A Lie Symmetry analysis of the heat equation through modified one-parameter local point transformation

by

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Abstract

Using a Lie symmetry group generator and a generalized form of Manale's formula for solving second order ordinary differential equations, we determine new symmetries for the one and two dimensional heat equations, leading to new solutions. As an application, we test a formula resulting from this approach on thin plate heat conduction.

Keywords: Heat equation; Partial differential equation; Lie Point Symmetry; Lie equivalence transformation; Invariant solution.

Declaration

Student number: 43216137

I declare that **Lie Symmetry analysis of the heat equation through modified one-parameter local point transformation** is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of a complete list of references.

$\overline{SIGNATURE \quad (Mr)}$ $\overline{DATE}$

Dedication

I would like to dedicate this dissertation to my parents Harry and Maggy Adams, my brothers and sister for their everlasting support and courage they showed me,and to the loving memory of my uncle Seun Mollepolle and my grandfather Grandwell Chathewa.

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- I wish to sincerely thank Dr J.M. Manale for his careful supervision of this project. For reading through and correcting this work meticulously and making his wide spectrum of books and articles available to me.

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Introduction

Lie group theory has been used in the study of ordinary differential (ODE) and Partial differential equations(PDE) . The method has developed into a useful tool to solve differential equations, to classify them and to preserve the set of solutions in the ODE [5],[12] [4], [7], [11]. A symmetry group of a system of differential equations is a group of transformations which maps any solution to another solution of the system.

If a system of partial differential equations is invariant under a Lie group of point transformations, one can find, constructively, special solutions, called similarity solutions or invariant solution which are invariant under some subgroup of the full group admitted by the system. These solutions result from solving a reduced system of differential equations with fewer independent variables.

A systematic approach is given in finding the invariant solutions to partial differential equations and,in particular the heat equation by the use of transformation groups. The new solutions of the heat equation are obtained [18].

The present work titled **A Lie Symmetry analysis of the heat equation through modified one-parameter local point transformation** seeks to explore the analysis of the one-dimensional and two-dimensional heat equations through determining the new symmetries and invariant solutions of some of these symmetries. The analysis is through a new method developed in [18]. Throughout the project we use Lie point symmetries.

The dissertation outline is as follows:

Chapter 1 presents the concept of Local one-parameter Point Symmetries. We introduce concepts of Local one-parameter point transformations, generator, prolongation formulas, determining equation and Lie algebras. These concepts serve as tools in the analysis of the Heat equation.

Chapter 2 presents the symmetry analysis of one-dimensional and two-dimensional heat equations and their invariant solutions (2.1) and(2.50) as outlined in [4],[11] and [14].

Chapter 3 is the core of the thesis. It introduces the symmetry analysis of equation (2.1) and(2.50) using the method developed through contributions in ([18]) and ([23]). The chapter details the symmetries of one-dimensional and two-dimensional heat equations (2.1) and(2.50) through modified Local one-parameter transformations.

Chapter 4 presents other areas where the method was successfully applied. The application includes heat conduction in thin plates.

Chapter 1

Local One-parameter Point Symmetries

The chapter presents the underlying theory of Lie symmetry analysis and the tools we will use in subsequent chapters. The most common of all symmetries in practice are local one-parameter point symmetries.

1.1 Local one-parameter point transformations

To begin, we consider the following definition.

Definition 1 Let

$$\bar{\mathbf{x}} = \mathbf{G}\left(\mathbf{x};\epsilon\right) \tag{1.1}$$

be a family of one-parameter $\epsilon \in R$ invertible transformations, of points $\mathbf{x} = (x^1, \cdots, x^N) \in \mathbf{R}^N$ into points $\bar{\mathbf{x}} = (\bar{x}^1, \cdots, \bar{x}^N) \in \mathbf{R}^{\mathbf{N}}$. These are known as one-parameter transformations, and subject to the conditions

$$\bar{\mathbf{x}}|_{\epsilon=0} = \mathbf{x}. \tag{1.2}$$

That is,

$$\mathbf{G}\left(\mathbf{x};\epsilon\right)\Big|_{\epsilon=0} = \mathbf{x}. \tag{1.3}$$

Expanding (1.1) about $\epsilon = 0$, in some neighborhood of $\epsilon = 0$, we get

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \left(\left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0} \right) + \frac{\epsilon^2}{2} \left(\left. \frac{\partial^2 \mathbf{G}}{\partial \epsilon^2} \right|_{\epsilon=0} \right) + \cdots = \mathbf{x} + \epsilon \left(\left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0} \right) + O\left(\epsilon^2\right). \quad (1.4)$$

Letting

$$\xi(\mathbf{x}) = \left. \frac{\partial \mathbf{G}}{\partial \epsilon} \right|_{\epsilon=0}, \quad (1.5)$$

reduces the expansion to

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \xi(\mathbf{x}) + O\left(\epsilon^2\right). \quad (1.6)$$

Definition 2 The expression

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon \xi(\mathbf{x}), \quad (1.7)$$

is called a local one-parameter point transformation.

The components of $\xi(\mathbf{x})$ are called the infinitesimals of (1.1) [4].

1.2 Local one-parameter point transformation groups

The set G of transformations

$$\bar{\mathbf{x}}_{\epsilon_i} = \mathbf{x} + \epsilon_i \left(\left. \frac{\partial \mathbf{G}}{\partial \epsilon_i} \right|_{\epsilon_i=0} \right) + \frac{\epsilon_i^2}{2} \left(\left. \frac{\partial^2 \mathbf{G}}{\partial \epsilon_i^2} \right|_{\epsilon_i=0} \right) + \cdots, \quad i = 1, 2, 3, \cdots, \quad (1.8)$$

becomes a group only when truncated at $O(\epsilon^2)$.

That is, G is a group since the following properties hold under binary operation $+$:

1. **Closure**. If $\bar{\mathbf{x}}_{\epsilon_1}, \bar{\mathbf{x}}_{\epsilon_2} \in G$ and $\epsilon_1, \epsilon_2 \in R$, then

$$\bar{\mathbf{x}}_{\epsilon_1} + \bar{\mathbf{x}}_{\epsilon_2} = (\epsilon_1 + \epsilon_2)\xi(\mathbf{x}) = \bar{\mathbf{x}}_{\epsilon_3} \in G, \quad \text{and} \quad \epsilon_3 = \epsilon_1 + \epsilon_2 \in R.$$

2. **Identity**. If $\bar{\mathbf{x}}_0 \equiv I \in G$ such that for any $\epsilon \in R$

$$\bar{\mathbf{x}}_0 + \bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon + \bar{\mathbf{x}}_0,$$

then $\bar{\mathbf{x}}_0$ is an identity in G.

3. **Inverses**. For $\bar{\mathbf{x}}_\epsilon \in G$, $\epsilon \in R$, there exists $\bar{\mathbf{x}}_\epsilon^{-1} \in G$, such that

$$\bar{\mathbf{x}}_\epsilon^{-1} + \bar{\mathbf{x}}_\epsilon \;=\; \bar{\mathbf{x}}_\epsilon + \bar{\mathbf{x}}_\epsilon^{-1}, \quad \bar{\mathbf{x}}_\epsilon^{-1} \;=\; \bar{\mathbf{x}}_{\epsilon^{-1}},$$

and $\epsilon^{-1} = -\epsilon \in D$, where $+$ is a binary composition of transformations and it is understood that $\bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon - \mathbf{x}$. Associativity follows from the closure property.

Example 1 :

Group of Rotations in the Plane

$$\bar{x}_1 \;=\; x_1 \cos\epsilon + x_2 \sin\epsilon,$$

$$\bar{x}_2 \;=\; x_2 \cos\epsilon - x_1 \sin\epsilon.$$

That is,

$$\bar{x}_1 \;=\; x_1 + x_2\epsilon,$$

$$\bar{x}_2 \;=\; x_2 - x_1\epsilon.$$

Example 2 : Group of Translations in the Plane

$$\bar{x}_i \;=\; x_i + \epsilon,$$

$$\bar{x}_j \;=\; x_j.$$

Example 3 :

Group of Scalings in the Plane

$$\bar{x}_i = (1+\epsilon)x_i,$$

$$\bar{x}_j = (1+\epsilon)^2 x_j. \qquad [4], [13].$$

1.3 The group generator

The local one-parameter point transformations in (1.7) can be rewritten in the form

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon\xi(\mathbf{x}) \cdot \nabla\, \mathbf{x}, \tag{1.9}$$

so that

$$\bar{\mathbf{x}} = (1 + \epsilon\xi(\mathbf{x}) \cdot \nabla)\, \mathbf{x}. \tag{1.10}$$

An operator,

$$G = \xi(\mathbf{x}) \cdot \nabla, \tag{1.11}$$

can then be introduced, so that (1.9) assumes the form

$$\bar{\mathbf{x}} = (1 + \epsilon G)\, \mathbf{x}. \tag{1.12}$$

The operator (1.11) has the expanded form

$$G = \sum_{k=1}^{N} \xi^k \frac{\partial}{\partial x^k}, \tag{1.13}$$

or simply

$$G = \xi^k \frac{\partial}{\partial x^k}. \qquad [4], [14] \tag{1.14}$$

1.4 Prolongations formulas

It often happens that the function F in (1.33) does not only depend on the point $\mathbf{x}$ alone, but also on the derivatives. When that is the case then we have to use the prolonged form of the operator G.

1.4.1 The case $N = 2$, with $x^1 = x$ and $x^2 = y$

The case $N = 2$, with $x^1 = x$ and $x^2 = y$ reduces (1.13) to

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{1.15}$$

In determining the prolongations, it is convenient to use the operator of total differentiation

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots, \tag{1.16}$$

where

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \quad \cdots. \tag{1.17}$$

The derivatives of the transformed point is then

$$\bar{y}' = \frac{d\bar{y}}{d\bar{x}}. \tag{1.18}$$

Since

$$\bar{x} = x + \epsilon\xi \quad \text{and} \quad \bar{y} = y + \epsilon\eta, \tag{1.19}$$

then

$$\bar{y}' = \frac{dy + \epsilon d\eta}{dx + \epsilon d\xi}. \tag{1.20}$$

That is,

$$\bar{y}' = \frac{dy/dx + \epsilon d\eta/dx}{dx/dx + \epsilon d\xi/dx}. \tag{1.21}$$

Now introducing the operator D:

$$\bar{y}' = \frac{y' + \epsilon D(\eta)}{1 + \epsilon D(\xi)} = \frac{(y' + \epsilon D(\eta))(1 - \epsilon D(\xi))}{1 - \epsilon^2(D(\xi))^2}. \tag{1.22}$$

Hence

$$\bar{y}' = \frac{y' - \epsilon(D(\eta) - y'D(\xi)) - \epsilon^2(D(\xi))}{1 - \epsilon^2(D(\xi))^2}. \tag{1.23}$$

That is,

$$\bar{y}' = y' + \epsilon(D(\eta) - y'D(\xi)), \tag{1.24}$$

or

$$\bar{y}' = y' + \epsilon\zeta^1, \tag{1.25}$$

with

$$\zeta^1 = D(\eta) - y'D(\xi). \tag{1.26}$$

It expands into

$$\zeta^1 = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y. \tag{1.27}$$

The first prolongation of G is then

$$G^{[1]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'}. \tag{1.28}$$

For the second prolongation, we have

$$\bar{y}'' = \frac{y'' + \epsilon D(\zeta^1)}{1 + \epsilon D(\xi)} \approx y'' + \epsilon\zeta^2, \tag{1.29}$$

with

$$\zeta^2 = D(\zeta^1) - y''D(\xi). \tag{1.30}$$

This expands into

$$\begin{aligned} \zeta^2 &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\ &- y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y''. \end{aligned} \tag{1.31}$$

The second prolongation of G is then

$$G^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'} + \zeta^2\frac{\partial}{\partial y''}. \tag{1.32}$$

Most applications involve up to second order derivatives. It is reasonable then to pause here, for this case [12] .

1.4.2 Invariant functions in R^2

Theorem 1 A function $F(x,y)$ is an invariant of the group of transformations (1.13) if for each point (x,y) it is constant along the trajectory determined by the totality of transformed points $(\bar{x},\bar{y})$:

$$F(\bar{x},\bar{y}) = F(x,y). \tag{1.33}$$

This requires that

$$GF = 0, \tag{1.34}$$

leading to the characteristic system

$$\frac{dx}{\xi} = \frac{dy}{\eta}. \tag{1.35}$$

1.4.3 Multi-dimensional cases

In dealing with the multi-dimensional cases, we may recast the generator (1.13) as

$$G = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}. \tag{1.36}$$

We consider the kth-order partial differential equation

$$F(x,u,u_{(1)},u_{(2)},...,u_{(k)}) \ \text{ where } \ x=(x_1\ldots x_n), \quad u_{(1)} = \frac{\partial u}{\partial x} \tag{1.37}$$

By definition of symmetry, the transformations (1.1) form a symmetry group G of the system (1.37) if the function $\bar{u}=\bar{u}(\bar{x})$ satisfies (1.50) whenever the function $u=u(x)$ satisfies (1.37). The transformed derivatives $\bar{u}_{(1)},\ldots,\bar{u}_{(k)}$ are found from (1.4) by using the formulae of change of variables in the derivatives, $D_i = D_i(f^j)\bar{D}_j$. [4]

Here

$$D_i = \frac{\partial}{\partial x^i} + u_i^a(\frac{\partial}{\partial u^a}) + u_{ij}^a(\frac{\partial}{\partial u_j^a} + \cdots) \tag{1.38}$$

is the total derivative operator w.r.t. x^i and $\bar{D}_j$ is given in terms of the transformed variables. The transformations (1.13) together with the transformations on $\bar{u}_{(1)}$ form a group, $G^{[1]}$, which is the first prolonged group which acts in the space $(x, u, \bar{u}_{(1)})$. Likewise, we obtain the prolonged groups $G^{[2]}$ and so on up to $G^{[k]}$.

The infinitesimal transformations of the prolonged groups are:

$$\begin{aligned} \bar{u}_i^a &\approx u_i^a + a\zeta_i^a(x, u, u_{(1)}) \,, \\ \bar{u}_{ij}^a &\approx u_{ij}^a + a\zeta_{ij}^a(x, u, u_{(1)}, u_{(2)}) \,, \\ &\vdots \\ \bar{u}_{i_1 \dots i_k}^a &\approx u_{i_1 \dots i_k}^a + a\zeta_{i_1 \dots i_k}^a(x, u, u_{(1)}, \dots, u_{(k)}) \,. \end{aligned} \tag{1.39}$$

The functions $\zeta_i^a(x, u, u_{(1)})$, $\zeta_{ij}^a(x, u, u_{(1)}, u_{(2)})$ and
$\zeta_{i_1 \dots i_k}^a(x, u, u_{(1)}, \dots, u_{(k)})$ are given, recursively, by the *prolongation formulas*:

$$\begin{aligned} \zeta_i^a &= D_i(\eta^a) - u_j^a D_i(\xi^j) \,, \\ \zeta_{ij}^a &= D_j(\zeta_i^a) - u_{il} D_j(\xi^l) \,, \\ &\vdots \\ \zeta_{i_1 \dots i_k}^a &= D_{i_k}(\zeta_{i_1 \dots i_{k-1}}^a) - u_{l i_1 \dots i_{k-1}} D_{i_k}(\xi^l) \,. \end{aligned} \tag{1.40}$$

The generator of the prolonged groups are:

$$\begin{aligned} G^{[1]} &= \xi^i(x,u)\tfrac{\partial}{\partial x^i} + \eta^a(x,u)\tfrac{\partial}{\partial u^a} + \zeta_i^a(x, u, u_{(1)})\tfrac{\partial}{\partial u_i^a} \,, \\ &\vdots \end{aligned} \tag{1.41}$$

$$\begin{aligned} G^{[k]} &= \xi^i(x,u)\tfrac{\partial}{\partial x^i} + \eta^a(x,u)\tfrac{\partial}{\partial u^a} + \zeta_i^a(x, u, u_{(1)})\tfrac{\partial}{\partial u_i^a} \\ &\quad + \cdots + \zeta_{i_1 \dots i_k}^a(x, u, \dots, u_{(k)})\tfrac{\partial}{\partial u_{i_1 \dots i_k}^a} \text{ .[4], [13]} \end{aligned} \tag{1.42}$$

Definition 3 A differential function $F(x, u, \dots, u_{(p)})$, $p \geq 0$, is a pth-order differential invariant of a group G if

$$F(x, u, \dots, u_{(p)}) = F(\bar{x}, \bar{u}, \dots, \bar{u}_{(p)}),$$

i.e. if F is invariant under the prolonged group $G^{[p]}$, where for $p=0$, $u_{(0)} \equiv u$ and $G^{[0]} \equiv G$.

