5.17)$$

The covariant derivative of the structure vector field  $\xi$  is given, for a quasi- $K$ -cosymplectic manifold, by,

$$\bar{\nabla}_X \xi = \varphi(\bar{\nabla}_{\varphi X} \xi), \quad \forall X, Y \in \Gamma(TM), \quad (5.18)$$

and for a quasi-Kenmotsu manifold, by

$$\bar{\nabla}_X \xi = -2\varphi^2 X + \varphi(\bar{\nabla}_{\varphi X} \xi), \quad \forall X, Y \in \Gamma(TM). \quad (5.19)$$

Note that, for both ambient almost contact manifolds, the following identities hold

$$\nabla_\xi \xi = 0 \quad \text{and} \quad h(\xi, \xi) = 0. \quad (5.20)$$

Now, we study the integrability of all the distributions involved in the definition of contact  $CR$ -submanifolds. First of all, we have:

**Lemma 5.1.4.** *For any  $X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,  $\varphi X = FX \in \Gamma(\mathcal{D} \oplus \{\xi\})$ .*

*Proof.* For any  $X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,  $X = hX + \eta(X)\xi$ . Applying  $\varphi$  to this equation, one has  $\varphi X = \varphi hX + \eta(X)\varphi\xi$ , i.e.  $\varphi X = FX$ . Therefore, for any  $Y \in \Gamma(TM)$ ,  $\bar{g}(\varphi X, vY) = -\bar{g}(X, \varphi vY)$ . Since  $\varphi vY \in \Gamma(\varphi\mathcal{D}^\perp) \subset \Gamma(TM^\perp)$ , we have  $\bar{g}(X, \varphi vY) = 0$ , that is,  $\bar{g}(\varphi X, vY) = 0$ , which completes the proof.  $\square$

The above lemma means that  $\mathcal{D} \oplus \{\xi\}$  is invariant under  $\varphi$ .

For some other considerations, the submanifold  $M$  may be considered to be of odd or even codimension, but whether the dimension of  $M$  is odd or even, the distribution  $\mathcal{D}$  is always of even dimension.

**Lemma 5.1.5.** *Let  $M$  be a contact CR-submanifold of an almost contact manifold  $\overline{M}$ . Then, if  $\overline{M}$  is quasi-K-cosymplectic, we have the following identities,*

$$\nabla_X \xi = F(\nabla_{FX} \xi) + fh(FX, \xi), \quad (5.21)$$

$$h(X, \xi) = N(\nabla_{FX} \xi) + nh(FX, \xi), \quad (5.22)$$

for any  $X \in \Gamma(\mathcal{D})$ .

Moreover, if  $\overline{M}$  is quasi-Kenmotsu, we have,

$$\begin{aligned} \nabla_X \xi &= 2\{X - \eta(X)\xi\} + F(\nabla_{FX} \xi) + fh(FX, \xi), \\ h(X, \xi) &= N(\nabla_{FX} \xi) + nh(FX, \xi), \end{aligned} \quad (5.23)$$

for any  $X \in \Gamma(\mathcal{D})$ .

*Proof.* If  $\overline{M}$  is a quasi-K-cosymplectic, from (5.18), one has, for any  $X \in \Gamma(\mathcal{D})$ ,

$$\begin{aligned} \nabla_X \xi + h(X, \xi) &= \overline{\nabla}_X \xi = \varphi(\overline{\nabla}_{FX} \xi) = \varphi(\nabla_{FX} \xi) + \varphi h(FX, \xi) \\ &= F(\nabla_{FX} \xi) + N(\nabla_{FX} \xi) + fh(FX, \xi) + nh(FX, \xi). \end{aligned}$$

On the other hand, if  $\overline{M}$  is a quasi-Kenmotsu, from (5.19), we get,

$$\begin{aligned} \nabla_X \xi + h(X, \xi) &= \overline{\nabla}_X \xi = -2\varphi^2 X + \varphi(\nabla_{FX} \xi) + \varphi h(FX, \xi) \\ &= -2\varphi^2 X + F(\nabla_{FX} \xi) + N(\nabla_{FX} \xi) + fh(FX, \xi) + nh(FX, \xi). \end{aligned}$$

Then, comparing tangential and normal components of both sides of these equations, we complete the proof.  $\square$

**Proposition 5.1.6.** *Let  $M$  be a contact CR-submanifold of an almost contact manifold  $\overline{M}$ . Then, the following assertions hold:*

- (i) *The distributions  $\mathcal{D}$ ,  $\mathcal{D}^\perp$  and  $\mathcal{D} \oplus \mathcal{D}^\perp$  are  $\xi$ -parallel if and only if  $h(\xi, FX) \in \Gamma(\nu)$ , for any  $X \in \Gamma(\mathcal{D})$ .*
- (ii) *If  $\overline{M}$  is quasi-K-cosymplectic (or quasi-Kenmotsu), then, for any  $X \in \Gamma(\mathcal{D})$ ,  $[X, \xi] \in \Gamma(\mathcal{D})$  if and only if  $h(\xi, FX) \in \Gamma(\nu)$ .*
- (iii) *If  $\overline{M}$  is quasi-Kenmotsu (or quasi-K-cosymplectic), then,  $[X, \xi] \in \Gamma(\mathcal{D}^\perp)$ , for any  $X \in \Gamma(\mathcal{D}^\perp)$ .*

*Proof.* (i) For  $X \in \Gamma(\mathcal{D})$ ,  $Y \in \Gamma(\mathcal{D}^\perp)$ , using (5.13), we get,

$$\begin{aligned} g(\nabla_\xi X, \xi) &= \xi(g(X, \xi)) - g(X, \nabla_\xi \xi) = 0, \\ g(\nabla_\xi X, Y) &= \xi(g(X, Y)) - g(X, \nabla_\xi Y) = g(F^2 X, \nabla_\xi Y) = -g(FX, F\nabla_\xi Y) \\ &= -g(FX, \nabla_\xi FY) + g(FX, A_{NY}\xi) + g(FX, fh(\xi, Y)) \\ &= g(h(\xi, FX), \varphi Y), \end{aligned}$$