Theorem 2 A differential function $F(x, u, \ldots, u_{(p)})$, $p \geq 0$, is a pth-order differential invariant of a group G if

$$G^{[p]}F = 0\,, \tag{1.43}$$

where $G^{[p]}$ is the pth prolongation of G and for $p=0$, $G^{[0]} \equiv G$.

The substitution of (1.41) and (1.42) into (1.49) gives rise to

$$E^{\sigma}(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(k)}) \approx E^{\sigma}(x, u, u_{(1)}, \ldots, u_{(k)}) + a(G^{[k]}E^{\sigma})\,, \tag{1.44}$$

$$\sigma = 1, \ldots, \tilde{m}\,.$$

Thus, we have

$$G^{[k]}E^{\sigma}(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \qquad \sigma = 1, \ldots, \tilde{m}, \tag{1.45}$$

whenever (1.37) is satisfied. The converse also applies.

1.4.4 Invariant functions in R^N

Theorem 3 A function $F(\mathbf{x})$ is an invariant of the group of transformations (1.13) if for each point $\mathbf{x}$ it is constant along the trajectory determined by the totality of transformed points $\bar{\mathbf{x}}$:

$$F(\bar{\mathbf{x}}) = F(\mathbf{x}). \tag{1.46}$$

This requires that

$$GF = 0, \tag{1.47}$$

leading to the characteristic system

$$\frac{dx^1}{\xi^1} = \cdots = \frac{dx^N}{\xi^N}. \tag{1.48}$$

1.5 Determining equations

Equations (1.45) are called the *determining equations*. They are written compactly as

$$G^{[k]}E^{\sigma}(x, u, u_{(1)}, \ldots, u_{(k)})\,\big|_{(1)} = 0, \qquad \sigma = 1, \ldots, \tilde{m}, \tag{1.49}$$

where $\big|_{(1)}$ means evaluated on the surface (1.37).

The determining equations are linear homogeneous partial differential equations of order k for the unknown functions $\xi^i(x, u)$ and $\eta^a(x, u)$. These are consequences of the prolongation formulae (1.40). Equations (1.49) also involve the derivatives $u_{(1)}, \ldots, u_{(k)}$ some of which are eliminated by the system (1.37). We then equate the coefficients of the remaining unconstrained partial derivatives of u to zero. In general, (1.49) decomposes into an overdetermined system of equations, that is, there are more equations than the $n+m$ unknowns ξ^i and η^a. There are computer algebra programs that can perform the task of solving determining equations (see, e.g. Baumann 1992).

Since the determining equations are linear homogeneous, their solutions form a *vector space* L.[4]

1.6 Lie algebras

There is another important property of the determining equations, viz. if the generators

$$G_1 = \xi_1^i(x, u)\frac{\partial}{\partial x^i} + \eta_1^a(x, u)\frac{\partial}{\partial u^a}$$

and

$$G_2 = \xi_2^i(x, u)\frac{\partial}{\partial x^i} + \eta_2^a(x, u)\frac{\partial}{\partial u^a}$$

satisfy the determining equations, so do their *commutator* $[G_1, G_2] = G_1G_2 - G_2G_1$

$$[G_1, G_2] = \left(G_1(\xi_2^i) - G_2(\xi_1^i)\right)\frac{\partial}{\partial x^i} + \left(G_1(\eta_2^a) - G_2(\eta_1^a)\right)\frac{\partial}{\partial u^a}$$

which obeys the properties of bilinearity, skew-symmetry and Jacobi's identity, viz.

1. **Bilinearity**. If $G_1, G_2, G_3 \in L$, then

$$[\alpha G_1 + \beta G_2, G_3] \;=\; \alpha [G_1, G_3] + \beta [G_2, G_3], \quad \alpha, \beta \text{ are scalars}$$

2. **Skew symmetry**. If $G_1, G_2 \in L$, then

$$[G_1, G_2] \;=\; -[G_2, G_1].$$

3. **Jacobi Identity**. If $G_1, G_2, G_3 \in L$, then

$$[[G_1, G_2], G_3] + [[G_2, G_3], G_1] + [[G_3, G_1], G_2] = 0.$$

The vector space L of all solutions of the determining equations forms a *Lie algebra* which generates a multi-parameter group admitted by (1.37).[13],[14]

1.7 Solvable Lie algebras

In this section, we will show that if $r = 1$, then the order of an ODE can be reduced constructively by one. If $n > 2$ and $r = 2$, the order can be reduced constructively by two. But if $n > 2$ and $r > 2$, it will not necessarily follow that the order can be reduced by more than one. However, if the r-dimensional Lie algebra of infinitesimal generators of an admitted r-parameter group has a q-dimensional solvable subalgebra, then the order of the ODE can be reduced constructively by q.

Definition 4 A subalgebra A of the Lie algebra L^r with dimension r, is called an ideal or normal subalgebra of L^r if $[G_\alpha, G_\beta] \in A$, for all $G_\alpha \in A$ and $G_\beta \in L^r$.

Definition 5 The Lie algebra L^r, with dimension r, is called an r-dimensional solvable Lie algebra if there exists a chain of subalgebras

$$A^1 \subset A^2 \subset \cdots \subset L^r,$$

with A^{k-1} being an ideal of A^k, and $k \leq r$.

Definition 6 A Lie algebra A is Abelian if $[G_\alpha, G_\beta] = 0$, if both G_α and G_β are in A. [14]

Theorem 4 An abelian algebra is solvable [14].

Theorem 5 A two-dimensional algebra is solvable.

1.8 Lie equations

One-parameter groups are obtained by their generators by means of *Lie's theorem*:

Theorem 6 Given the infinitesimal transformations $\bar{x^i} = x^i + \epsilon\xi^i(x)$, $\bar{u^\alpha} = u^\alpha + \epsilon\eta^\alpha(x)$ or its symbol G, the corresponding one-parameter group G is obtained by solution of the *Lie equations*

$$\frac{d\bar{x}^i}{d\epsilon} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{d\epsilon} = \eta^\alpha(\bar{x}, \bar{u}),$$

subject to the initial conditions

$$\bar{x}^i|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha|_{\epsilon=0} = u^\alpha. \quad [16] \tag{1.50}$$

1.9 Canonical Parameter

If in the group property 1., discussed above, the expression $\varphi(\epsilon_1, \epsilon_2)$ can be written as

$$\varphi(\epsilon_1, \epsilon_2) = \epsilon_1 + \epsilon_2,$$

then the parameter a is said to be *canonical.* In general, a canonical parameter exists whenever φ exists. That is, one has the following theorem:

Theorem 7 : For any $\varphi(a, b)$, there exists the canonical parameter

$$\tilde{a} = \int_0^a \frac{da'}{A(a')},$$

where

$$A(a) = \frac{\varphi(a,b)}{b}|_{b=0}\,. \qquad [16]$$

This system, with a as the canonical parameter below, transforms form-invariantly in variables $t, x, y, z, u, v, w, p, \mu$ (see Ibragimov and Ünal 1994) under

$$\bar{t} = t\exp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],\ \bar{x} = x\exp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],\ \bar{y} = y\exp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],$$

$$\bar{z} = z\exp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],\ \bar{u} = u\exp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],\ \bar{v} = v\exp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],$$

$$\bar{w} = w\exp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],\ \bar{p} = p\ \exp\left[\int_0^a \frac{da'}{\bar{\mu}F'(\mu)}\right],\ F(\bar{\mu}) = a + F(\mu),$$

where

$$F(\mu) = \frac{1}{\mu F'(\mu)}, [14]$$

1.10 Canonical variables

Theorem 8 : Every one-parameter group of transformations ($\bar{x} = f(x,y,\epsilon)$, $\bar{y} = g(x,y,\epsilon)$ reduces to a group of translations $\bar{t} = t+\epsilon$, $\bar{u} = u$ with the generator

$$X = \frac{\partial}{\partial t}$$

by a suitable change of variables

$$t = t(x,y), \quad u = u(x,y).$$

The variables t, u are called canonical variables.

Theorem 9 : By a suitable choice of the basis G_1, G_2, any two-dimensional Lie algebra can be reduced to one of the four different types, which are determined by the following canonical structural relations:

I.

$$[G_1, G_2] = 0, \quad G_1 \vee G_2 \neq 0; \tag{1.51}$$

II.

$$[G_1, G_2] = 0, \quad G_1 \vee G_2 = 0; \tag{1.52}$$

III.

$$[G_1, G_2] \neq 0, \quad G_1 \vee G_2 \neq 0; \tag{1.53}$$

IV.

$$[G_1, G_2] \neq 0, \quad G_1 \vee G_2 = 0, \tag{1.54}$$

where

$$G_1 \vee G_2 = \xi_1 \eta_2 - \eta_1 \xi_2,$$

and

$$G_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}, \quad G_1 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}. \qquad [13].$$

Type I.

$$[G_1, G_2] = 0, \quad G_1 \vee G_2 \neq 0.$$

This condition reduces

$$y'' = f(y'),$$

to

$$\int \frac{dy'}{f(y')} = x + C_1,$$

with C_1 being the integration constant.

Type II.

$$[G_1, G_2] = 0, \quad G_1 \vee G_2 = 0;$$

This condition reduces

$$y'' = f(x),$$

to

$$y = \int \left(\int f(x)dx \right) dx + C_1 x + C_2.$$

with C_1 and C_2 being the integration constants.

Type III.

$$[G_1, G_2] \neq 0, \quad G_1 \vee G_2 \neq 0;$$

This condition reduces

$$y'' = \frac{1}{x} f(y'),$$

to

$$\int \frac{dy'}{f(y')} = \ln(x) + C_1,$$

with C_1 being the integration constant.

Type IV.

$$[G_1, G_2] \neq 0, \quad G_1 \vee G_2 = 0,$$

This condition reduces

$$y'' = y' f(x),$$

to

$$y = C_1 \int e^{\int f(x)dx} dx + C_2.$$

Theorem 10 : The basis of an algebra L_r can be reduced by a suitable change of variable to one of the following forms:

I.

$$G_1 = \frac{\partial}{\partial x}, \quad G_2 = \frac{\partial}{\partial y};$$

II.

$$G_1 = \frac{\partial}{\partial y}, \quad G_2 = x\frac{\partial}{\partial y};$$

III.

$$G_1 = \frac{\partial}{\partial y}, \quad G_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y};$$

IV.

$$G_1 = \frac{\partial}{\partial y}, \quad G_2 = y\frac{\partial}{\partial y}.$$

The variables x and y are called canonical variables.

1.11 One Dependent and Two Independent Variables.

We consider the equation

$$u_t = u_{xx}. \tag{1.55}$$

In order to generate point symmetries for equation (2.1), we first consider a change of variables from t, x and u to t^*, x^* and u^* involving an infinitesimal parameter ϵ. A Taylor's series expansion in ϵ near $\epsilon = 0$ yields

$$\left.\begin{array}{rcl} \bar{t} & \approx & t + \epsilon T(t, x, u) \\ \bar{x} & \approx & x + \epsilon \xi(t, x, u) \\ \bar{u} & \approx & u + \epsilon \zeta t, x, u) \end{array}\right\} \tag{1.56}$$

where

$$\left.\begin{array}{rcl} \frac{\partial \bar{t}}{\partial \epsilon}|_{\epsilon=0} & = & T(t, x, u) \\ \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0} & = & \xi(t, x, u) \\ \frac{\partial \bar{u}}{\partial \epsilon}|_{\epsilon=0} & = & \zeta t, x, u) \end{array}\right\}. \tag{1.57}$$

The tangent vector field (1.63) is associated with an operator

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u}, \tag{1.58}$$

called a symmetry generator. This in turn leads to the invariance condition

$$G^{[2]}\left[F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})\right]|_{\{F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})=0\}} = 0, \tag{1.59}$$

where $G^{[2]}$ is the second prolongation of G. It is obtained from the formulas:

$$\begin{aligned} G^{[2]} &= G + \zeta_t^1 \frac{\partial}{\partial u_t} + \zeta_x^1 \frac{\partial}{\partial u_x} + \zeta_{tt}^2 \frac{\partial}{\partial u_{tt}} \\ &\quad + \zeta_{tx}^2 \frac{\partial}{\partial u_{tx}} + \zeta_{xx}^2 \frac{\partial}{\partial u_{xx}}, \end{aligned}$$

where

$$\zeta_t^1 = \tfrac{\partial g}{\partial t} + u\tfrac{\partial f}{\partial t} + \left[f - \tfrac{\partial T}{\partial t}\right] u_t - \tfrac{\partial \xi}{\partial x} u_x, \tag{1.60}$$

$$\zeta_x^1 = \tfrac{\partial g}{\partial x} + u\tfrac{\partial f}{\partial x} + \left[f - \tfrac{\partial \xi}{\partial x}\right] u_x - \tfrac{\partial T}{\partial t} u_t, \tag{1.61}$$

$$\begin{aligned} \zeta_{tt}^2 &= \tfrac{\partial^2 g}{\partial t^2} + u\tfrac{\partial^2 f}{\partial t^2} + \left[2\tfrac{\partial f}{\partial t} - \tfrac{\partial^2 T}{\partial t^2}\right] u_t - \tfrac{\partial^2 \xi}{\partial t^2} u_x \\ &\quad + \left[f - 2\tfrac{\partial T}{\partial t}\right] u_{tt} - 2\tfrac{\partial \xi}{\partial t} u_{tx}, \end{aligned}$$

$$\begin{aligned} \zeta_{xx}^2 &= \tfrac{\partial^2 g}{\partial x^2} + u\tfrac{\partial^2 f}{\partial x^2} + \left[2\tfrac{\partial f}{\partial x} - \tfrac{\partial^2 \xi}{\partial x^2}\right] u_x - \tfrac{\partial^2 T}{\partial x^2} u_t \\ &\quad + \left[f - 2\tfrac{\partial T}{\partial x}\right] u_{xx} - 2\tfrac{\partial T}{\partial x} u_{tx}, \end{aligned}$$

and

$$\begin{aligned} \zeta_{tx}^2 &= \tfrac{\partial^2 g}{\partial t \partial x} + u\tfrac{\partial^2 f}{\partial t \partial x} + \left[2\tfrac{\partial f}{\partial x} - \tfrac{\partial^2 T}{\partial t \partial x}\right] u_t \\ &\quad + \left[2\tfrac{\partial f}{\partial t} - \tfrac{\partial^2 \xi}{\partial t \partial x}\right] u_x - \left[f - \tfrac{\partial T}{\partial t} - \tfrac{\partial \xi}{\partial x}\right] u_{tx} \\ &\quad - \tfrac{\partial T}{\partial x} u_{tt} - \tfrac{\partial \xi}{\partial t} u_{xx}. \end{aligned}$$

1.12 One Dependent and Three Independent Variables.

In order to generate point symmetries for equation (1.68), we first consider a change of variables from t, x and u to t^*, x^* and u^* involving an infinitesimal parameter ϵ. A Taylor's series expansion in ϵ near $\epsilon = 0$ yields

$$\left.\begin{aligned} \bar{t} &\approx t + \epsilon T(t,x,u) \\ \bar{x} &\approx x + \epsilon \xi(t,x,u) \\ \bar{u} &\approx u + \epsilon \zeta t, x, u) \end{aligned}\right\} \tag{1.62}$$

where

$$\left.\begin{array}{rcl} \frac{\partial \bar{t}}{\partial \epsilon}|_{\epsilon=0} & = & T(t,x,u) \\ \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0} & = & \xi(t,x,u) \\ \frac{\partial \bar{u}}{\partial \epsilon}|_{\epsilon=0} & = & \zeta t,x,u) \end{array}\right\}. \tag{1.63}$$

The tangent vector field (1.63) is associated with an operator

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u}, \tag{1.64}$$

called a symmetry generator. This in turn leads to the invariance condition

$$G^{[2]}\left[F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})\right]|_{\{F(t,x,u_t,u_x,u_{tx},u_{tt},u_{xx})=0\}} = 0, \tag{1.65}$$

where $G^{[2]}$ is the second prolongation of G. It is obtained from the formulas:

$$\begin{array}{rcl} G^{[2]} & = & G + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} \\ & & + \zeta_{tx}^2\frac{\partial}{\partial u_{tx}} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}}, \end{array}$$

where

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x, \tag{1.66}$$

$$\zeta_x^1 = \frac{\partial g}{\partial x} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t, \tag{1.67}$$

$$\begin{array}{rcl} \zeta_{tt}^2 & = & \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x \\ & & + \left[f - 2\frac{\partial T}{\partial t}\right]u_{tt} - 2\frac{\partial \xi}{\partial t}u_{tx}, \end{array}$$

$$\begin{array}{rcl} \zeta_{xx}^2 & = & \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right]u_x - \frac{\partial^2 T}{\partial x^2}u_t \\ & & + \left[f - 2\frac{\partial T}{\partial x}\right]u_{xx} - 2\frac{\partial T}{\partial x}u_{tx}, \end{array}$$

and

$$\begin{array}{rcl} \zeta_{tx}^2 & = & \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 f}{\partial t\partial x} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t\partial x}\right]u_t \\ & & + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right]u_x - \left[f - \frac{\partial T}{\partial t} - \frac{\partial \xi}{\partial x}\right]u_{tx} \\ & & - \frac{\partial T}{\partial x}u_{tt} - \frac{\partial \xi}{\partial t}u_{xx}. \end{array}$$

Looking at two dimensional heat equation respectively .

$$u_t = u_{xx} + u_{yy} \tag{1.68}$$

In order to generate point symmetries for equation (1.68), we first consider a change of variables from t, x, y and u to t^*, x^*, y^* and u^* involving an infinitesimal parameter ϵ. A Taylor's series expansion in ϵ near $\epsilon = 0$ yields

$$\left.\begin{array}{rcl} \bar{t} & \approx & t + \epsilon T(t,x,y,u) \\ \bar{x} & \approx & x + \epsilon \xi(t,x,y,u) \\ \bar{y} & \approx & y + \epsilon \varphi(t,x,y,u) \\ \bar{u} & \approx & u + \epsilon \zeta t,x,y,u) \end{array}\right\} \tag{1.69}$$

where

$$\left.\begin{array}{rcl} \frac{\partial \bar{t}}{\partial \epsilon}|_{\epsilon=0} & = & T(t,x,y,u) \\ \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0} & = & \xi(t,x,y,u) \\ \frac{\partial \bar{y}}{\partial \epsilon}|_{\epsilon=0} & = & \varphi(t,x,y,u) \\ \frac{\partial \bar{u}}{\partial \epsilon}|_{\epsilon=0} & = & \zeta t,x,y,u) \end{array}\right\}. \tag{1.70}$$

The tangent vector field (1.70) is associated with an operator

$$G = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial u}, \tag{1.71}$$

called a symmetry generator. This in turn leads to the invariance condition

$$G^{[2]}\left[F(t,x,u_t,u_x,u_y,u_{tx},u_{ty},u_{tt},u_{xx},u_{xy},u_{yy})\right]|_{\{F(t,x,u_t,u_x,u_y,u_{tx},u_{ty},u_{tt},u_{xx},u_{xy},u_{yy})=0\}} = 0, \tag{1.72}$$

where $G^{[2]}$ is the second prolongation of G. It is obtained from the formulas:

$$\begin{array}{rcl} G^{[2]} & = & G + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_y^1\frac{\partial}{\partial u_y} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} \\ & & + \zeta_{tx}^2\frac{\partial}{\partial u_{tx}} + \zeta_{ty}^2\frac{\partial}{\partial u_{ty}} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}} + \zeta_{xy}^2\frac{\partial}{\partial u_{xy}} + \zeta_{yy}^2\frac{\partial}{\partial u_{yy}}, \end{array}$$

where

$$\zeta_t^1 = \frac{\partial g}{\partial t} + u\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t}\right]u_t - \frac{\partial \xi}{\partial x}u_x - u_y\frac{\partial \varphi}{\partial t}, \tag{1.73}$$

$$\zeta_x^1 = \frac{\partial g}{\partial x} + u\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x}\right]u_x - \frac{\partial T}{\partial t}u_t - u_y\frac{\partial \varphi}{\partial x}, \tag{1.74}$$

$$\zeta_y^1 = \frac{\partial g}{\partial y} + u\frac{\partial f}{\partial y} + \left[f - \frac{\partial \varphi}{\partial x}\right]u_y - \frac{\partial \varphi}{\partial t}u_t - u_x\frac{\partial \xi}{\partial y} - u_t\frac{\partial T}{\partial y}, \tag{1.75}$$

$$\begin{array}{rcl} \zeta_{tt}^2 & = & \frac{\partial^2 g}{\partial t^2} + u\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2}\right]u_t - \frac{\partial^2 \xi}{\partial t^2}u_x - \frac{\partial^2 \xi}{\partial t^2}u_y \\ & & + \left[f - 2\frac{\partial T}{\partial t}\right]u_{tt} - 2\frac{\partial \xi}{\partial t}u_{tx} - 2\frac{\partial \varphi}{\partial t}u_{yt}, \end{array}$$

$$
\begin{aligned}
\zeta^2_{tx} \quad = \quad & \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 f}{\partial t\partial x} + \left[\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t\partial x}\right] u_t \\
& + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right] u_x - u_y\frac{\partial^2 \varphi}{\partial t\partial x} - \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial x}\right] u_{tx} \\
& - \left[\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\right] u_{xx} - \left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right] u_{xy}.
\end{aligned}
$$

$$
\begin{aligned}
\zeta^2_{ty} \quad = \quad & \frac{\partial^2 g}{\partial t\partial x} + u\frac{\partial^2 f}{\partial t\partial x} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t\partial x}\right] u_t \\
& + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t\partial x}\right] u_x - \left[f - \frac{\partial T}{\partial t} - \frac{\partial \xi}{\partial x}\right] u_{tx} \\
& - \frac{\partial T}{\partial x} u_{tt} - \frac{\partial \xi}{\partial t} u_{xx}.
\end{aligned}
$$

$$
\begin{aligned}
\zeta^2_{xx} \quad = \quad & \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right] u_x - \frac{\partial^2 \varphi}{\partial x^2} u_y - \frac{\partial^2 T}{\partial x^2} u_t \\
& + \left[f - 2\frac{\partial \xi}{\partial x}\right] u_{xx} - 2\frac{\partial \varphi}{\partial x} u_{xy}, -2\frac{\partial T}{\partial x} u_{tx},
\end{aligned}
$$

$$
\begin{aligned}
\zeta^2_{ty} \quad = \quad & \frac{\partial^2 g}{\partial t\partial y} + u\frac{\partial^2 f}{\partial t\partial y} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial t\partial y}\right] u_y - \frac{\partial^2 \xi}{\partial t\partial y} u_x - \frac{\partial^2 T}{\partial y^2} u_t \\
& - \left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\right] u_{yy} - 2\left[\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial y}\right] u_{xy} + \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial y}\right] u_{yt},
\end{aligned}
$$

$$
\begin{aligned}
\zeta^2_{yy} \quad = \quad & \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 f}{\partial^2 y} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right] u_y - \frac{\partial^2 \xi}{\partial y^2} u_x - \frac{\partial^2 T}{\partial y^2} u_t \\
& - \left[f - 2\frac{\partial \varphi}{\partial y}\right] u_{yy} - 2\left[\frac{\partial \xi}{\partial y}\right] u_{xy} + \left[2f - 2\frac{\partial T}{\partial y}\right] u_{yt},
\end{aligned}
$$

1.13 One Dependent and n Independent Variables.

The local one-parameter point transformations

$$
\begin{aligned}
\bar{x} \quad &= \quad X_i(x, u, \epsilon) = x_i + \epsilon\xi(x, u) + 0\epsilon^2 && (1.76) \\
\bar{u} \quad &= \quad U(x, u, \epsilon) = u + \epsilon\eta(x, u) + 0\epsilon^2 \quad i = 1, 2, \ldots, n && (1.77)
\end{aligned}
$$

acting on (x, u) - space has generator

$$
X = \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u}
$$

The kth extended *infinitesimals* are given by

$$
\xi(x, u), \eta(x, u), \eta^{(1)}(x, u, \partial u), \ldots, \eta^{(1)}(x, u, \partial u, \ldots, \partial u^{(1)}), \qquad (1.78)
$$

and the corresponding kth extended generator is

$$
X^{(k)} = X_i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial u} + \zeta^1_i\frac{\partial}{\partial u_i} + \ldots + \zeta^k_i\frac{\partial}{\partial u_{i_1 i_2 \ldots i_l}} \quad i = 1, 2, \ldots, n \;\; l = 1, 2, \ldots, k \;\; k \geq 1[4]. \qquad (1.79)
$$

Theorem 11 The extended infinitesimals satisfy the recursive relations

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j, \quad i = 1, 2, \ldots, n \tag{1.80}$$

$$\zeta^k_{i_1 i_2 \ldots i_k} = D_{i_k}\zeta^{k-1}_{i_1 i_2 \ldots i_{k-1}} - (D_i\xi_j)u_{i_1 i_2 \ldots i_{k-1}}j \tag{1.81}$$

$i = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k \geq 2$

Proof. Let A be an $n \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} D_1X_1 & \cdots & D_1X_n \\ \vdots & \cdots & \vdots \\ D_nX_1 & \cdots & D_nX_n \end{bmatrix}$$

and assume that A^{-1} exists. From equation (1.76) and the matrix A we have that

$$\mathbf{A} = \begin{bmatrix} D_1(x_1 + \epsilon\xi_1) & D_1(x_2 + \epsilon\xi_2) & \cdots & D_1(x_n + \epsilon\xi_n) \\ D_2(x_1 + \epsilon\xi_1) & D_2(x_2 + \epsilon\xi_2) & \cdots & D_2(x_2 + \epsilon\xi_2) \\ \vdots & \cdots & \cdots & \vdots \\ D_n(x_1 + \epsilon\xi_1) & D_n(x_2 + \epsilon\xi_2) & \cdots & D_n(x_n + \epsilon\xi_n) \end{bmatrix} + 0(\epsilon^2) = I + \epsilon B + 0(\epsilon^2)$$

where I is the identity matrix and

$$\mathbf{B} = \begin{bmatrix} D_1\xi_1 & D_1\xi_2 & \cdots & D_1\xi_n \\ D_2\xi_1 & D_2\xi_2 & \cdots & D_2\xi_n \\ \vdots & \vdots & \cdots & \vdots \\ D_n\xi_1 & D_n\xi_2 & \cdots & D_n\xi_n \end{bmatrix}$$

Then $A^{-1} = I - \epsilon B + 0(\epsilon^2)$ Using some transformations we arrive at that

$$\begin{bmatrix} \zeta^k_{i_1 i_2 \ldots i_k}1 \\ \zeta^k_{i_1 i_2 \ldots i_k}2 \\ \vdots \\ \zeta^k_{i_1 i_2 \ldots i_k}n \end{bmatrix} = \begin{bmatrix} D_1\zeta^{k-1}_{i_1 i_2 \ldots} \\ D_2\zeta^{k-1}_{i_1 i_2 \ldots} \\ \vdots \\ D_n\zeta^{k-1}_{i_1 i_2 \ldots} \end{bmatrix} - B \begin{bmatrix} u_{i_1 i_2 \ldots i_{k-1}}1 \\ u_{i_1 i_2 \ldots i_{k-1}}2 \\ \vdots \\ u_{i_1 i_2 \ldots i_{k-1}}n \end{bmatrix}$$

$i = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k \geq 2$ and this leads to (1.81). Hence the proof. The details of the proof are contained in [4].

1.14 m Dependent and n Independent Variables.

We consider the case of n independent variables $x = (x^1 \ldots . \ldots x^n)$ and m dependent variables $u(x) = u^1(x) \ldots u^n(x)$. Partial derivatives are denoted by $u_i^\mu = \frac{\partial u^\mu}{\partial x^i}$. The notation

$$\partial u \equiv \partial^1 u = u_1^1(x) \ldots u_n^1(x) \ldots u_1^m(x) \ldots u_n^m(x)$$

denotes the set of all first-order partial derivatives

$$\begin{aligned} \partial^p u &= \{u^p_{i_1 \ldots i_p} | \mu = 1 \ldots m : i_1 \ldots i_p = 1 \ldots n\} \\ &= \{\frac{\partial^p u^p(x)}{\partial x^{i_1} \ldots \partial x^{i_p}} | \mu = 1 \ldots m : i_1 \ldots i_p = 1 \ldots n\} \end{aligned}$$

denotes the set of all partial derivatives of order p. Point transformations of the form

$$\bar{x} = f(x, u) \tag{1.82}$$

$$\bar{u} = g(x, u) \tag{1.83}$$

acting on the $n + m$ dimensional space (x, u) has as its pth extended transformation

$$(\bar{x})^i = f^i(x, u) \tag{1.84}$$

$$(\bar{u^\mu}) = g^\mu(x, u) \tag{1.85}$$

$$(\bar{u_i^\mu}) = h_i^\mu(x, u, \partial u) \tag{1.86}$$

$$\vdots \tag{1.87}$$

$$(\bar{u^\mu_{i_1 \ldots i_p}}) = h^\mu_{i_1 \ldots i_p}(x, u, \partial u \ldots \partial^p u) \tag{1.88}$$

with $i, i_1, \ldots, i_p = 1, \ldots, n;\quad \mu = 1 \ldots m;\quad \frac{\partial(\bar{u^\mu})}{\partial((\bar{x})i}$. The transformed components of the first-order derivatives are determined by

$$\begin{bmatrix} (\bar{u})_1^\mu \\ (\bar{u})_2^\mu \\ \vdots \\ (\bar{u})_n^\mu \end{bmatrix} = \begin{bmatrix} h_1^\mu \\ h_2^\mu \\ \vdots \\ h_n^\mu \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 g^\mu \\ D_2 g^\mu \\ \vdots \\ D_n g^\mu \end{bmatrix}$$

where A^{-1} is the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} D_1 f^1 & \cdots & D_1 f^n \\ \vdots & \cdots & \vdots \\ D_n f^1 & \cdots & D_n f^n \end{bmatrix}$$

in terms of the total derivative operators

$$D_i = \frac{\partial}{\partial x^i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ii_1}^\mu \frac{\partial}{\partial u_{i_1}^\mu} + \ldots,$$