$\nabla_\xi X \in \Gamma(\mathcal{D}^\perp)$  if and only if  $h(\xi, FX) \in \nu$ . Similarly, we can proceed for  $\mathcal{D}^\perp$ . Finally, if  $\mathcal{D}$  and  $\mathcal{D}^\perp$  are  $\xi$ -parallel, then so is  $\mathcal{D} \oplus \mathcal{D}^\perp$ . (ii) If  $M$  is a contact  $CR$ -submanifold of a quasi- $K$ -cosymplectic manifold  $\overline{M}$ , then, by (5.21) and (5.13), and for any  $X \in \Gamma(TM)$  and  $Y \in \Gamma(\mathcal{D}^\perp)$ , we have,

$$g(\nabla_X \xi, \xi) = 0, \quad (5.24)$$

$$\begin{aligned} g(\nabla_X \xi, Y) &= g(\overline{\nabla}_X \xi, Y) - g(h(X, \xi), Y) = g(\varphi(\overline{\nabla}_{\varphi X} \xi), Y) \\ &= -g(h(\varphi X, \xi), \varphi Y). \end{aligned} \quad (5.25)$$

Then,  $\nabla_X \xi \in \Gamma(\mathcal{D})$  if and only if  $g(h(\varphi X, \xi) \in \nu$ . Consequently,  $[X, \xi] = \nabla_X \xi - \nabla_\xi X \in \Gamma(\mathcal{D})$  if and only if  $g(h(\varphi X, \xi) \in \nu$ . On the other hand, if  $M$  is a contact  $CR$ -submanifold of a quasi-Kenmotsu manifold  $\overline{M}$ , for any  $X \in \Gamma(TM)$  and  $Y \in \Gamma(\mathcal{D}^\perp)$  and since  $\varphi\mathcal{D}^\perp \subset TM^\perp$ , we have,

$$\begin{aligned} g(\nabla_X \xi, Y) &= g(\overline{\nabla}_X \xi, Y) - g(h(X, \xi), Y) = -2g(\varphi^2 X, Y) + g(\varphi(\overline{\nabla}_{\varphi X} \xi), Y) \\ &= 2g(X, Y) + g(\varphi(\nabla_{\varphi X} \xi), Y) + g(\varphi h(\varphi X, \xi), Y) \\ &= 2g(X, Y) - g(h(\varphi X, \xi), \varphi Y). \end{aligned} \quad (5.26)$$

The latter vanishes if and only if  $g(h(\varphi X, \xi) \in \nu$ , for any  $X \in \Gamma(\mathcal{D})$ . Thus,  $[X, \xi] = \nabla_X \xi - \nabla_\xi X \in \Gamma(\mathcal{D})$  if and only if  $g(h(\varphi X, \xi) \in \nu$ ,  $\forall X \in \Gamma(\mathcal{D})$ . The assertion (iii) is obvious, using the defining relations of quasi- $K$ -cosymplectic and quasi-Kenmotsu, which completes the proof.  $\square$

The differential of the fundamental form  $\phi$  gives, for any  $X, Y, Z \in \Gamma(TM)$ ,

$$\begin{aligned} 3d\phi(X, Y, Z) &= X(\phi(Y, Z)) + Y(\phi(Z, X)) + Z(\phi(X, Y)) \\ &\quad - \phi([X, Y], Z) - \phi([Z, X], Y) - \phi([Y, Z], X). \end{aligned} \quad (5.27)$$

Using this differential, we have, for any  $Y, Z \in \Gamma(\mathcal{D}^\perp)$ ,

$$3d\phi(X, Y, Z) = -\phi([Y, Z], X) = -g([Y, Z], \varphi X) = g(\varphi[Y, Z], X). \quad (5.28)$$

So,  $d\phi(X, Y, Z) = 0$  if and only if  $[Y, Z] \in \ker(F) = \mathcal{D}^\perp \oplus \{\xi\}$ . This is equivalent to

$$[Y, Z] = v[Y, Z] + \eta([Y, Z])\xi.$$

But,

$$\eta([Y, Z]) = g(\xi, \overline{\nabla}_Y Z) - g(\xi, \overline{\nabla}_Z Y) = g(\overline{\nabla}_Z \xi, Y) - g(\overline{\nabla}_Y \xi, Z).$$

If  $\overline{M}$  is a quasi- $K$ -cosymplectic manifold, we have,  $\overline{\nabla}_Z \xi = \varphi(\overline{\nabla}_{\varphi Z} \xi)$  and this implies that, for any  $Y, Z \in \Gamma(\mathcal{D}^\perp)$ ,

$$g(\overline{\nabla}_Z \xi, Y) = -g(\nabla_{\varphi Z} \xi, \varphi Y) - g(h(\varphi Z, \xi), \varphi Y) = -g(A_{\varphi Y} \xi, \varphi Z) = 0, \quad (5.29)$$

since  $A_{\varphi Y} \xi \in \Gamma(TM)$  and  $\varphi Z \in \Gamma(\varphi \mathcal{D}^\perp)$ . Consequently,  $\eta([Y, Z]) = 0$  and  $[Y, Z] \in \mathcal{D}^\perp$ . On the other hand, if  $\overline{M}$  is a quasi-Kenmotsu manifold, then, by its definition,  $0 = 2d\eta(Y, Z) = -\eta([Y, Z])$  and  $[Y, Z] \in \mathcal{D}^\perp$ . Therefore, we have:

**Lemma 5.1.7.** *Let  $M$  be a contact CR-submanifold of a quasi- $K$ -cosymplectic (or quasi-Kenmotsu) manifold  $\overline{M}$ . The distribution  $\mathcal{D}^\perp$  is integrable if and only if  $d\phi(X, Y, Z) = 0$ , for any  $X$  tangent to  $M$  and  $Y, Z \in \Gamma(\mathcal{D}^\perp)$ .*

From this lemma, we deduce:

**Theorem 5.1.8.** *Let  $M$  be a contact CR-submanifold of a quasi-Kenmotsu manifold  $\overline{M}$ . Then, the distribution  $\mathcal{D}^\perp$  is always integrable .*

*Proof.* For any  $X, Y, Z \in \Gamma(\mathcal{D}^\perp)$ ,  $3d\phi(X, Y, Z) = -g([Y, Z], \varphi X) = 0$ , since  $[Y, Z] \in \Gamma(TM)$  and  $\varphi X \in \Gamma(\varphi \mathcal{D}^\perp) \subset \Gamma(TM^\perp)$ . By Lemma 5.1.7, we complete the proof.  $\square$

Finally, we characterize the integrability of  $\mathcal{D} \oplus \{\xi\}$ .