$i = 1, \ldots, n$ [5]. The transformed components of the higher-order derivatives are determined by

$$\begin{bmatrix} (\bar{u})^\mu_{i_1 \ldots i_p} 1 \\ (\bar{u})^\mu_{i_1 \ldots i_p} 2 \\ \vdots \\ (\bar{u})^\mu_{i_1 \ldots i_p} n \end{bmatrix} = \begin{bmatrix} h^\mu_{i_1 \ldots i_p} 1 \\ h^\mu_{i_1 \ldots i_p} 2 \\ \vdots \\ h^\mu_{i_1 \ldots i_p} n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 h^\mu_{i_1 \ldots i_{p-1}} 1 \\ D_2 h^\mu_{i_1 \ldots i_{p-1}} 2 \\ \vdots \\ D_n h^\mu_{i_1 \ldots i_{p-1}} n \end{bmatrix}$$

The situation where the point transformation (1.82,1.83) is a one-parameter group of transformation given by

$$\bar{x}^i = f^i(x,u,\epsilon) = x^i + \epsilon \xi^i(x,u) + 0(\epsilon^2), \quad i = 1, ...n \tag{1.89}$$

$$\bar{u}^\mu = g^\mu(x,u,\epsilon) = u^\mu + \epsilon \xi^i(x,u) + 0(\epsilon^2), \quad \mu = 1, ...m \tag{1.90}$$

will have the corresponding generator given by

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu} \qquad [5] \tag{1.91}$$

Chapter 2

Lie group analysis of the heat equation

In this chapter we consider the symmetry analysis as presented in the books of Bluman,Kumei and Anco [3],[4] . We present the Lie symmetries of one-dimensional and two-dimensional heat equations. We also find the invariant solutions under certain symmetry generators of this equations.We use method from the above authors since they can be easily related to the symmetry method that Manale used in his new formula to find the new symmetries of the heat equation

2.1 One-dimensional heat equation

The heat equation is given by

$$u_{xx} - u_t = 0 \tag{2.1}$$

2.1.1 Prolongation formulas

Let x and t be two independent variables, and u a dependent variable. The total derivatives are defined by

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots$$

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots$$

The infinitesimal generator is given by

$$X = T(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.2}$$

where $X^{[2]}$ is the second prolongation of X given by

$$X^{[2]} = X + \zeta_x^1\frac{\partial}{\partial u_x} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_{xx}^2\frac{\partial}{\partial u_{xx}} + \zeta_{xt}^2\frac{\partial}{\partial u_{xt}} + \zeta_{tt}^2\frac{\partial}{\partial u_{tt}} \tag{2.3}$$

The coefficients ζ_x, ζ_t, ζ_{xx} and ζ_{tt} are given by

$$\begin{aligned}
\zeta_x^1 &= D_x(\eta) - u_xD_x(\xi) - u_tD_x(T)\\
&= g_x + f_xu + u_x(f-\xi_x) - u_tT_x,
\end{aligned}$$

$$\begin{aligned}
\zeta_t^1 &= D_t(\eta) - u_xD_t(\xi) - u_tD_t(T)\\
&= g_t + f_tu + u_t(f-T_t) - u_x\xi_t,
\end{aligned}$$

$$\begin{aligned}
\zeta_{xx}^2 &= D_x(\zeta_x^1) - u_{xx}D_x(\xi) - u_{xt}D_x(T)\\
&= g_{xx} + uf_{xx} + u_x(2f_x - \xi_{xx}) - u_tT_{xx} + u_{xx}(f-2\xi_x) - u_{xx}(f-2\xi_x) - 2u_{xt}T_x,
\end{aligned}$$

$$\begin{aligned}
\zeta_{xt}^2 &= D_t(\zeta_x^1) - u_{xx}D_t(\xi) - u_{xt}D_x(T)\\
&= g_{xt} + f_{xt}u + u_x(f_t - \xi_{xt}) - u_t(f_x - T_{xt}) - \xi_tu_{xx} + (f - \xi_x - T_t)u_{xt} - T_xu_{tt},
\end{aligned}$$

$$\begin{aligned}
\zeta_{tt}^2 &= D_t(\zeta_t^1) - u_{xt}D_t(\xi) - u_{tt}D_t(T)\\
&= g_{tt} + f_{tt}u + u_x(\xi_{tt}) - u_t(2f_t - T_{tt}) - 2u_{xt}\xi_t + u_{tt}(f - 2T_t),
\end{aligned}$$

2.1.2 Determining the symmetries of the one-dimensional heat equation

The determining equation is obtained from invariance condition

$$\begin{gathered}
\big(T(t,x,u)\tfrac{\partial}{\partial t} + \xi(t,x,u)\tfrac{\partial}{\partial x} + \eta(t,x,u)\tfrac{\partial}{\partial u} + \zeta_x^1\tfrac{\partial}{\partial u_x} + \zeta_t^1\tfrac{\partial}{\partial u_t} + \zeta_{xx}^2\tfrac{\partial}{\partial u_{xx}} + \zeta_{xt}^2\tfrac{\partial}{\partial u_{xt}}\\
+\zeta_{tt}^2\tfrac{\partial}{\partial u_{tt}}\big)\big(u_t - u_{xx}\big)|_{u_t=u_{xx}} = 0,
\end{gathered}$$

which is

$$\big(\zeta_t^1 - \zeta_{xx}^2\big)|_{u_t=u_{xx}} = 0, \tag{2.4}$$

After substituting ζ_t^1,ζ_{xx}^2 and $u_t = u_{xx}$ in equation (2.4), we have

$$g_t + f_t u + u_t(f - T_t) - u_x\xi_t - \left[g_{xx} + uf_{xx} + u_x(2f_x - \xi_{xx}) - u_t T_{xx} + u_t(f - 2\xi_x) - 2u_{xt}T_x\right] = 0, \tag{2.5}$$

Separating coefficients in (2.5) yields the following monomials

$$C \ : \ g_t - g_{xx} = 0, \tag{2.6}$$

$$u \ : \ f_t - f_{xx} = 0, \tag{2.7}$$

$$u_t \ : \ T_t - T_{xx} - 2\xi_x = 0, \tag{2.8}$$

$$u_x \ : \ \xi_t - \xi_{xx} + 2f_x = 0, \tag{2.9}$$

$$u_{xt} \ : \ T_x = 0, \tag{2.10}$$

Integration equation(2.10) with respect to x result into

$$T = a(t) \tag{2.11}$$

Substituting T in (2.8) and integrating with respect to x we obtain

$$\xi = \frac{1}{2}a_t x + b(t) \tag{2.12}$$

Differentiating (2.12) with respect to t we have

$$\xi_t = \frac{1}{2}a_{tt}x + b_t \tag{2.13}$$

Substituting ξ_t in (2.9) and integrating with respect with respect to x yields

$$f = -\frac{1}{8}a_{tt}x^2 - \frac{1}{2}b_t x + c(t) \tag{2.14}$$

Substituting f in (2.7) we obtain

$$-\frac{1}{8}a_{ttt}x^2 - \frac{1}{2}b_{tt}x + c_t + \frac{1}{4}a_{tt} = 0 \tag{2.15}$$

Splitting equation (2.15) with respect to the powers of x we obtain

$$x^2 \ : \ a_{ttt} = 0, \tag{2.16}$$

$$x^1 \ : \ b_{tt} = 0, \tag{2.17}$$

$$x^0 \ : \ c_t + \frac{1}{4}a_{tt} = 0, \tag{2.18}$$

Integrating (2.26),(2.27) and (2.28) with respect to t yields

$$a(t) = \frac{A_1}{2}t^2 + A_2 t + A_3 \tag{2.19}$$

$$b(t) = A_4 t + A_5 \tag{2.20}$$

$$c(t) = -\frac{1}{4}A_1 t + A_6 \tag{2.21}$$

The Infinitesimals :

$$T = \alpha_1 t^2 + 2\alpha_2 t + \alpha_3 \tag{2.22}$$

$$\xi = \alpha_1 tx + \alpha_2 x + \alpha_4 t + \alpha_5 \tag{2.23}$$

$$f = -\frac{1}{4}\alpha_1 x^2 - \frac{1}{2}\alpha_4 x - \alpha_1 t + \alpha_6 \tag{2.24}$$

2.1.3 Symmetries

$$X = T\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + (fu+g)\frac{\partial}{\partial u} \tag{2.25}$$

Therefore corresponding symmetries are given by

$$X_1 = \frac{\partial}{\partial x}, \tag{2.26}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{2.27}$$

$$X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \tag{2.28}$$

$$X_4 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (\frac{1}{4}x^2 + \frac{1}{2}u)u\frac{\partial}{\partial u}, \tag{2.29}$$

$$X_5 = t\frac{\partial}{\partial x} - \frac{1}{2}xu\frac{\partial}{\partial u}, \tag{2.30}$$

$$X_6 = u\frac{\partial}{\partial u}, \tag{2.31}$$

$$X_\infty = g\frac{\partial}{\partial u}, \tag{2.32}$$

2.1.4 Invariant Solutions

A useful tool of symmetry group that conserves the set of solutions in the differential equations admitting this group. That is, the symmetry transformations merely permute the integral curves among themselves. Such integral curves are termed invariant solutions.

Theorem 12 *A function $F(x,y)$ is called an invariant of the group G if and only if it solves the following first-order linear partial differential equation*

$$XF = \xi(x,y)\frac{\partial F}{\partial x} + \eta(x,y)\frac{\partial F}{\partial y} \tag{2.33}$$

$$X(x,t,u)\frac{\partial u}{\partial x} + T(x,t,u)\frac{\partial u}{\partial t} = \eta(x,t,u) \tag{2.34}$$

is the general partial differential equation of an invariant surface,with the following characteristic equations

$$\frac{dx}{X(x,t,u)} = \frac{dt}{T(x,t,u)} = \frac{du}{\eta(x,t,u)} \tag{2.35}$$

For the symmetry equation (2.30) characteristic equation is

$$\frac{xdx}{2t} + \frac{du}{u} = 0 \tag{2.36}$$

Integrating (2.36) we obtain

$$u = \beta(t)e^{-\frac{x^2}{4t}} \tag{2.37}$$

Differentiating (2.37) with respect t and twice with x respectively we obtained

$$u_t = \beta'(t)e^{-\frac{x^2}{4t}} + \beta(t)e^{-\frac{x^2}{4t}}\frac{x^2}{4t^2} \tag{2.38}$$

$$u_{xx} = \frac{x^2}{4t^2}e^{-\frac{x^2}{4t}}\beta(t) - \frac{1}{2t}e^{-\frac{x^2}{4t}}\beta(t) \tag{2.39}$$

substituting u_t and u_x in (2.1) yields

$$\beta'(t) + \frac{1}{2}\beta(t) = 0 \tag{2.40}$$

Then the solution becomes

$$u = \frac{k}{\sqrt{t}}e^{-\frac{x^2}{4t}} \tag{2.41}$$

Finding the invariant solution the symmetry equation (2.29) is used since it contains (x,t,u) then the invariance condition becomes

$$xtu_x + t^2u_t = -(\frac{1}{4}x^2 + \frac{1}{2})u \tag{2.42}$$

The corresponding characteristic equations are given by

$$\frac{dx}{xt} = \frac{dt}{t^2} = \frac{du}{-(\frac{1}{4}x^2 + \frac{1}{2}t)u} \tag{2.43}$$

by separation of variables and integration, the solution of the characteristic equations yields two invariants of X_4

$$\zeta = \frac{x}{t}$$

and $v = \sqrt{t}e^{x^2/4t}u$ Then the solution of the invariant surface condition(2.42) is given by the invariant form

$$\sqrt{t}e^{x^2/4t}u = \phi(\frac{x}{t})$$

solving for the following equation is obtained

$$u = \theta(x,t) = \frac{1}{\sqrt{t}}e^{-x^2/4t}\phi(\zeta) \tag{2.44}$$

Finding the solution:

$$u_t = \frac{x^2u}{4t^2} - \frac{u}{2t} - \dot{\phi}(\frac{x}{t})(\frac{xu}{t^2}) \tag{2.45}$$

$$u_{xx} = \ddot{\phi}(\frac{x}{t})\frac{u}{t^2} - \dot{\phi}(\frac{x}{t})(\frac{xu}{t^2}) + (\frac{x^2}{4t^2} - \frac{1}{2t})u \tag{2.46}$$

substituting u_t and u_{xx} in (2.1) we obtain

$$\ddot{\phi}(\frac{x}{t})\frac{u}{t^2} = 0 \tag{2.47}$$

then

$$\ddot{\phi} = 0 \tag{2.48}$$

Solving the differential equation yields

$$u = \frac{1}{\sqrt{t}}[C_1 + C_2\frac{x}{t}]e^{-\frac{x^2}{4t}} \tag{2.49}$$

2.2 Two-dimensional heat equation

The two dimensional heat equation is given by the equation

$$u_t - u_{xx} - u_{yy} = 0 \tag{2.50}$$

in which the dependent variable is u and independent variables are t, x and y. The infinitesimal generator is given by

$$X = T(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \varphi(t,x,y,u)\frac{\partial}{\partial y} + \zeta(t,x,y,u)\frac{\partial}{\partial u}, \tag{2.51}$$