**Theorem 5.1.9.** *Let  $M$  be a contact CR-submanifold of a quasi- $K$ -cosymplectic manifold  $\overline{M}$ . If the horizontal distribution  $\mathcal{D} \oplus \{\xi\}$  is integrable, then,*

$$h(FX, Y) = h(X, FY), \quad \forall X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\}). \quad (5.30)$$

*Proof.* For any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ , we have,

$$\begin{aligned} \varphi[\varphi X, \varphi Y] &= \overline{\nabla}_{\varphi X} \varphi^2 Y - (\overline{\nabla}_{\varphi X} \varphi) \varphi Y - \overline{\nabla}_{\varphi Y} \varphi^2 X + (\overline{\nabla}_{\varphi Y} \varphi) \varphi X \\ &= -\overline{\nabla}_{\varphi X} Y + \varphi X(\eta(Y))\xi + \eta(Y)\overline{\nabla}_{\varphi X} \xi - (\overline{\nabla}_{\varphi X} \varphi) \varphi Y \\ &\quad + \overline{\nabla}_{\varphi Y} X - \varphi Y(\eta(X))\xi - \eta(X)\overline{\nabla}_{\varphi Y} \xi + (\overline{\nabla}_{\varphi Y} \varphi) \varphi X \\ &= \overline{\nabla}_{\varphi Y} X - \overline{\nabla}_{\varphi X} Y + \{\varphi X(\eta(Y)) - \varphi Y(\eta(X))\}\xi + \eta(Y)(\overline{\nabla}_{\varphi X} \xi) \\ &\quad - (\overline{\nabla}_{\varphi X} \varphi) \varphi Y - \eta(X)(\overline{\nabla}_{\varphi Y} \xi) + (\overline{\nabla}_{\varphi Y} \varphi) \varphi X. \end{aligned} \quad (5.31)$$

Likewise, using the Gauss equation, we get, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\varphi[X, Y] = \nabla_X \varphi Y - \nabla_Y \varphi X + (\overline{\nabla}_Y \varphi) X - (\overline{\nabla}_X \varphi) Y + h(X, \varphi Y) - h(\varphi X, Y). \quad (5.32)$$

Since  $\overline{M}$  is a quasi- $K$ -cosymplectic manifold. Then, we have,

$$(\overline{\nabla}_X \varphi) Y + (\overline{\nabla}_{\varphi X} \varphi) \varphi Y = \eta(Y)(\overline{\nabla}_{\varphi X} \xi),$$

and the relation (5.31) becomes,

$$\begin{aligned} \varphi[\varphi X, \varphi Y] &= \nabla_{\varphi Y} X - \nabla_{\varphi X} Y + h(X, \varphi Y) - h(\varphi X, Y) \\ &\quad + \{\varphi X(\eta(Y)) - \varphi Y(\eta(X))\}\xi + (\overline{\nabla}_X \varphi) Y - (\overline{\nabla}_Y \varphi) X. \end{aligned} \quad (5.33)$$

Adding (5.33) and (5.32), one obtains,

$$\begin{aligned} & \varphi[\varphi X, \varphi Y] + \varphi[X, Y] + \{\varphi Y(\eta(X)) - \varphi X(\eta(Y))\}\xi \\ & = \nabla_{\varphi Y} X - \nabla_{\varphi X} Y + \nabla_X \varphi Y - \nabla_Y \varphi X + 2\{h(X, \varphi Y) - h(\varphi X, Y)\}. \end{aligned}$$

If  $\mathcal{D} \oplus \{\xi\}$  is integrable and since  $\varphi X = FX$ , for any  $X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ , the terms on the left hand-side are tangential to  $M$ . Then, equating normal components in the above equation, we obtain the desired relation.  $\square$

## 5.2 Properties of $CR$ -submersions

A vector field  $X$  on  $M$  is said to be basic if  $X \in \Gamma(\mathcal{D}_x \oplus \{\xi\})$  and  $X$  is  $\pi$ -related to a vector field on  $M'$ , i.e., there exists a vector field  $X_* \in TM'$  such that  $\pi_*(X_x) = X_{*\pi(x)}$ , for each  $x \in M$ . The condition (2),  $\pi_*(\xi_x) = \xi'_{\pi(x)}$ , in the definition of submersion preceding lemma 5.1.1, shows that the structural vector field  $\xi$  is a basic vector field.

**Lemma 5.2.1 ([38]).** *Let  $X$  and  $Y$  be basic vector fields on  $M$ . Then*

- (i)  $g(X, Y) = g'(X_*, Y_*) \circ \pi$ ,
- (ii) *the component  $h([X, Y]) + \eta([X, Y])\xi$  of  $[X, Y]$  is a basic vector field and corresponds to  $[X_*, Y_*]$ , i.e.,  $\pi_*(h([X, Y]) + \eta([X, Y])\xi) = [X_*, Y_*]$ ,*
- (iii)  $[U, X] \in \mathcal{D}^\perp$ , for any  $U \in \mathcal{D}^\perp$ ,
- (iv)  $h(\nabla_X Y) + \eta(\nabla_X Y)\xi$  is a basic vector field corresponding to  $\nabla_{X_*}^* Y_*$ , where  $\nabla^*$  denotes the Levi-Civita connection on  $M'$ .

Note that the above Lemma 5.2.1 is the analogous of Proposition 2.1.1 of O'Neill.

For basic vector fields on  $M$ , we define the operator  $\tilde{\nabla}^*$  corresponding to  $\nabla^*$  by setting, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\tilde{\nabla}_X^* Y = h(\nabla_X Y) + \eta(\nabla_X Y)\xi. \quad (5.34)$$

By (iv) of Lemma 5.2.1,  $\tilde{\nabla}_X^* Y$  is a basic vector field, and we have

$$\pi_*(\tilde{\nabla}_X^* Y) = \nabla_{X_*}^* Y_*. \quad (5.35)$$

Define the tensor field  $C$  by, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\nabla_X Y = \tilde{\nabla}_X^* Y + C(X, Y), \quad (5.36)$$

where  $C(X, Y)$  is the vertical part of  $\nabla_X Y$ . It is known that  $C$  is skew-symmetric and satisfies

$$C(X, Y) = \frac{1}{2}v[X, Y], \quad X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\}). \quad (5.37)$$

Next, we want to examine the influence of a given structure defined on the ambient  $\overline{M}$  on the determination of the corresponding structure on the contact  $CR$ -submanifold  $M$  and the base space  $M'$ .