Where $X^{[2]}$is the second prolongation of X given by

$$\begin{aligned} X^{[2]} &= X + \zeta_t^1 \tfrac{\partial}{\partial u_t} + \zeta_x^1 \tfrac{\partial}{\partial u_x} + \zeta_y^1 \tfrac{\partial}{\partial u_y} + \zeta_{tt}^2 \tfrac{\partial}{\partial u_{tt}} \\ &+ \zeta_{tx}^2 \tfrac{\partial}{\partial u_{tx}} + \zeta_{ty}^2 \tfrac{\partial}{\partial u_{ty}} + \zeta_{xx}^2 \tfrac{\partial}{\partial u_{xx}} + \zeta_{xy}^2 \tfrac{\partial}{\partial u_{xy}} + \zeta_{yy}^2 \tfrac{\partial}{\partial u_{yy}}, \end{aligned}$$

with the invariance condition

$$X^{[2]}(u_t - u_{xx} - u_{yy})|_{(u_{yy}=u_t-u_{xx})} = 0 \tag{2.52}$$

that yields

$$\zeta_t^1 - \zeta_{xx}^{(2)} - \zeta_{yy}^2|_{(u_{yy}=u_t-u_{xx})} = 0 \tag{2.53}$$

where $\zeta_1, \zeta_{xx}, \zeta_{yy}$ are substituted in (2.53)

$$\zeta_t^1 = \tfrac{\partial g}{\partial t} + u\tfrac{\partial f}{\partial t} + \left[f - \tfrac{\partial T}{\partial t}\right] u_t - \tfrac{\partial \xi}{\partial x} u_x - u_y \tfrac{\partial \varphi}{\partial t}, \tag{2.54}$$

$$\zeta_x^1 = \tfrac{\partial g}{\partial x} + u\tfrac{\partial f}{\partial x} + \left[f - \tfrac{\partial \xi}{\partial x}\right] u_x - \tfrac{\partial T}{\partial t} u_t - u_y \tfrac{\partial \varphi}{\partial x}, \tag{2.55}$$

$$\zeta_y^1 = \tfrac{\partial g}{\partial y} + u\tfrac{\partial f}{\partial y} + \left[f - \tfrac{\partial \varphi}{\partial x}\right] u_y - \tfrac{\partial \varphi}{\partial t} u_t - u_x \tfrac{\partial \xi}{\partial y} - u_t \tfrac{\partial T}{\partial y}, \tag{2.56}$$

$$\begin{aligned} \zeta_{tt}^2 &= \tfrac{\partial^2 g}{\partial t^2} + u\tfrac{\partial^2 f}{\partial t^2} + \left[2\tfrac{\partial f}{\partial t} - \tfrac{\partial^2 T}{\partial t^2}\right] u_t - \tfrac{\partial^2 \xi}{\partial t^2} u_x - \tfrac{\partial^2 \xi}{\partial t^2} u_y \\ &+ \left[f - 2\tfrac{\partial T}{\partial t}\right] u_{tt} - 2\tfrac{\partial \xi}{\partial t} u_{tx} - 2\tfrac{\partial \varphi}{\partial t} u_{yt}, \end{aligned}$$

$$\begin{aligned} \zeta_{tx}^2 &= \tfrac{\partial^2 g}{\partial t \partial x} + u\tfrac{\partial^2 f}{\partial t \partial x} + \left[\tfrac{\partial f}{\partial x} - \tfrac{\partial^2 T}{\partial t \partial x}\right] u_t \\ &+ \left[2\tfrac{\partial f}{\partial t} - \tfrac{\partial^2 \xi}{\partial t \partial x}\right] u_x - u_y \tfrac{\partial^2 \varphi}{\partial t \partial x} - \left[2f - 2\tfrac{\partial T}{\partial t} - \tfrac{\partial T}{\partial x}\right] u_{tx} \\ &- \left[\tfrac{\partial \xi}{\partial t} + \tfrac{\partial \xi}{\partial x}\right] u_{xx} - \left[2\tfrac{\partial \varphi}{\partial t} + \tfrac{\partial \varphi}{\partial x}\right] u_{xy}. \end{aligned}$$

$$\begin{aligned} \zeta_{ty}^2 &= \tfrac{\partial^2 g}{\partial t \partial x} + u\tfrac{\partial^2 f}{\partial t \partial x} + \left[2\tfrac{\partial f}{\partial x} - \tfrac{\partial^2 T}{\partial t \partial x}\right] u_t \\ &+ \left[2\tfrac{\partial f}{\partial t} - \tfrac{\partial^2 \xi}{\partial t \partial x}\right] u_x - \left[f - \tfrac{\partial T}{\partial t} - \tfrac{\partial \xi}{\partial x}\right] u_{tx} \\ &- \tfrac{\partial T}{\partial x} u_{tt} - \tfrac{\partial \xi}{\partial t} u_{xx}. \end{aligned}$$

$$\begin{aligned} \zeta_{xx}^2 &= \tfrac{\partial^2 g}{\partial x^2} + u\tfrac{\partial^2 f}{\partial x^2} + \left[2\tfrac{\partial f}{\partial x} - \tfrac{\partial^2 \xi}{\partial x^2}\right] u_x - \tfrac{\partial^2 \varphi}{\partial x^2} u_y - \tfrac{\partial^2 T}{\partial x^2} u_t \\ &+ \left[f - 2\tfrac{\partial \xi}{\partial x}\right] u_{xx} - 2\tfrac{\partial \varphi}{\partial x} u_{xy}, -2\tfrac{\partial T}{\partial x} u_{tx}, \end{aligned}$$

$$\zeta^2_{ty} = \frac{\partial^2 g}{\partial t\partial y} + u\frac{\partial^2 f}{\partial t\partial y} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial t\partial y}\right]u_y - \frac{\partial^2 \xi}{\partial t\partial y}u_x - \frac{\partial^2 T}{\partial y^2}u_t$$
$$-\left[2\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y}\right]u_{yy} - 2\left[\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial t} - \frac{\partial T}{\partial y}\right]u_{yt},$$
$$\zeta^2_{yy} = \frac{\partial^2 g}{\partial y^2} + u\frac{\partial^2 f}{\partial^2 y} + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}\right]u_y - \frac{\partial^2 \xi}{\partial y^2}u_x - \frac{\partial^2 T}{\partial y^2}u_t$$
$$+\left[f - 2\frac{\partial \varphi}{\partial y}\right]u_{yy} - 2\left[\frac{\partial \xi}{\partial y}\right]u_{xy} + \left[2f - 2\frac{\partial T}{\partial y}\right]u_{yt},$$

2.2.1 Determining equation of two dimensional heat equation

$$\zeta^1_t = \zeta^2_{xx} + \zeta^2_{yy} = 0 \tag{2.57}$$

$$\begin{aligned}
&\frac{\partial g}{\partial t} + \frac{\partial f}{\partial t}u + u_t(f - \frac{\partial \tau}{\partial t}) - u_x\frac{\partial \xi}{\partial t} - u_y\frac{\partial \varphi}{\partial t} \\
&= \frac{\partial^2 g}{\partial x^2} + u\frac{\partial^2 f}{\partial x^2} + u_x(2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}) - u_y\frac{\partial^2 \varphi}{\partial x^2} - u_t\frac{\partial^2 \tau}{\partial x^2} + u_{xx}(f - 2\frac{\partial \xi}{\partial x}) - 2u_{xy}\frac{\partial \varphi}{\partial x} - 2u_{xt}\frac{\partial \tau}{\partial x} \\
&+ \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial y^2}u + u_y(2\frac{\partial f}{\partial y} - \frac{\partial^2 \varphi}{\partial y^2}) - u_x\frac{\partial^2 \xi}{\partial y^2} - u_t\frac{\partial^2 \tau}{\partial y^2} + [u_t - u_{xx}](f - 2\frac{\partial \varphi}{\partial y}) - 2u_{xy}\frac{\partial \xi}{\partial y} - 2u_{yt}\frac{\partial \tau}{\partial y}
\end{aligned} \tag{2.58}$$

The defining equations from the monomials,

$$C \quad : \quad g_t - g_{xx} - g_{yy} = 0 \tag{2.59}$$

$$u \quad : \quad f_t - f_{xx} - f_{yy} = 0 \tag{2.60}$$

$$u_t \quad : \quad T_t - T_{xx} - T_{yy} - 2\varphi_y = 0 \tag{2.61}$$

$$u_x \quad : \quad \xi_t - \xi_{xx} - \xi_{yy} + 2f_x = 0 \tag{2.62}$$

$$u_y \quad : \quad \varphi_t - \varphi_{xx} - \varphi_{yy} + 2f_y = 0 \tag{2.63}$$

$$u_{xx} \quad : \quad \xi_x - \varphi_y = 0 \tag{2.64}$$

$$u_{xy} \quad : \quad \varphi_x + \xi_y = 0 \tag{2.65}$$

$$u_{xt} \quad : \quad T_x = 0 \tag{2.66}$$

$$u_{yt} \quad : \quad T_y = 0 \tag{2.67}$$

Integrating (2.66) and (2.67) with respect to x and y respectively results into

$$T = a(t) \tag{2.68}$$

Substituting T in (2.61) and integrate with respect to y we obtain

$$\varphi = \frac{1}{2}a_t y + b(t,x) \tag{2.69}$$

Differentiating φ twice with respect to x and y respectively we get

$$\varphi_{xx} = b_{xx}(t,x), \quad \varphi_{yy} = 0 \tag{2.70}$$

Differentiating φ with respect to t results to

$$\varphi_t = \frac{1}{2}a_{tt}y + b_t(t,x) \tag{2.71}$$

Substituting φ_y in (2.64) and integrate with respect to x yields

$$\xi = \frac{1}{2}a_t x + c(t,y) \tag{2.72}$$

Differentiating ξ with respect to t and twice with respect to x and y respectively we obtain the following equations

$$\xi_t = \frac{1}{2}a_{tt}x + c_t(t,y) \tag{2.73}$$

$$\xi_{xx} = 0, \quad \xi_{yy} = c_{yy}(t,y) \tag{2.74}$$

Differentiating (2.65) with respect to x and y respectively we obtain the following equations

$$\varphi_{xx} = 0 = b_{xx}, \quad \xi_{yy} = 0 = c_{yy} \tag{2.75}$$

Integrating (2.75) with respect to x and y respectively to obtain

$$b = A_1 x + A_2 \tag{2.76}$$

$$c = A_1 y + A_4 \tag{2.77}$$

Substituting ξ_t,ξ_{xx},ξ_{yy} in (2.62) andφ_t,φ_{xx},φ_{yy} in (2.63) then integrate respectively with respect to x and y yields

$$f = -\frac{1}{8}a_{tt}x^2 - \frac{1}{8}a_{tt}y^2 - \frac{1}{2}c_t(t,y)x - \frac{1}{2}b_t(t,x)y + d(t,x) + e(t,y) \tag{2.78}$$

Differentiating f with respect to t and twice with respect to x and y respectively we obtain

$$f_t = -\frac{1}{8}a_{ttt}x^2 - \frac{1}{8}a_{ttt}y^2 - \frac{1}{2}c_{tt}(t,y)x - \frac{1}{2}b_{tt}(t,x)y + d_t(t,x) + e_t(t,y) \tag{2.79}$$

$$f_{xx} = -\frac{1}{4}a_{tt} + d_{xx}(t,x), \qquad f_{yy} = -\frac{1}{4}a_{tt} + e_{yy}(t,y) \tag{2.80}$$

Substituting f_t,f_{xx},f_{yy} in (2.60)

$$-\frac{1}{8}a_{ttt}x^2 - \frac{1}{8}a_{ttt}y^2 - \frac{1}{2}c_{tt}(t,y)x - \frac{1}{2}b_{tt}(t,x)y + d_t(t,x) + e_t(t,y) + \frac{1}{2}a_{tt} - e_{yy}(t,y) - d_{xx}(t,x) = 0, \tag{2.81}$$

Splitting (2.81) we obtain

$$: \quad a_{ttt}(t) = 0, \tag{2.82}$$

$$: \quad d_{xx}(t,x) = 0, \tag{2.83}$$

$$: \quad e_{yy}(t,y) = 0, \tag{2.84}$$

$$: \quad d_t(t,x) = 0, \tag{2.85}$$

$$: \quad e_t(t,y) = 0, \tag{2.86}$$

Integrating with respect to t,x,y in (2.85),(2.86) and (2.84) respectively yields

$$a(t) = \frac{1}{2}A_5t^2 + A_6t + A_7 \tag{2.87}$$

$$d(t,x) = A_8x + A_9 \tag{2.88}$$

$$e(t,y) = A_{10}y + A_{11} \tag{2.89}$$

The Infinitesimals :

$$T = \beta_1t^2 + 2\beta_2t + \beta_3 \tag{2.90}$$

$$\xi = \beta_1tx + \beta_2x + \beta_4y + \beta_5 \tag{2.91}$$

$$\varphi = \beta_1ty + \beta_2y + \beta_4x + \beta_6 \tag{2.92}$$

$$f = -\frac{1}{4}\beta_1(x^2 + y^2) + \beta_7x + \beta_8y + \beta_9 \tag{2.93}$$

Symmetries:

$$X_1 = t\frac{\partial}{\partial t}, \tag{2.94}$$

$$X_2 = \frac{\partial}{\partial x}, \tag{2.95}$$

$$X_3 = \frac{\partial}{\partial y}, \tag{2.96}$$

$$X_4 = u\frac{\partial}{\partial u}, \tag{2.97}$$

$$X_5 = xu\frac{\partial}{\partial u}, \tag{2.98}$$

$$X_6 = yu\frac{\partial}{\partial u}, \tag{2.99}$$

$$X_7 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{2.100}$$

$$X_8 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{2.101}$$

$$X_9 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - \frac{1}{4}(x^2 + y^2)u\frac{\partial}{\partial u}, \tag{2.102}$$

$$X_\infty = g\frac{\partial}{\partial u}, \tag{2.103}$$

2.2.2 Invariant Solution

$$X_3 = \frac{\partial}{\partial y} \tag{2.104}$$

The invariant condition is given by

$$X_3 I = \frac{\partial I}{\partial y}, \tag{2.105}$$

the characteristic equation is

$$\frac{dy}{1} = \frac{dx}{0} = \frac{du}{0} = \frac{dt}{0} \tag{2.106}$$

this implies that $dy = 0$, thus the invariant solution $u = \phi(x)$ or $u = \phi(t)$ for $u = \phi(t)$, we substitute this in the original equation $u_t = u_{xx} + u_{yy}$ to get $\phi'(t) = 0$ thus $\phi(t) = k$ Similarly for $u = \phi(x)$ we have $u_{xx} = \phi''(x) = 0$ which implies $\phi(x) = C_1 x + C_2$ then the invariant solution becomes

$$u = C_1 x + C_2 \tag{2.107}$$

For

$$X_2 = \frac{\partial}{\partial x} \tag{2.108}$$

We similarly obtain the invariant solution

$$u = A_1 y + A_2 \tag{2.109}$$

For

$$X_7 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{2.110}$$

The invariance condition is given by

$$X_7 I = y\frac{\partial I}{\partial x} + x\frac{\partial I}{\partial y} \tag{2.111}$$

The characteristic equation is given by

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{0} = \frac{dt}{0} \tag{2.112}$$

Then

$$C = y^2 - x^2 \tag{2.113}$$

the first invariant $\psi_1 = y^2 - x^2$ and the second invariant is $\psi_2 = \phi(t)$ The invariant solution is $u = \phi(t)(y^2 - x^2)$ substituting this solution in the original equation yields

$$\phi'(t)(y^2 - x^2) = 0 \quad \phi'(t) = 0, \qquad \phi(t) = C \tag{2.114}$$

Thus invariant solution

$$u = C(y^2 - x^2) \tag{2.115}$$

$$X_9 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - \frac{1}{4}(x^2 + y^2)u\frac{\partial}{\partial u} \tag{2.116}$$

The invariance condition is given by

$$X_9 I = t^2\frac{\partial I}{\partial t} + tx\frac{\partial I}{\partial x} + ty\frac{\partial I}{\partial y} - \frac{1}{4}(x^2 + y^2)u\frac{\partial I}{\partial u} \tag{2.117}$$