The curvature tensors  $R, R^*$  of the connection  $\nabla, \nabla^*$  on  $M$  and  $M'$ , respectively, are related as from the work of Papaghiuc [38]. We have for any  $X, Y, Z, W \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\begin{aligned} R(X, Y, Z, W) &= R^*(X_*, Y_*, Z_*, W_*) - g(C(Y, Z), C(X, W)) \\ &\quad + g(C(X, Z), C(Y, W)) + 2g(C(X, Y), C(Z, W)), \end{aligned} \quad (5.38)$$

where,  $\pi_*X = X_*$ ,  $\pi_*Y = Y_*$ ,  $\pi_*Z = Z_*$  and  $\pi_*W = W_*$ .

We now pay attention to the different ambient manifolds involved, namely, quasi- $K$ -cosymplectic and quasi-Kenmotsu manifolds. First of all, we have, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y) = \nabla_X Y + ph(X, Y) + qh(X, Y) \\ &= \tilde{\nabla}_X^* Y + C(X, Y) + ph(X, Y) + qh(X, Y). \end{aligned} \quad (5.39)$$

Using this, we have,

$$\varphi(\overline{\nabla}_X Y) = \varphi\tilde{\nabla}_X^* Y + \varphi C(X, Y) + \varphi ph(X, Y) + \varphi qh(X, Y). \quad (5.40)$$

Replacing  $Y$  by  $\varphi Y$  into the relation (5.39), we obtain

$$\overline{\nabla}_X \varphi Y = \tilde{\nabla}_X^* \varphi Y + C(X, \varphi Y) + ph(X, \varphi Y) + qh(X, \varphi Y). \quad (5.41)$$

If  $\overline{M}$  is a quasi- $K$ -cosymplectic manifold, we find

$$(\overline{\nabla}_X \varphi)Y = \overline{\nabla}_X \varphi Y - \varphi(\overline{\nabla}_X Y) = -(\overline{\nabla}_X \varphi)\varphi Y + \eta(Y)(\overline{\nabla}_{\varphi X} \xi). \quad (5.42)$$

Substituting (5.40) and (5.41) in (5.42), one obtains

$$\begin{aligned} &\tilde{\nabla}_X^* \varphi Y + C(X, \varphi Y) + ph(X, \varphi Y) + qh(X, \varphi Y) - \varphi\tilde{\nabla}_X^* Y \\ &\quad - \varphi C(X, Y) - \varphi ph(X, Y) - \varphi qh(X, Y) \\ &= -(\overline{\nabla}_{\varphi X} \varphi)\varphi Y + \eta(Y)(\overline{\nabla}_{\varphi X} \xi). \end{aligned} \quad (5.43)$$

On the other hand, if  $\overline{M}$  is a quasi-Kenmotsu manifold, we get,

$$(\overline{\nabla}_X \varphi)Y = \varphi((\overline{\nabla}_{\varphi X} \varphi)Y) + g(\varphi X, Y)\xi - 2\eta(Y)\varphi X. \quad (5.44)$$

Putting (5.40) and (5.41) in (5.44), with  $(\overline{\nabla}_X \varphi)Y = \overline{\nabla}_X \varphi Y - \varphi(\overline{\nabla}_X Y)$ , one has,

$$\begin{aligned} &\tilde{\nabla}_X^* \varphi Y + C(X, \varphi Y) + ph(X, \varphi Y) + qh(X, \varphi Y) - \varphi\tilde{\nabla}_X^* Y \\ &\quad - \varphi C(X, Y) - \varphi ph(X, Y) - \varphi qh(X, Y) \\ &= \varphi((\overline{\nabla}_{\varphi X} \varphi)Y) + g(\varphi X, Y)\xi - 2\eta(Y)\varphi X. \end{aligned} \quad (5.45)$$

We have the following results.

**Theorem 5.2.2.** *Let  $\pi : M \longrightarrow M'$  be a submersion of a contact CR-submanifold of a manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$ . Then,*

(i) *If  $\overline{M}$  is quasi-K-cosymplectic, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,*

$$(\tilde{\nabla}_X^* \varphi)Y + (\tilde{\nabla}_{\varphi X}^* \varphi)\varphi Y = \eta(Y)\tilde{\nabla}_{\varphi X}^* \xi, \quad (5.46)$$

$$C(X, \varphi Y) - C(\varphi X, Y) = f\{h(X, Y) + h(\varphi X, \varphi Y)\}, \quad (5.47)$$

$$q\{h(X, \varphi Y) - h(\varphi X, Y)\} = n\{h(X, Y) + h(\varphi X, \varphi Y)\}, \quad (5.48)$$

$$p\{h(X, \varphi Y) - h(\varphi X, Y)\} = \varphi\{C(X, Y) + C(\varphi X, \varphi Y)\}. \quad (5.49)$$

(ii) *If  $\overline{M}$  is quasi-Kenmotsu, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,*

$$(\tilde{\nabla}_X^* \varphi)Y - \varphi((\tilde{\nabla}_{\varphi X}^* \varphi)Y) = g(\varphi X, Y)\xi - 2\eta(Y)\varphi X, \quad (5.50)$$

$$C(X, \varphi Y) - C(\varphi X, Y) = fh(X, Y), \quad (5.51)$$

$$C(X, Y) = -C(\varphi X, \varphi Y), \quad (5.52)$$

$$ph(X, \varphi Y) = \varphi qh(X, Y). \quad (5.53)$$

*Proof.* (i) If  $\overline{M}$  is a quasi-K-cosymplectic manifold, we have,

$$\overline{\nabla}_{\varphi X} \xi = \nabla_{\varphi X} \xi + h(\varphi X, \xi) = \tilde{\nabla}_{\varphi X}^* \xi + C(\varphi X, \xi) + h(\varphi X, \xi), \quad (5.54)$$

and

$$\begin{aligned} (\overline{\nabla}_{\varphi X} \varphi)\varphi Y &= \overline{\nabla}_{\varphi X} \varphi^2 Y - \varphi(\overline{\nabla}_{\varphi X} \varphi Y) \\ &= \nabla_{\varphi X} \varphi^2 Y + h(\varphi X, \varphi^2 Y) - \varphi(\nabla_{\varphi X} \varphi Y + h(\varphi X, \varphi Y)) \\ &= \tilde{\nabla}_{\varphi X}^* \varphi^2 Y + C(\varphi X, \varphi^2 Y) + h(\varphi X, \varphi^2 Y) - \varphi(\tilde{\nabla}_{\varphi X}^* \varphi Y) \\ &\quad - \varphi C(\varphi X, \varphi Y) - \varphi h(\varphi X, \varphi Y) \\ &= (\tilde{\nabla}_{\varphi X}^* \varphi)\varphi Y - C(\varphi X, Y) + \eta(Y)C(\varphi X, \xi) - h(\varphi X, Y) \\ &\quad + \eta(Y)h(\varphi X, \xi) - \varphi C(\varphi X, \varphi Y) - \varphi h(\varphi X, \varphi Y). \end{aligned} \quad (5.55)$$

for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ . Putting the pieces (5.54) and (5.55) into (5.43), we have,

$$\begin{aligned} &(\tilde{\nabla}_X^* \varphi)Y + C(X, \varphi Y) + ph(X, \varphi Y) + qh(X, \varphi Y) - \varphi C(X, Y) \\ &\quad - \varphi nh(X, Y) - \varphi qh(X, Y) \\ &= -(\tilde{\nabla}_{\varphi X}^* \varphi)\varphi Y + \eta(Y)\tilde{\nabla}_{\varphi X}^* \xi + C(\varphi X, Y) + h(\varphi X, Y) \\ &\quad + \varphi C(\varphi X, \varphi Y) + \varphi h(\varphi X, \varphi Y). \end{aligned} \quad (5.56)$$

Comparing the components of  $\mathcal{D} \oplus \{\xi\}$ ,  $\mathcal{D}^\perp$ ,  $\varphi\mathcal{D}^\perp$  and  $\nu$ , respectively, on both sides of (5.56), we find

$$\begin{aligned} &(\tilde{\nabla}_X^* \varphi)Y + (\tilde{\nabla}_{\varphi X}^* \varphi)\varphi Y = \eta(Y)\tilde{\nabla}_{\varphi X}^* \xi, \\ &C(X, \varphi Y) - C(\varphi X, Y) = \varphi p\{h(X, Y) + h(\varphi X, \varphi Y)\}, \\ &q\{h(X, \varphi Y) - h(\varphi X, Y)\} = \varphi q\{h(X, Y) + h(\varphi X, \varphi Y)\}, \\ &p\{h(X, \varphi Y) - h(\varphi X, Y)\} = \varphi\{C(X, Y) + C(\varphi X, \varphi Y)\}. \end{aligned}$$

(ii) Suppose that  $\overline{M}$  is a quasi-Kenmotsu manifold. Using the fact that  $C$  is vertical, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\begin{aligned}\varphi(\overline{\nabla}_{\varphi X}\varphi)Y &= \varphi(\overline{\nabla}_{\varphi X}\varphi Y) - \varphi^2(\overline{\nabla}_{\varphi X}Y) \\ &= \varphi(\widetilde{\nabla}_{\varphi X}^*\varphi Y) - \varphi^2(\widetilde{\nabla}_{\varphi X}^*Y) + \varphi C(\varphi X, \varphi Y) - \varphi^2(C(\varphi X, Y)) \\ &= \varphi((\widetilde{\nabla}_{\varphi X}^*\varphi)Y) + \varphi C(\varphi X, \varphi Y) + C(\varphi X, Y).\end{aligned}\tag{5.57}$$

Putting (5.57) in (5.45), we get,

$$\begin{aligned}(\widetilde{\nabla}_X^*\varphi)Y + C(X, \varphi Y) + ph(X, \varphi Y) + qh(X, \varphi Y) - \varphi C(X, Y) \\ - \varphi ph(X, Y) - \varphi qh(X, Y) \\ = \varphi((\widetilde{\nabla}_{\varphi X}^*\varphi)Y) + g(\varphi X, Y)\xi - 2\eta(Y)\varphi X + \varphi C(\varphi X, \varphi Y) \\ + C(\varphi X, Y).\end{aligned}\tag{5.58}$$

Also, comparing the components of  $\mathcal{D} \oplus \{\xi\}$ ,  $\mathcal{D}^\perp$ ,  $\varphi\mathcal{D}^\perp$  and  $\nu$ , respectively, on both sides of (5.58), we have  $(\widetilde{\nabla}_X^*\varphi)Y - \varphi((\widetilde{\nabla}_{\varphi X}^*\varphi)Y) = g(\varphi X, Y)\xi - 2\eta(Y)\varphi X$ ,  $C(X, \varphi Y) - C(\varphi X, Y) = \varphi ph(X, Y)$ ,  $C(X, Y) = -C(\varphi X, \varphi Y)$  and  $ph(X, \varphi Y) = \varphi qh(X, Y)$ , which complete the proof.  $\square$

Following the nature of ambient manifold, that is, if  $\overline{M}$  is a quasi- $K$ -cosymplectic manifold, for any  $X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$C(X, \varphi X) = \frac{1}{2}\varphi p\{h(X, X) + h(\varphi X, \varphi X)\},\tag{5.59}$$

and if  $\overline{M}$  is a quasi-Kenmotsu manifold, for any  $X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$C(X, \varphi X) = \frac{1}{2}\varphi ph(X, X).\tag{5.60}$$

**Lemma 5.2.3.** *Let  $\pi : M \longrightarrow M'$  be a submersion of a contact CR-submanifold of a quasi- $K$ -cosymplectic manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$ . Then,  $C(\xi, \xi) = 0$ ,  $h(\xi, \xi) = 0$  and  $C(X, \xi) = \varphi ph(\varphi X, \xi)$ ,  $\forall X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ .*

*Proof.* Putting  $Y = \xi$  in the relation (iv) of Theorem 5.2.2 and using the fact that  $\varphi\xi = 0$ , we get,  $ph(\varphi X, \xi) = -\varphi C(X, \xi)$ . Applying  $\varphi$  to this equation, we have, for any  $X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,  $\varphi ph(\varphi X, \xi) = -\varphi^2 C(X, \xi) = C(X, \xi) - \eta(C(X, \xi))\xi = C(X, \xi)$ , since  $\eta(C(X, \xi)) = 0$  because of the fact that  $C$  is vertical and  $\xi$  is a basic vector, and this proves the last relation. The first two relations are obvious.  $\square$