Then characteristic equation then is given by the equation

$$\frac{dt}{t^2} = \frac{dx}{tx} = \frac{dy}{ty} = \frac{du}{-\frac{1}{4}(x^2 + y^2)u} \tag{2.118}$$

from the characteristic equation (2.118) we have

$$\frac{dt}{t^2} = \frac{dx}{tx} \tag{2.119}$$

Integrating the equations leads to

$$\frac{x}{t} = \varphi_1 \tag{2.120}$$

Then from (2.118) we obtain

$$\frac{dt}{t^2} + \frac{du}{-\frac{1}{4}(x^2+y^2)u} = 0 \tag{2.121}$$

Integration yields

$$u = F(\frac{x}{t})e^{\frac{(x^2+y^2)}{4t}} \tag{2.122}$$

Differentiating (2.122) with respect to t and twice with respect x and y respectively we obtain

$$\begin{aligned} u_t &= -F\frac{x}{t^2}e^{\frac{(x^2+y^2)}{4t}} \\ &\quad -Fe^{\frac{(x^2+y^2)}{4t}}\frac{x^2+y^2}{4t^2} \end{aligned} \tag{2.123}$$

$$\begin{aligned} u_{xx} &= F''\frac{1}{t^2}e^{\frac{(x^2+y^2)}{4t}} + F'\frac{x}{t^2}e^{\frac{(x^2+y^2)}{4t}} \\ &\quad +Fe^{\frac{(x^2+y^2)}{4t}}\frac{1}{2t} \end{aligned} \tag{2.124}$$

$$\begin{aligned} u_{yy} &= Fe^{\frac{(x^2+y^2)}{4t}}(\frac{1}{2t}) \\ &\quad +Fe^{\frac{(x^2+y^2)}{4t}}\frac{y^2}{4t^2} \end{aligned} \tag{2.125}$$

substituting u_t,u_{xx} and u_{yy} in (2.50) leads to

$$F''\frac{1}{t^2} + F'\frac{2x}{t^2} + F\left(\frac{1}{t} + \frac{x^2}{4t^2} + \frac{y^2}{4t^2} + e^{(\frac{x^2+y^2}{4t^2})}\right) = 0, \tag{2.126}$$

the second order differential equation (2.126) reduces to

$$F''\alpha + F'\beta + F\lambda = 0, \tag{2.127}$$

using the characteristic equation (2.118)

$$\frac{dx}{tx} = \frac{dy}{ty} \tag{2.128}$$

integration yields

$$\frac{x}{y} = \varphi_2 \tag{2.129}$$

from (2.118) we obtain

$$\frac{dy}{ty} + \frac{du}{\frac{1}{4}(x^2+y^2)u} = 0 \tag{2.130}$$

then integration yields

$$u = R(x/y)e^{-(\frac{2x^2lny+y^2}{8t})}, \tag{2.131}$$

Differentiating (2.131) with respect to t and twice with respect to x and y respectively we obtain

$$u_t = Re^{-(\frac{2x^2lny+y^2}{8t})}\left(\frac{2x^2lny+y^2}{8t^2},\right) \tag{2.132}$$

$$\begin{aligned} u_{xx} &= R''\frac{1}{y^2}e^{-(\frac{2x^2lny+y^2}{8t})} - R'\left(\frac{1}{y}e^{-(\frac{2x^2lny+y^2}{8t})}\frac{xlny}{ty}\right) \\ &\quad -R\left(e^{-(\frac{2x^2lny+y^2}{8t})}\frac{lny}{2t} + e^{-(\frac{2x^2lny+y^2}{8t})}\frac{x^2ln^2y}{4t^2}\right), \end{aligned}$$

$$\begin{aligned} u_{yy} &= R''\left(\frac{x^2}{y^4}e^{-(\frac{2x^2lny+y^2}{8t})}\right) + R'\left(\frac{2x}{y^3}e^{-(\frac{2x^2lny+y^2}{8t})} + \frac{x}{y^2}e^{-(\frac{2x^2lny+y^2}{8t})}\frac{x^2+y^2}{2t}\right) \\ &\quad +R\left(e^{-(\frac{2x^2lny+y^2}{8t})}(\frac{x^2+y^2}{4ty})^2 + e^{-(\frac{2x^2lny+y^2}{8t})}\frac{1}{4t}\frac{x+y^2}{y^2}\right), \end{aligned}$$

Substituting u_t, u_{xx} and u_{yy} in (2.50) yields

$$\begin{aligned} 0 &= R''\left(\frac{1}{y^2} - \frac{x^2}{y^2}\right) - R'\left(\frac{xlny}{ty^2} - \frac{2x}{y^3} - \frac{x(x^2+y^2)}{2ty^2}\right) \\ &\quad -R\left(\frac{2x^2lny+y^2}{8t} + \frac{lny}{2t} + \frac{x^2ln^2y}{4t^2} - (\frac{x^2+y^2}{4ty})^2 - \frac{x+y^2}{4ty^2}\right), \end{aligned} \tag{2.133}$$

which reduces to

$$R''\delta - R'\kappa - R\gamma = 0, \tag{2.134}$$

From[14] for two-dimension Lie algebra spanned by X, Y has the invariants $r = \sqrt{x^2+y^2}$ and $v = u^{-kt}$. looking at for invariant solutions in the form $v = \phi(r)$,whence $u = \phi(r)e^{kt}$.substituting in (2.50) and multiplying by r the resulting equation becomes

$$r\phi'' + \phi' - kr\phi = 0 \tag{2.135}$$

assuming $k < 0$.Then setting $k = -\alpha^2$ and $\tilde{r} = \alpha r$ the equation becomes Bessel function $J_0(\tilde{r})$ of order zero:

$$\tilde{r}\phi'' + \phi' + \tilde{r}\phi = 0 \qquad (2.136)$$

where $\phi = J_0(\tilde{r}$ and the invariant solution is given by

$$u = J_0(\alpha r)e^{-\alpha^2 t} \qquad (2.137)$$

Then the solution of (2.134) and (2.127) should be similar to (2.136)

Chapter 3

Introduction of new method

This chapter presents the new symmetries of one-dimensional and two-dimensional heat equations. The new symmetries of one-dimensional heat equation was presented in the paper by Manale[18] . We determine the new symmetries of one dimensional and two dimensional with an infinitesimal $\omega \to 0$. We determine invariant solutions for one operator in each case. Graphical solutions are presented for the calculated solutions.

3.1 New symmetries of one-dimensional heat equation

We determine the new Lie point symmetries obtained by Manale[18] for the one dimensional heat equation

$$u_t - u_{xx} = 0, \tag{3.1}$$

in which the dependent variable is u and independent variables are t and u. This equation admits the one-parameter Lie group of transformation with infinitesimal generator

$$G = T(t,x,u)\frac{\partial}{\partial u} + \xi(t,x,u)\frac{\partial}{\partial u} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3.2}$$

with the invariance condition

$$G^{[2]}\left(u_{xx}-u_t\right)|_{\{u_{xx}=u_t\}}=0, \tag{3.3}$$

where $G^{[2]}$ is the second prolongation of G. Obtained from the formulas:

$$\begin{aligned} G^{[2]} &= G+\zeta_t^{(1)}\frac{\partial}{\partial u_t}+\zeta_x^{(1)}\frac{\partial}{\partial u_x}+\zeta_{tt}^{(2)}\frac{\partial}{\partial u_{tt}} \\ &\quad +\eta_{tx}^{(2)}\frac{\partial}{\partial u_{tx}}+\eta_{xx}^{(2)}\frac{\partial}{\partial u_{xx}}, \end{aligned}$$

where

$$\eta_t^{(1)} = \frac{\partial g}{\partial t}+u\frac{\partial f}{\partial t}+\left[f-\frac{\partial T}{\partial t}\right]u_t-\frac{\partial \xi}{\partial x}u_x, \tag{3.4}$$

$$\eta_x^{(1)} = \frac{\partial g}{\partial x}+u\frac{\partial f}{\partial x}+\left[f-\frac{\partial \xi}{\partial x}\right]u_x-\frac{\partial T}{\partial t}u_t, \tag{3.5}$$

$$\begin{aligned} \eta_{tt}^{(2)} &= \frac{\partial^2 g}{\partial t^2}+u\frac{\partial^2 f}{\partial t^2}+\left[2\frac{\partial f}{\partial t}-\frac{\partial^2 T}{\partial t^2}\right]u_t-\frac{\partial^2 \xi}{\partial t^2}u_x \\ &\quad +\left[f-2\frac{\partial T}{\partial t}\right]u_{tt}-2\frac{\partial \xi}{\partial t}u_{tx}, \end{aligned}$$

$$\begin{aligned} \eta_{xx}^{(2)} &= \frac{\partial^2 g}{\partial x^2}+u\frac{\partial^2 f}{\partial x^2}+\left[2\frac{\partial f}{\partial x}-\frac{\partial^2 \xi}{\partial x^2}\right]u_x-\frac{\partial^2 T}{\partial x^2}u_t \\ &\quad +\left[f-2\frac{\partial T}{\partial x}\right]u_{xx}-2\frac{\partial T}{\partial x}u_{tx}, \end{aligned}$$

and

$$\begin{aligned} \eta_{tx}^{(2)} &= \frac{\partial^2 g}{\partial t\partial x}+u\frac{\partial^2 f}{\partial t\partial x}+\left[2\frac{\partial f}{\partial x}-\frac{\partial^2 T}{\partial t\partial x}\right]u_t \\ &\quad +\left[2\frac{\partial f}{\partial t}-\frac{\partial^2 \xi}{\partial t\partial x}\right]u_x-\left[f-\frac{\partial T}{\partial t}-\frac{\partial \xi}{\partial x}\right]u_{tx} \\ &\quad -\frac{\partial T}{\partial x}u_{tt}-\frac{\partial \xi}{\partial t}u_{xx}. \end{aligned}$$

The determining equation is given by
$\frac{\partial^2 g}{\partial x^2}+u\frac{\partial^2 f}{\partial x^2}+\left[2\frac{\partial f}{\partial x}-\frac{\partial^2 \xi}{\partial x^2}\right]u_x-\frac{\partial^2 T}{\partial x^2}u_t+\left[f-2\frac{\partial T}{\partial x}\right]u_t-2\frac{\partial T}{\partial x}u_{tx}-\frac{\partial g}{\partial t}-u\frac{\partial f}{\partial t}-\left[f-\frac{\partial T}{\partial t}\right]u_t+\frac{\partial \xi}{\partial x}u_x=0,$
The defining equations from the determining equation are :

$$C \;:\; g_t-g_{xx}=0, \tag{3.6}$$

$$u \;:\; f_t-f_{xx}=0, \tag{3.7}$$

$$u_t \;:\; T_t-T_{xx}-2\xi_x=0, \tag{3.8}$$

$$u_x \;:\; \xi_t-\xi_{xx}+2f_x=0, \tag{3.9}$$

$$u_{xt} \;:\; T_x=0, \tag{3.10}$$

The first defining equation $T_x = 0$, suggests T depends on both t and x near $\epsilon = 0$, but not at $\epsilon = 0$. Differentiating this defining equation with respect to t, gives

$$T_{tx} = 0. \tag{3.11}$$

This can then be used to simplify the second defining equation. When the latter is differentiated with respect to x, we get

$$T_{xt} - 2\xi_{xx} = 0. \tag{3.12}$$

Because the function T is analytic everywhere, Euler's mixed derivatives theorem holds, meaning $T_{xt} = T_{tx}$. This then reduces (3.12) into

$$\xi_{xx} = 0, \tag{3.13}$$

which then integrates into

$$\xi = k_1 + xk_2, \tag{3.14}$$

where $k_1 = k_1(t)$ and $k_2 = k_2(t)$. from Manale's formula[18] the solution becomes

$$\xi = \frac{k_1\phi\cos(\omega x/i) + k_2\sin(\omega x/i)}{\omega/i}, \tag{3.15}$$

where $\phi = \sin(\omega/i)$. It is clear that (3.14) reduces to (3.13) when $\omega \to 0$. The second defining equation, $T_t - 2\xi_x = 0$, then leads to

$$T = \frac{-2\dot{k}_1\phi\sin(\omega x/i) + 2\dot{k}_2\cos(\omega x/i)}{\omega} + A_0, \tag{3.16}$$

where A_0 is a constant. Thus T now appears to also depend on x, but we know this is subject to $\omega = 0$. Substituting ξ and T from equations (3.15) and (3.16) into the third defining equation, $2f_x = \eta_{xx} - \eta_t$, leads to

$$\begin{aligned} 2f_x &= -\frac{\dot{k}_1\phi}{i}\frac{w}{i}cos(\omega x/i) - \dot{k}_2\frac{\omega}{i}sin(\omega x/i) \\ &\quad -\ddot{k}_1\frac{\phi}{\omega}cos(\omega x/i) - \frac{i\ddot{k}_2}{\omega}sin(\omega x/i), \end{aligned} \tag{3.17}$$

Integrating this with respect to x gives

$$\begin{aligned} f &= -\left(\dot{k}_1 + \ddot{k}_1\right)\frac{\phi}{2}sin(\omega x/i) \\ &\quad + \left(\dot{k}_2 - \ddot{k}_2\right)\frac{1}{2}cos(\omega x/i) + \frac{B_0}{2}, \end{aligned} \tag{3.18}$$

where B_0 is a constant. We now substitute this into the fourth defining equation to establish the functions a and b. First we differentiate (3.18) once with respect to t:

$$\begin{aligned} f_t &= -\left(\ddot{k_1} + {k_1}^{(3)}\right)\frac{\phi}{2}sin(\omega x/i) \\ &\quad + \left(\ddot{k_2} - {k_2}^{(3)}\right)\frac{1}{2}cos(\omega x/i), \end{aligned} \tag{3.19}$$

then twice with respect to x:

$$\begin{aligned} f_{xx} &= -\left(\dot{k_1} + \ddot{k_1}\right)\frac{\phi}{2}\omega^2 sin(\omega x/i) \\ &\quad + \left(\dot{k_2} - \ddot{k_2}\right)\frac{\omega^2}{2}cos(\omega x/i). \end{aligned} \tag{3.20}$$

The substitution leads to

$$\left(\dot{k_1} + \ddot{k_1}\right)\omega^2 = \ddot{k_1} + {k_1}^{(3)}, \tag{3.21}$$

and

$$\left(\dot{k_2} - \ddot{k_2}\right)\omega^2 = \ddot{k_1} - {k_1}^{(3)}. \tag{3.22}$$

To solve (3.21), we note it can be written in the form

$$\frac{\ddot{k_1} + {k_1}^{(3)}}{\dot{k_1} + \ddot{k_1}} = \omega^2. \tag{3.23}$$

That is,

$$\dot{k_1} + \ddot{k_1} = C_0 e^{\omega^2 t}. \tag{3.24}$$

Subsequently,

$$k_1 = \frac{C_0}{\omega^2}\frac{1}{\omega^2+1}e^{\omega^2 t} + C_1 + C_2 e^{-t}. \tag{3.25}$$

Similarly, solving equation (3.22) yields

$$k_2 = \frac{D_0}{\omega^2}\frac{1}{\omega^2-1}e^{\omega^2 t} + D_1 + D_2 e^{t}, \tag{3.26}$$

for some constants C_0, C_1, C_2, D_0, D_1 and D_2.

3.1.1 Infinitesimals

The linearly independent solutions of the defining equations (2.6) to (2.10) lead to the infinitesimals

$$
\begin{aligned}
T &= -2\phi\left(\frac{C_0}{\omega^4(\omega^2+1)}e^{\omega^2 t}\right)\sin(\omega x/i) \\
&\quad -2\phi\left(C_1 t - C_2 e^{-t}\right)\sin(\omega x/i) \\
&\quad +2\left(\frac{D_0}{\omega^4(\omega^2-1)}e^{\omega^2 t}\right)\cos(\omega x/i) \\
&\quad +2\left(D_1 t + D_2 e^{t}\right)\cos(\omega x/i) + A_0,
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
\xi &= \frac{i\phi}{\omega}\left(\frac{C_0}{\omega^2}\frac{1}{\omega^2+1}e^{\omega^2 t}\right)\cos(\omega x/i) \\
&\quad +\frac{i\phi}{\omega}\left(C_1 + C_2 e^{-t}\right)\cos(\omega x/i) \\
&\quad +\frac{i}{\omega}\left(\frac{D_0}{\omega^2}\frac{1}{\omega^2-1}e^{\omega^2 t}\right) sin(\omega x/i) \\
&\quad +\frac{i}{\omega}\left(D_1 + D_2 e^{t}\right) sin(\omega x/i)
\end{aligned}
\tag{3.28}
$$

and

$$
\begin{aligned}
f &= -C_0\frac{\phi e^{\omega^2 t}}{2}\sin(\omega x/i) \\
&\quad -D_0\frac{e^{\omega^2 t}}{2}\cos(\omega x/i) + \frac{B_0}{2}.
\end{aligned}
\tag{3.29}
$$

3.1.2 The symmetries

According to (3.4), the infinitesimals: (3.27), (3.28) and (3.29), lead to the generators

$$
\begin{aligned}
X_1 &= \frac{2e^{\omega^2 t}}{\omega^4(\omega^2-1)}\cos(\omega x/i)\frac{\partial}{\partial t} \\
&\quad +\frac{ie^{\omega^2 t}}{\omega^3(\omega^2-1)}\sin(\omega x/i)\frac{\partial}{\partial x} \\
&\quad - \frac{e^{\omega^2 t}}{2}\cos(\omega x/i)u\frac{\partial}{\partial u},
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
X_2 &= -\frac{2\phi e^{\omega^2 t}}{\omega^4(\omega^2+1)}\sin(\omega x/i)\frac{\partial}{\partial t} \\
&\quad +\frac{i\phi e^{\omega^2 t}}{\omega^3(\omega^2+1)}\cos(\omega x/i)\frac{\partial}{\partial x} \\
&- \frac{\phi e^{\omega^2 t}}{2}\sin(\omega x/i)u\frac{\partial}{\partial u},
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
X_3 &= -2\phi\ t\ \sin(\omega x/i)\frac{\partial}{\partial t} \\
&\quad +\frac{i\phi}{\omega}\cos(\omega x/i)\frac{\partial}{\partial x},
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
X_4 &= 2\ t\ \cos(\omega x/i)\frac{\partial}{\partial t} \\
&\quad +\frac{i}{\omega}\sin(\omega x/i)\frac{\partial}{\partial x},
\end{aligned} \tag{3.33}
$$

$$
\begin{aligned}
X_5 &= 2\phi\ e^{-t}\ \sin(\omega x/i)\frac{\partial}{\partial t} \\
&\quad +\frac{i\phi}{\omega}\ e^{-t}\ \cos(\omega x/i)\frac{\partial}{\partial x},
\end{aligned} \tag{3.34}
$$

$$
\begin{aligned}
X_6 &= 2\ e^{t}\ \cos(\omega x/i)\frac{\partial}{\partial t} \\
&\quad +\frac{i}{\omega}\ e^{t}\ \sin(\omega x/i)\frac{\partial}{\partial x},
\end{aligned} \tag{3.35}
$$

$$
X_7 = \frac{\partial}{\partial t}, \tag{3.36}
$$

$$
X_8 = u\frac{\partial}{\partial u}. \tag{3.37}
$$

The last defining equation leads to an infinite symmetry generator.

$$
X_\infty = g(t,x)\frac{\partial}{\partial u}. \tag{3.38}
$$

3.2 Construction of invariant solutions for (2.1)

The symmetries X_7, X_8 and X_∞ are not different from (2.27),(2.31) and (2.32) obtained by Bluman and others, as such unlikely to lead to anything not already known. We limit our construction of invariant solutions to X_1 and X_2, as they appear to be broader and more encompassing than X_3, X_4, X_5, and X_6. What is certain is that X_3 and X_4 are automatically addressed.

3.2.1 Invariant solutions through the symmetry X_1

The characteristic equations that arise from the symmetry X_1:

$$\begin{aligned}\frac{\omega^4(\omega^2-1)e^{-\omega^2 t}dt}{2\cos(\omega x/i)} &= \frac{i\omega^3(\omega^2-1)e^{-\omega^2 t}dx}{\sin(\omega x/i)} \\ &= \frac{2e^{-\omega^2 t}du}{\cos(\omega x/i)u},\end{aligned} \tag{3.39}$$

lead to

$$\frac{\omega^4(\omega^2-1)e^{-\omega^2 t}dt}{\cos(\omega x/i)} = 2\frac{i\omega^3(\omega^2-1)e^{-\omega^2 t}dx}{\sin(\omega x/i)}, \tag{3.40}$$

and

$$\frac{\omega^4(\omega^2-1)e^{-\omega^2 t}dt}{2\cos(\omega x/i)} = \frac{2e^{-\omega^2 t}du}{\cos(\omega x/i)u}. \tag{3.41}$$

Equation (3.40) becomes

$$\omega^2 dt = -2\frac{(\omega/i)\cos(\omega x/i)dx}{\sin(\omega x/i)}, \tag{3.42}$$

so that

$$\lambda = -\omega^2 t - 2\ln|\sin(\omega x/i)|. \tag{3.43}$$

Hence,

$$\eta = e^{\frac{\omega^2}{2}t}\ |\sin(\omega x/i)| \tag{3.44}$$

where $\eta = \exp(-\lambda/2)$.

Equation (3.41) becomes

$$\frac{\omega^4(\omega^2-1)dt}{4} = \frac{du}{u}, \tag{3.45}$$

so that the invariant solution has the form

$$u = e^{(\omega^4(\omega^2-1))t/4}\phi(\eta). \tag{3.46}$$

This means

$$\begin{aligned} u_t &= \frac{\omega^4(\omega^2-1)}{4}e^{(\omega^4(\omega^2-1))t/4}\phi \\ &\quad + e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}\eta_t. \end{aligned} \tag{3.47}$$

That is,

$$\begin{aligned} u_t &= \frac{\omega^4(\omega^2-1)}{4}e^{(\omega^4(\omega^2-1))t/4}\phi \\ &\quad + \frac{\omega^2}{2}\eta e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}. \end{aligned} \tag{3.48}$$

On the other hand,

$$u_x = e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}\eta_x, \tag{3.49}$$

so that

$$\begin{aligned} u_{xx} &= e^{(\omega^4(\omega^2-1))t/4}\ddot{\phi}\ (\eta_x)^2 \\ &\quad + e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}\ \eta_{xx}. \end{aligned} \tag{3.50}$$

That is,

$$\begin{aligned} & u_{xx} = e^{(\omega^4(\omega^2-1))t/4}\ddot{\phi} \\ & \times \left(\pm e^{\frac{\omega^2}{2}t}\ (-\omega/i)\ \frac{\cos(\omega x/i)}{\omega}\right)^2 \\ & - e^{(\omega^4(\omega^2-1))t/4}\dot{\phi} \\ & \times \left(\mp e^{-\frac{\omega^2}{2}t}\ (-\omega/i)^2\ \frac{\sin(\omega x/i)}{\omega}\right), \end{aligned} \tag{3.51}$$

or

$$\begin{aligned} u_{xx} &= \omega^2 e^{(\omega^4(\omega^2-1))t/4}\ddot{\phi}\ \left(e^{\omega^2 t} - \eta^2\right) \\ &\quad + \omega^2\eta e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}. \end{aligned} \tag{3.52}$$

Substituting the expression for u_t from equation (3.48) and the one for u_{xx} from equation (3.52) into (2.1), give

$$\begin{aligned}&\omega^2\ddot{\phi}\left(e^{\omega^2 t}-\eta^2\right)+\omega^2\eta\dot{\phi}\\&=\frac{\omega^4(\omega^2-1)}{4}\phi+\frac{\omega^2}{2}\eta\dot{\phi}.\end{aligned} \tag{3.53}$$

In the limit ω approaching zero, this equation reduces to

$$\left(1-\eta^2\right)\ddot{\phi}+\frac{\eta}{2}\dot{\phi}=0. \tag{3.54}$$

That is,

$$\frac{\ddot{\phi}}{\dot{\phi}}=\frac{1}{2}\frac{\eta}{\eta^2-1}, \tag{3.55}$$

so that

$$\int_{\eta_1}^{\eta_2}\frac{d}{d\eta}\left(\ln\dot{\phi}\right)d\eta=\frac{1}{2}\int_{\eta_1}^{\eta_2}\frac{\tilde{\eta}}{\tilde{\eta}^2-1}d\tilde{\eta}. \tag{3.56}$$

The integral on the left evaluates easily. Hence,

$$\ln\dot{\phi}=\tilde{F}_0+\frac{1}{2}\int_{\eta_1}^{\eta_2}\frac{\tilde{\eta}}{\tilde{\eta}^2-1}d\tilde{\eta}, \tag{3.57}$$

where $\tilde{F}_0$ is a constant. The other requires letting $\eta_1=\eta$ and $\eta_2=\eta+\omega$ then invoking L'hopital's principle. That is,

$$\ln\dot{\phi}=\tilde{F}_0+\frac{\frac{\omega}{2}\frac{d\eta}{d\omega}\frac{d}{d\eta}\int_{\eta}^{\eta+\omega}\frac{\tilde{\eta}}{\tilde{\eta}^2-1}d\tilde{\eta}}{\frac{d}{d\omega}\omega}. \tag{3.58}$$

Evaluating $d\eta/d\omega$:

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\omega}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)\\&\times e^{\frac{\omega^2}{2}t}\frac{d}{d\eta}\int_{\eta}^{\eta+\omega}\frac{\tilde{\eta}}{\tilde{\eta}^2-1}d\tilde{\eta}.\end{aligned} \tag{3.59}$$

The fundamental theorem of calculus ensures that the derivative removes the integral, simplifying the equation to

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\omega}{2}\left(e^{\frac{\omega^2}{2}t}\,|\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)\\&\times e^{\frac{\omega^2}{2}t}\frac{\eta}{\eta^2-1}.\end{aligned} \tag{3.60}$$

A further simplification on the right gives

$$\begin{aligned}
&\ln\dot{\phi} = \tilde{F}_0 + \\
&\frac{\omega}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)| \pm (x/i)\cos(\omega x/i)\right) \\
&\times\frac{\omega e^{\frac{\omega^2}{2}t}\ \frac{|\sin(\omega x/i)|}{\omega}}{\eta^2 e^{-\frac{\omega^2}{2}t} - e^{-\frac{\omega^2}{2}t}}.
\end{aligned} \tag{3.61}$$

That is,

$$\begin{aligned}
&\ln\dot{\phi} = \tilde{F}_0 + \\
&\frac{\omega^2}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)| \pm (x/i)\cos(\omega x/i)\right) \\
&\times\frac{e^{\frac{\omega^2}{2}t}\ (\pm x/i)\cos(\omega x/i)}{\eta^2 e^{-\frac{\omega^2}{2}t} - e^{-\frac{\omega^2}{2}t}},
\end{aligned} \tag{3.62}$$

so that

$$\begin{aligned}
&\ln\dot{\phi} = \tilde{F}_0 + \\
&\frac{\frac{\omega^2}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)| \pm (x/i)\cos(\omega x/i)\right)}{|\sin(\omega x/i)|^2 e^{\frac{\omega^2}{2}t} - e^{-\frac{\omega^2}{2}t}} \\
&\times e^{\frac{\omega^2}{2}t}\ (\pm x/i)\cos(\omega x/i).
\end{aligned} \tag{3.63}$$

That is,

$$\begin{aligned}
&\ln\dot{\phi} = \tilde{F}_0 + \\
&\frac{\frac{\omega^2}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)| \pm (x/i)\cos(\omega x/i)\right)}{(-\cos(\omega x/i))^2 e^{\frac{\omega^2}{2}t} + e^{\frac{\omega^2}{2}t} - e^{-\frac{\omega^2}{2}t}} \\
&\times e^{\frac{\omega^2}{2}t}\ (\pm x/i)\cos(\omega x/i).
\end{aligned} \tag{3.64}$$

The trigonometric and hyperbolic identities ensure that there are further simplifications in the denominator. Hence,

$$\begin{aligned}
&\ln\dot{\phi} = \tilde{F}_0 + \\
&\frac{\frac{\omega^2}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)| \pm (x/i)\cos(\omega x/i)\right)}{(-\cos(\omega x/i))^2 e^{\frac{\omega^2}{2}t} + 2i\sin(\frac{\omega^2}{2i}t)} \\
&\times e^{\frac{\omega^2}{2}t}\ (\pm x/i)\cos(\omega x/i).
\end{aligned} \tag{3.65}$$

Evaluating the limits:

$$\ln \dot{\phi} = \tilde{F}_0 + \frac{-x^2}{4t}. \tag{3.66}$$

That is,

$$\dot{\phi} = F_0 e^{\frac{-x^2}{4t}}, \tag{3.67}$$

with $F_0 = \exp(\tilde{F}_0)$. Hence,

$$\phi = F_0 \int_{\eta_1}^{\eta_2} e^{\frac{-x^2}{4t}} d\tilde{\eta}. \tag{3.68}$$

The solutions for (2.1) follows from (3.46). The above expression then leads to

$$u = e^{(\omega^4(\omega^2-1))t/4} \left[F_0 \int_{\eta_1}^{\eta_2} e^{\frac{-x^2}{4t}} d\tilde{\eta} \right]. \tag{3.69}$$

The first solution through X_1

When $F_0 = -iA/\omega$ and $\omega = 0$ inside the integral in (3.69), we get

$$u = Ae^{(\omega^4(\omega^2-1))t/4} \int_{x_1}^{x_2} e^{\frac{-x^2}{4t}} dx. \tag{3.70}$$

The plot of this result is given in Figure 3.1. What is in Figure 3.2 is the same solution obtained through other means by Fassari and Rinaldi [6].

The second solutions through X_1: Olsen's result.

The second solution for (2.1) follows from a slight modification of invariant ϕ's coefficient $e^{(\omega^4(\omega^2-1))t/4}$ in (3.68). It is replaced by the expression developed in Appendix B, given in (4.28). Hence,

$$u = \frac{1}{\sqrt{(\omega^2-1)t}} \frac{F_0 \int_{\eta_1}^{\eta_2} e^{\frac{-x^2}{4t}} d\tilde{\eta}}{\omega^2}. \tag{3.71}$$

This then invokes L'hopital's principle in the limit ω going to zero with $\eta_1 = \eta$ and $\eta_2 = \eta + \omega$. That is,

$$u = \frac{1}{\sqrt{(\omega^2-1)t}} \frac{\frac{d\eta}{d\omega}\frac{d}{d\eta} F_0 \int_{\eta}^{\eta+\omega} e^{\frac{-x^2}{4t}} d\tilde{\eta}}{2\omega}. \tag{3.72}$$

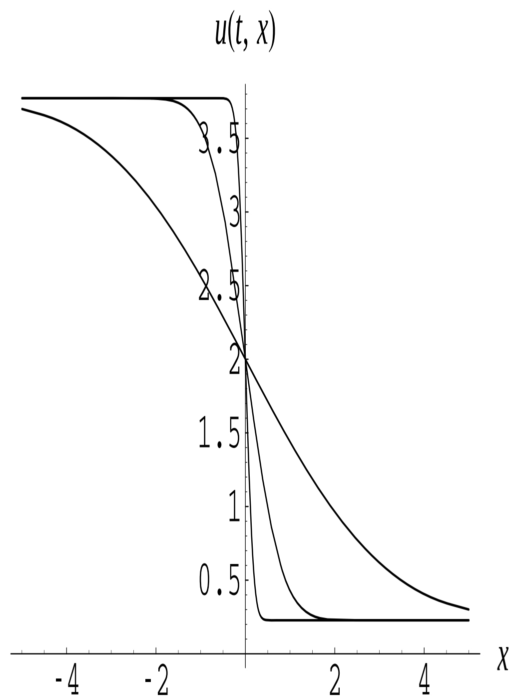

Figure 3.1: Plot of the solution in (3.70) for equation (2.1).

That is,

$$u = \frac{1}{\sqrt{(\omega^2-1)t}} \frac{\frac{d\eta}{d\omega}\frac{d}{d\eta}F_0\int_{\eta}^{\eta+\omega} e^{\frac{-x^2}{4t}}\,d\tilde{\eta}}{2\omega}. \tag{3.73}$$

Hence,

$$u = F_0 \frac{x}{2\sqrt{t}} e^{\frac{-x^2}{4t}}. \tag{3.74}$$

This solution is sketched in Figure 3.3. A similar result by Richards and Abrahamsen [26] is in Figure 3.4.

The third solution through X_1: Bluman's result.

Another solution is possible out of (3.73), and is made possible by the factor $e^{\omega^2 t/2}$ in η with $\mu = \omega^2 t/2$. It takes the form

$$u = F_0 \frac{x}{2t^{3/2}} e^{\frac{-x^2}{4t}}. \tag{3.75}$$

This result is the same as the second component in Bluman's solution with $C_2 = F_0/2$.