**Theorem 5.2.4.** *Let  $\pi : M \longrightarrow M'$  be a submersion of a contact CR-submanifold of a manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$ . Then,*

- (1) *If  $\overline{M}$  is quasi- $K$ -cosymplectic, then,  $M'$  is also a quasi- $K$ -cosymplectic manifold.*
- (2) *If  $\overline{M}$  is quasi-Kenmotsu, then,  $M$  is  $\mathcal{D} \oplus \{\xi\}$ -totally geodesic and  $M'$  is also a quasi-Kenmotsu manifold.*

*Proof.* (1) Using (i) of Theorem 5.2.2, we have, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$(\tilde{\nabla}_X^* \varphi)Y + (\tilde{\nabla}_{\varphi X}^* \varphi)Y = \eta(Y)\tilde{\nabla}_{\varphi X}^* \xi.$$

Applying  $\pi_*$  to the above equation and using Lemma 5.2.1, equation (5.35), we derive

$$\pi_*((\tilde{\nabla}_X^* \varphi)Y) + \pi_*((\tilde{\nabla}_{\varphi X}^* \varphi)Y) = \pi_*(\eta(Y)\tilde{\nabla}_{\varphi X}^* \xi).$$

That is,

$$(\nabla_{X_*}^* \varphi')Y_* + (\nabla_{\varphi' X_*}^* \varphi')\varphi'Y_* = \eta'(Y_*)\nabla_{\varphi' X_*}^* \xi',$$

which proves that  $M'$  is a quasi- $K$ -cosymplectic manifold. (2) From (5.53), we have  $ph(X, Y) = 0$  and  $qh(X, Y) = 0$ , and therefore,  $h(X, Y) = 0, \forall X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ . This proves that  $M$  is  $\mathcal{D} \oplus \{\xi\}$ -totally geodesic. The last assertion follows from (5.50), mimicking the techniques used in (1).  $\square$

By Proposition 2.4.8, we deduce the following results.

**Theorem 5.2.5.** *Let  $\pi : M \longrightarrow M'$  be a submersion of type II of contact CR-submanifold of a quasi- $K$ -cosymplectic (or quasi-Kenmotsu) manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$  with  $\dim \mathcal{D}^\perp = 2k$ . Then, the base space  $M'$  is a  $(1, 2)$ -symplectic manifold.*

It is known that by a result of Chen [10] that the anti-invariant distribution  $\mathcal{D}^\perp$  of a CR-submanifold of a Kähler manifold is always integrable. This is still true for CR-submanifold of locally conformal Kähler manifold [17]. Now, we have:

**Theorem 5.2.6.** *Let  $\pi : M \longrightarrow M'$  be a submersion of a contact CR-submanifold of a quasi- $K$ -cosymplectic (or quasi-Kenmotsu) manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$ . If the horizontal distribution  $\mathcal{D} \oplus \{\xi\}$  is integrable and the vertical distribution  $\mathcal{D}^\perp$  is parallel, then,  $M$  is CR-product.*

*Proof.* Since the horizontal distribution  $\mathcal{D} \oplus \{\xi\}$  is integrable, then, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ , we have  $[X, Y] \in \Gamma(\mathcal{D} \oplus \{\xi\})$ . Therefore,  $v[X, Y] = 0$ . Now, using the equation (5.37), we have  $C(X, Y) = 0, \forall X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ . Putting the value of  $C(X, Y)$  in (5.36), we have  $\nabla_X Y = \tilde{\nabla}_X^* Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ , which shows that  $\mathcal{D} \oplus \{\xi\}$  is parallel. Since the horizontal distribution  $\mathcal{D} \oplus \{\xi\}$  and vertical distribution  $\mathcal{D}^\perp$  are both parallel, thus, using De Rham's theorem [30, Thm 6.2], it follows that  $M$  is the product  $M_1 \times M_2$ , where  $M_1$  is invariant submanifold of  $\overline{M}$  and  $M_2$  is totally real submanifold of  $\overline{M}$ . Hence,  $M$  is a CR-product.  $\square$

Next, we discuss the holomorphic sectional curvature of quasi- $K$ -cosymplectic, quasi-Kenmotsu manifold  $\overline{M}$  and  $M'$ , respectively.

Let  $\pi : M \longrightarrow M'$  be a submersion of a contact CR-submanifold of a manifold  $\overline{M}$ . For any manifold  $\overline{M}$  and putting  $Y = \varphi X, Z = \varphi Y, W = Y$  in Gauss equation

$$\begin{aligned} \overline{R}(X, Y, Z, W) &= R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) \\ &\quad + g(h(X, Z), h(Y, W)), \end{aligned} \tag{5.61}$$

to obtain the following equation, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\begin{aligned} \bar{R}(X, \varphi X, \varphi Y, Y) &= R(X, \varphi X, \varphi Y, Y) - g(h(X, Y), h(\varphi X, \varphi Y)) \\ &\quad + g(h(X, \varphi Y), h(\varphi X, Y)). \end{aligned} \quad (5.62)$$

Substituting  $h = ph + qh$ , in the above equation and using (5.38), we derive

$$\begin{aligned} \bar{R}(X, \varphi X, \varphi Y, Y) &= R(X, \varphi X, \varphi Y, Y) - g(h(X, Y), h(\varphi X, \varphi Y)) \\ &\quad + g(h(X, \varphi Y), h(\varphi X, Y)) \\ &= R^*(X_*, \varphi'X_*, \varphi'Y_*, Y_*) - g(C(X, Y), C(\varphi X, \varphi Y)) \\ &\quad + g(C(X, \varphi Y), C(\varphi X, Y)) + 2g(C(X, \varphi X), C(\varphi Y, Y)) \\ &\quad - g(ph(X, Y), ph(\varphi X, \varphi Y)) - g(qh(X, Y), qh(\varphi X, \varphi Y)) \\ &\quad + g(ph(X, \varphi Y), ph(\varphi X, Y)) + g(qh(X, \varphi Y), qh(\varphi X, Y)). \end{aligned} \quad (5.63)$$

Suppose that the distribution  $\mathcal{D} \oplus \{\xi\}$  is integrable. Then, we have

$$C(X, Y) = \frac{1}{2}v[X, Y] = 0, \quad (5.64)$$

for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ . Thus, from the definition of  $C$ , we have  $\nabla_X Y = \tilde{\nabla}_X^* Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ , i.e.,  $\mathcal{D} \oplus \{\xi\}$  is parallel. By relation (5.30) and since  $\varphi X = FX$ ,  $h(\varphi X, \xi) = 0$  which implies that  $h(X, \xi) = 0$ , since  $h(\xi, \xi) = 0$ . Taking  $Y = \varphi Y$  in (5.30), one obtains,

$$h(\varphi X, \varphi Y) = -h(X, Y), \quad \forall X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\}). \quad (5.65)$$

Using this, the relation (5.63) becomes, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\begin{aligned} \bar{R}(X, \varphi X, \varphi Y, Y) &= R^*(X_*, \varphi'X_*, \varphi'Y_*, Y_*) - g(ph(X, Y), ph(\varphi X, \varphi Y)) \\ &\quad - g(qh(X, Y), qh(\varphi X, \varphi Y)) + g(ph(X, \varphi Y), ph(\varphi X, Y)) \\ &\quad + g(qh(X, \varphi Y), qh(\varphi X, Y)) \\ &= R^*(X_*, \varphi'X_*, \varphi'Y_*, Y_*) + \|ph(X, Y)\|^2 + \|qh(X, Y)\|^2 \\ &\quad + \|ph(X, \varphi Y)\|^2 + \|qh(X, \varphi Y)\|^2. \end{aligned} \quad (5.66)$$

It is easy to check that, for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,

$$\|h(X, Y)\|^2 = \|ph(X, Y)\|^2 + \|qh(X, Y)\|^2.$$

Therefore, we have,

$$\bar{R}(X, \varphi X, \varphi Y, Y) = R^*(X_*, \varphi'X_*, \varphi'Y_*, Y_*) + \|h(X, Y)\|^2 + \|h(X, \varphi Y)\|^2, \quad (5.67)$$

which implies that

$$\bar{H}(X) = H'(X_*) + \|h(X, X)\|^2 + \|h(X, \varphi X)\|^2, \quad (5.68)$$

where  $\bar{H}(X) = \bar{R}(X, \varphi X, \varphi X, X)$  and  $H'(X_*) = R^*(X_*, \varphi'X_*, \varphi'X_*, X_*)$  are the holomorphic sectional curvatures of  $\bar{M}$  and  $M'$ , respectively.

**Theorem 5.2.7.** *Let  $\pi : M \longrightarrow M'$  be a submersion of a contact CR-submanifold of a quasi-K-cosymplectic manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$  with integrable  $\mathcal{D} \oplus \{\xi\}$ . Then, the holomorphic sectional curvatures  $\overline{H}$  and  $H^*$  of  $\overline{M}$  and  $M'$ , respectively, satisfy*

$$\overline{H}(X) \geq H'(X_*), \quad \forall X \in \Gamma(\mathcal{D} \oplus \{\xi\}), \quad \|X\| = 1, \quad \pi_*X = X_*, \quad (5.69)$$

and the equality holds if and only if  $M$  is  $\mathcal{D} \oplus \{\xi\}$ -totally geodesic.

*Proof.* The first assertion holds from (5.68). The equality holds if and only if  $h(X, X) = 0$  and  $h(X, \varphi X) = 0$ , for any  $X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,  $\|X\| = 1$ . From  $h(X, X) = 0$ ,  $X \in \Gamma(\mathcal{D} \oplus \{\xi\})$ ,  $\|X\| = 1$  and linearity of  $h$  it follows immediately that  $h(X, Y) = 0$ , for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$  and proves that  $M$  is  $\mathcal{D} \oplus \{\xi\}$ -totally geodesic.  $\square$

This result is similar of the one found in [16] for CR-submanifold of a quasi-Kähler manifold onto an almost Hermitian manifold.

When the ambient manifold  $\overline{M}$  is quasi-Kenmotsu, then, using (5.52), (5.63), (5.60) and (2) in Theorem 5.2.4, the curvature tensors  $\overline{R}$  and  $R^*$  are related as,

$$\begin{aligned} \overline{R}(X, \varphi X, \varphi Y, Y) &= R^*(X_*, \varphi'X_*, \varphi'Y_*, Y_*) - g(C(X, Y), C(\varphi X, \varphi Y)) \\ &\quad + g(C(X, \varphi Y), C(\varphi X, Y)) + 2g(C(X, \varphi X), C(\varphi Y, Y)) \\ &= R^*(X_*, \varphi'X_*, \varphi'Y_*, Y_*) + \|C(X, Y)\|^2 + \|C(X, \varphi Y)\|^2 \\ &\quad + 2g(C(X, \varphi X), C(\varphi Y, Y)) \\ &= R^*(X_*, \varphi'X_*, \varphi'Y_*, Y_*) + \|C(X, Y)\|^2 + \|C(X, \varphi Y)\|^2, \end{aligned} \quad (5.70)$$

for any  $X, Y \in \Gamma(\mathcal{D} \oplus \{\xi\})$ , since  $C(X, \varphi X) = \frac{1}{2}\varphi ph(X, X) = 0$ . The relation (5.70) reduces to,

$$\overline{H}(X) = H'(X_*) + \|C(X, X)\|^2 + \|C(X, \varphi X)\|^2. \quad (5.71)$$

**Theorem 5.2.8.** *Let  $\pi : M \longrightarrow M'$  be a submersion of a contact CR-submanifold of a quasi-Kenmotsu manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$ . Then, the holomorphic sectional curvatures  $\overline{H}$  and  $H^*$  of  $\overline{M}$  and  $M'$ , respectively, satisfy*

$$\overline{H}(X) \geq H'(X_*), \quad \forall X \in \Gamma(\mathcal{D} \oplus \{\xi\}), \quad \|X\| = 1, \quad \pi_*X = X_*, \quad (5.72)$$

and the equality holds if and only if the distribution  $\mathcal{D} \oplus \{\xi\}$  is integrable.

*Proof.* The inequality follows from the relation (5.71). If  $\overline{H}(X) = H'(X_*)$  if and only if the skew-symmetric tensor  $C$  vanishes which means that the distribution  $\mathcal{D} \oplus \{\xi\}$  is parallel and this completes the proof.  $\square$

Also, we have,

**Theorem 5.2.9.** *Let  $\pi : M \longrightarrow M'$  be a submersion of a contact CR-submanifold of a quasi-Kenmotsu manifold  $\overline{M}$  onto an almost contact metric manifold  $M'$  such that the holomorphic sectional curvatures  $\overline{H}$  and  $H^*$  of  $\overline{M}$  and  $M'$ , respectively, coincide on  $\mathcal{D} \oplus \{\xi\}$ . Then,  $M$  is locally a product  $M^* \times C$ , where  $M^*$  is a totally geodesic leaf of  $\mathcal{D} \oplus \{\xi\}$  and  $C$  is a curve tangent to the distribution  $\mathcal{D}^\perp$ .*

*Proof.* By Theorem 5.2.8, we have, the distribution  $\mathcal{D} \oplus \{\xi\}$  is integrable. We deduce that  $\mathcal{D} \oplus \{\xi\}$  determines a foliation and if  $M^*$  is a leaf of  $\mathcal{D} \oplus \{\xi\}$ , it is totally geodesic. By Theorem 5.1.8, the distribution  $\mathcal{D}^\perp$  is integrable and then, it defines a foliation. So being  $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\}$ , we complete the proof.  $\square$

Let us give a short conclusion and some open problems indicating how to pursue, in the future, this study.

## CONCLUSION

In the study of submersions, it can be noted that the first step is to find manifolds to use as total and base space of fibration. Next, one tests the compatibility with the Riemannian structure and then establishes relevant properties. It is known that, almost contact metric submersions are Riemannian submersions between manifolds equipped with special structures.

Concerning manifolds, among 4,096 classes of almost contact metric structures, we have examined only a few number of them, less than 100. It must be of interest to pursue the study with new manifolds. What will be needed is the defining relations of some new almost contact metric manifolds. The new manifolds can be constructed using the direct sums of the irreducible Chinea-Gonzalez  $C_i$ -manifolds [15]. Other manifolds can be obtained by the use of warped product following the formalism of Kenmotsu [29]. In this case, we have constructed and studied 9 classes such as: nearly Kenmotsu, quasi Kenmotsu,  $G_i$ -Kenmotsu where  $i \in \{1, 2\}$ , generalized Kenmotsu [49] and some others.

Recall that in our research, we are interested by the following steps among others:

Let

$$F \longrightarrow M \xrightarrow{\pi} M'$$

be an almost contact metric submersion of type I. It is known that the fibres,  $F$ , are almost Hermitian manifolds.

(1) We deduce the defining relations of the total space  $M$  and deduce the structure of the fibres;

(2) With the structure of the fibres, we interrelate the curvature properties of  $M$  with that of the fibres  $F$ ;

(3) Given the defining relations of the base space  $M'$ , we have shown how can be transferred this structure to the total space  $M$  and what can be the role of the fibres?

### **New directions of research.**

We think that it is possible to introduce the concept of *conjugaison* in contact and paracontact geometries and obtain various types of almost contact metric submersions. Let us say that a  $(\varphi, \xi, \eta)$ -structure is said to be

1. Conjugated almost contact structure if  $\varphi^2 = -I - \eta \otimes \xi$ .
2. Almost paracontact structure if  $\varphi^2 = I - \eta \otimes \xi$ .
3. Conjugated almost paracontact structure if  $\varphi^2 = I + \eta \otimes \xi$ .
4. Almost para-Hermitian structure if  $J^2 = I$ , where  $J$  is a an almost para complex structure.

From Gündüzalp and Sahin [23], we have the following defining relations for some remarkable classes in the case of paracontact structures.

An almost paracontact manifold is said to be:

- (1) *para-normal* if  $N_\varphi - 2d\eta \otimes \xi = 0$ ,
- (2) *para-contact* if  $\phi = d\eta$ ,
- (3) *K-para-contact* if it is para-contact and  $\xi$  is Killing,
- (4) *para-cosymplectic* if  $\nabla\eta = 0$  and  $\nabla\phi = 0$ ,
- (5) *almost para-cosymplectic* if  $d\phi = 0$  and  $d\eta = 0$ ,
- (6) *weakly para-cosymplectic* if  $d\phi = 0$ ,  $d\eta = 0$  and  $[R(D, E), \varphi] = R(D, E)\varphi - \varphi R(D, E) = 0$ ,
- (7) *para-Sasakian* if  $\phi = d\eta$  and  $M$  is para-normal,
- (8) *quasi- para-Sasakian* if  $d\phi = 0$  and  $M$  is para-normal.

Note that Gündüzalp and Sahin have introduced the study of submersions considering that the total and the base spaces are paracontact.

Our purpose will be to consider the case where the total space remains an almost contact metric manifold; the base space being conjugated almost contact metric, almost paracontact, conjugated almost paracontact or almost para-Hermitian manifolds. In this case, regarding the fundamental properties, related to the structure of the fibres, we can conjecture the following two elementaries results.

**Theorem A.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$  be a Riemannian submersion whose total space  $M$  is an almost contact metric manifold. If the base space  $M'$  is a conjugated almost contact manifold, an almost paracontact manifold or a conjugated almost paracontact manifold, then the fibres are almost Hermitian manifolds.*

**Theorem B.** *Let  $\pi : M^{2m+1} \longrightarrow M'^{2m'}$  be a Riemannian submersion whose total space  $M$  is an almost contact metric manifold. If the base space  $M'$  is an almost Hermitian or an almost para-Hermitian manifold, then the fibres are almost contact metric manifolds.*

Theorem A shows that conjugated almost contact, almost paracontact and conjugated almost paracontact manifolds have some common properties which force the fibres to lie in the class of almost Hermitian manifolds. This result resembles to that of Gündüzalp and Sahin [23, Prop 3.5]. Suppose that, in this case, the base space is a weakly para-cosymplectic manifold, what can be the structure of the fibres?

In the same manner, Theorem B shows that almost Hermitian and almost para-Hermitian manifolds have some common properties which force the fibres to lie in the class of almost contact manifolds.

We hope that, in the future, the study can be extended in this way by adding some other new objects.

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