3.2.2 Invariant solutions through the symmetry X_2

Determining solutions through X_2 is very much the same as through X_1, because the two symmetries are very much alike. The invariants have similar forms. That is,

$$\eta = e^{\frac{\omega^2}{2}t}\ |\cos(\omega x/i)| \tag{3.76}$$

and

$$u = e^{(\omega^4(\omega^2+1))t/4}\phi(\eta). \tag{3.77}$$

This means

$$\begin{aligned} u_t &= \frac{\omega^4(\omega^2+1)}{4} e^{(\omega^4(\omega^2+1))t/4}\phi \\ &\quad + e^{(\omega^4(\omega^2+1))t/4}\dot{\phi}\eta_t. \end{aligned} \tag{3.78}$$

That is,

$$
\begin{aligned}
u_t &= \frac{\omega^4(\omega^2+1)}{4} e^{(\omega^4(\omega^2+1))t/4}\phi \\
&\quad + \frac{\omega^2}{2}\eta e^{(\omega^4(\omega^2+1))t/4}\dot{\phi}.
\end{aligned} \tag{3.79}
$$

On the other hand,

$$
u_x = e^{(\omega^4(\omega^2+1))t/4}\dot{\phi}\eta_x, \tag{3.80}
$$

so that

$$
\begin{aligned}
u_{xx} &= e^{(\omega^4(\omega^2+1))t/4}\ddot{\phi}\ (\eta_x)^2 \\
&\quad + e^{(\omega^4(\omega^2+1))t/4}\dot{\phi}\ \eta_{xx}.
\end{aligned} \tag{3.81}
$$

That is,

$$
\begin{aligned}
&u_{xx} = e^{(\omega^4(\omega^2+1))t/4}\ddot{\phi} \\
&\times \left(-e^{-\frac{\omega^2}{2}t}\ (-\omega/i)\ \frac{\sin(\omega x/i)}{\omega} \right)^2 \\
&- e^{(\omega^4(\omega^2+1))t/4}\dot{\phi} \\
&\times \left(e^{-\frac{\omega^2}{2}t}\ (-\omega/i)^2\ \frac{\cos(\omega x/i)}{\omega} \right),
\end{aligned} \tag{3.82}
$$

or

$$
\begin{aligned}
u_{xx} &= \omega^2 e^{(\omega^4(\omega^2+1))t/4}\ddot{\phi}\ \left(e^{-\omega^2 t} - \eta^2\right) \\
&\quad + \omega^2\eta e^{(\omega^4(\omega^2+1))t/4}\dot{\phi}.
\end{aligned} \tag{3.83}
$$

Substituting the expression for u_t from equation (3.79) and the one for u_{xx} from equation (3.83) into (2.1), give

$$
\begin{aligned}
&\omega^2\ddot{\phi}\ \left(e^{-\omega^2 t} - \eta^2\right) + \omega^2\eta\dot{\phi} \\
&= \frac{\omega^4(\omega^2+1)}{4}\phi + \frac{\omega^2}{2}\eta\dot{\phi}.
\end{aligned} \tag{3.84}
$$

In the limit ω approaching zero, this equation reduces to

$$
\left(1-\eta^2\right)\ddot{\phi} + \frac{\eta}{2}\dot{\phi} = 0. \tag{3.85}
$$

That is,

$$\frac{\ddot{\phi}}{\dot{\phi}} = \frac{1}{2}\frac{\eta}{\eta^2 - 1}, \tag{3.86}$$

so that

$$\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta = F_0 + \frac{1}{2}\int \frac{\eta}{\eta^2 - 1} d\eta, \tag{3.87}$$

where F_0 is a constant.

The Gaussian function solution follows from reversing the integral on the right side of equation (3.87). This requires the use of L'hopital's principle. We induce this by introducing ω to the denominator, and another one in the numerator for balance:

$$\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta = \omega \frac{\frac{1}{2}\int \frac{\eta}{\eta^2-1} d\eta}{\omega}. \tag{3.88}$$

Next, we use the result

$$\lim_{\omega \to 0}\left[\int_{\eta}^{\eta+\omega} \frac{\tilde{\eta}}{\tilde{\eta}^2 - 1} d\tilde{\eta}\right] = 0, \tag{3.89}$$

in conjunction with L'hopital's principle on (3.88), to yield

$$\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta = \omega \frac{\frac{1}{2}\frac{d}{d\omega}\int \frac{\eta}{\eta^2-1} d\eta}{\frac{d\omega}{d\omega}}, \tag{3.90}$$

so that

$$\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta = \frac{\omega}{2}\frac{d\eta}{d\omega}\frac{d}{d\eta}\int \frac{\eta}{\eta^2 - 1} d\eta. \tag{3.91}$$

Now introducing the value for η to the coefficient:

$$\begin{aligned}
&\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta \\
&= \frac{\omega}{2}\left(\omega |\cos(\omega x/i)| - (\pm x/i)\sin(\omega x/i)\right) \\
&\times e^{\frac{\omega^2}{2}t}\frac{d}{d\eta}\int \frac{\eta}{\eta^2 - 1} d\eta.
\end{aligned} \tag{3.92}$$

The fundamental theorem of calculus ensures that the derivative removes the integral, simplifying the equation to

$$\begin{aligned}&\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta \\ &= \frac{\omega}{2}\left(\omega|\cos(\omega x/i)| - (\pm x/i)\sin(\omega x/i)\right) \\ &\times e^{\frac{\omega^2}{2}t}\ \frac{\eta}{\eta^2 - 1}.\end{aligned} \tag{3.93}$$

A further simplification on the right gives

$$\begin{aligned}&\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta \\ &= \frac{\omega}{2}\left(\omega t|\cos(\omega x/i)| - (\pm x/i)\sin(\omega x/i)\right) \\ &\times \frac{|\cos(\omega x/i)|}{\eta^2 e^{-\frac{\omega^2}{2}t} - e^{-\frac{\omega^2}{2}t}}.\end{aligned} \tag{3.94}$$

Since ω is small, we get

$$\begin{aligned}&\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta \\ &= \frac{\omega}{2}\left(\omega t|\cos(\omega x/i)| - (\pm x/i)\sin(\omega x/i)\right) \\ &\times \frac{|\cos(\omega x/i)|}{\eta^2 e^{-\frac{\omega^2}{2}t} - e^{-\frac{\omega^2}{2}t}},\end{aligned} \tag{3.95}$$

so that

$$\begin{aligned}&\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta \\ &= \frac{\frac{\omega}{2}\left(\omega t|\cos(\omega x/i)| - (\pm x/i)\sin(\omega x/i)\right)}{\left(\cos(\omega x/i)\right)^2 e^{\frac{\omega^2}{2}t} - e^{-\frac{\omega^2}{2}t}} \\ &\times|\cos(\omega x/i)|.\end{aligned} \tag{3.96}$$

That is,

$$\begin{aligned}&\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta \\ &= \frac{\frac{\omega}{2}\left(\omega t|\cos(\omega x/i)| - (\pm x/i)\sin(\omega x/i)\right)}{\left(-\sin(\omega x/i)\right)^2 e^{\frac{\omega^2}{2}t} + e^{\frac{\omega^2}{2}t} - e^{-\frac{\omega^2}{2}t}} \\ &\times|\cos(\omega x/i)|.\end{aligned} \tag{3.97}$$

The first solution is

$$\begin{aligned}
&\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta \\
&= \frac{\frac{\omega^2}{2}\left(t|\cos(\omega x/i)| - (\pm x/i)\frac{\sin(\omega x/i)}{\omega}\right)}{\left(-\sin(\omega x/i)\right)^2 e^{\frac{\omega^2}{2}t} + 2i\sin\left(\frac{\omega^2}{2i}t\right)} \\
&\times|\cos(\omega x/i)|, \qquad (3.98)
\end{aligned}$$

so that

$$\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta = -\frac{x^2}{4t}. \qquad (3.99)$$

That is,

$$\dot{\phi} = F_0 e^{\frac{-x^2}{4t}}, \qquad (3.100)$$

or

$$\phi = F_1 + F_0 \int e^{\frac{-x^2}{4t}} d\eta, \qquad (3.101)$$

where F_0 is a constant. The expression for u then assumes the form

$$u = e^{(\omega^4(\omega^2-1))t/4}\left[F_1 + F_0 \int e^{\frac{-x^2}{4t}} d\eta\right]. \qquad (3.102)$$

The first solution through X_2

When $\omega = 0$ and $F_0 = -A/\omega$ in (3.102), we get

$$u = \left[F_1 + F_0 \int e^{\frac{-x^2}{4t}} x dx\right], \qquad (3.103)$$

so that

$$u = F_1 + Ae^{\frac{-x^2}{4t}}. \qquad (3.104)$$

This solution is plotted in Figure 3.5.

The second solution through X_2: Bluman's second result.

The second solution through X_2 follows a similar procedure as was for X_1, leading to

$$u = \frac{A}{2\sqrt{t}} e^{\frac{-x^2}{4t}}. \tag{3.105}$$

This result is the same as the first component in Bluman's solution with $C_1 = 1/2$. It is sketched in Figure 3.7. A similar result by Balluffi, Allen and Carter [2] is in Figure 3.8.

The third solution through X_2: Ibragimov's result.

Like the second solution, a third solution takes the form

$$u = \frac{A}{2t^{3/2}} e^{\frac{-x^2}{4t}}. \tag{3.106}$$

This result is the same is a special case of Ibragimov's solution with $n = 3$.

Other solutions through X_2

More solutions follow from evaluating the limits in (3.93) by following a different path, leading to

$$\begin{aligned} &\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta \\ &= \frac{\frac{\omega^2}{2}\{-(\pm x/i)(x/i)\}}{\left(-\sin(\omega x/i)\right)^2 e^{\frac{\omega^2}{2}t} + 2i\sin\left(\frac{\omega^2}{2i}t\right)} \\ &\times \cos(\omega x/i)|\cos(\omega x/i)| \end{aligned} \tag{3.107}$$

so that

$$\int \frac{d}{d\eta}\left(\ln \dot{\phi}\right) d\eta = \mp \frac{x^2}{2(x^2 - t^2)}. \tag{3.108}$$

Hence,

$$u = F_1 + e^{\omega^4(\omega^2+1)} F_0 \int_{\eta_1}^{\eta_2} e^{\pm\frac{x^2}{2(x^2-t^2)}} d\tilde{\eta}. \tag{3.109}$$

A first couple of solutions through X_2

A simple pair of solutions results from (3.109) when ω goes to zero requires. That is,

$$u = F_1 + F_0 \int_{\eta_1}^{\eta_2} e^{-\frac{x^2}{2(x^2-t^2)}} d\tilde{\eta} \tag{3.110}$$

and

$$u = F_1 + F_0 \int_{\eta_1}^{\eta_2} e^{\frac{x^2}{2(x^2-t^2)}} d\tilde{\eta}. \tag{3.111}$$

A second couple of solutions through X_2

Setting $\eta_1 = \eta$ and $\eta_2 = \eta + \omega$ in (3.109) and letting ω go to zero requires that $F_0 = -A/\omega$ for some constant A. This invokes L'hopital's principle, so that

$$u = F_1 + Ae^{-\frac{x^2}{2(x^2-t^2)}} \tag{3.112}$$

and

$$u = F_1 + Ae^{\frac{x^2}{2(x^2-t^2)}}. \tag{3.113}$$

A third couple of solutions through X_2

As was the case for X_1, the limits in Appendix B can be used to create more solutions. The following pair results:

$$u = F_1 + \frac{A}{\sqrt{t}} e^{-\frac{x^2}{2(x^2-t^2)}} \tag{3.114}$$

and

$$u = F_1 + \frac{A}{\sqrt{t}} e^{\frac{x^2}{2(x^2-t^2)}}. \tag{3.115}$$

A fourth couple of solutions through X_2

Continuing with the argument started in the preceding section leads to the fourth couple of solutions:

$$u = F_1 + \frac{A}{t^{3/2}} e^{-\frac{x^2}{2(x^2-t^2)}} \tag{3.116}$$

and

$$u = F_1 + \frac{A}{t^{3/2}} e^{\frac{x^2}{2(x^2-t^2)}}. \tag{3.117}$$

It is apparent from these calculations that though X_2 are largely of the family

$$u = f(t,x) e^{\pm\frac{x^2}{2(x^2-t^2)}}. \tag{3.118}$$

3.3 New symmetries of two-dimensional heat equation

The defining equation $T_y = 0$, suggests T depends on t and y near $\epsilon = 0$ not at $\epsilon = 0$. Differentiating the defining equation $T_y = 0$ with respect to t yields $T_{ty} = 0$ Substituting T_{ty} in

$$T_t - T_{xx} - T_{yy} - \varphi_y = 0, \tag{3.119}$$

yields $T_{ty} = \varphi_{yy}$ and since T is analytical everywhere, Euler's mixed derivatives theorem holds, meaning $T_{yt} = T_{ty}$

$$\varphi_{yy} = 0, \tag{3.120}$$

Integrating (3.120) and using the new formula by Manale we obtain

$$\varphi = \frac{a(t,x)\phi\cos(\omega y/i) + b(t,x)\sin(\omega y/i)}{w/i}, \tag{3.121}$$

Using the defining equation $\xi_x - \varphi_y = 0$ and differentiating with respect to x yields

$$\xi_{xx} - \varphi_{xy} = 0, \qquad \varphi_{xy} = 0 \tag{3.122}$$

then

$$\xi_{xx} = 0, \tag{3.123}$$

Integrating (3.123) and applying the new formula by Manale yields

$$\xi = \frac{c(t,y)\phi\cos(\omega x/i) + d(t,y)\sin(\omega x/i)}{w/i}, \tag{3.124}$$

From the defining equation $T_t - T_{xx} - T_{yy} - 2\varphi_y = 0$ then integrating with respect to t ads to

$$T = \frac{-2\dot{a}(t,x)\phi\sin(\omega y/i) + 2\dot{b}(t,x)\cos(\omega y/i)}{w} + R_0, \tag{3.125}$$

Differentiating ξ with respect t and twice with respect x and y we obtain

$$\xi_t = \frac{\dot{c}(t,y)\phi\cos(\omega x/i) + \dot{d}(t,y)\sin(\omega x/i)}{w/i}, \tag{3.126}$$

$$\xi_{xx} = 0 \tag{3.127}$$

$$\xi_{yy} = \frac{\ddot{c}(t,y)\phi\cos(\omega y/i) + \ddot{b}(t,x)\sin(\omega y/i)}{w/i}, \tag{3.128}$$

Substituting ξ_t,ξ_{xx} and ξ_{yy} in the defining equation $\xi_t - \xi_{xx} - \xi_{yy} + 2f_x$ we obtain

$$\begin{aligned} f_x &= \frac{1}{2}\left(\frac{\ddot{c}(t,y)\phi cos(\omega x/i) + \ddot{d}(t,y)sin(\omega x/i)}{w/i}\right) \\ &\quad -\frac{1}{2}\left(\frac{\dot{c}(t,y)\phi cos(\omega x/i) + \dot{d}(t,y)sin(\omega x/i)}{w/i}\right), \end{aligned} \tag{3.129}$$

Differentiating φ with respect to t twice with respect to x and y we obtain

$$\varphi_t = \frac{\dot{a}(t,x)\phi cos(wy/i) + \dot{b}(t,x)sin(\omega y/i)}{\omega/i} \tag{3.130}$$

$$\varphi_{yy} = 0 \tag{3.131}$$

$$\varphi_{xx} = \frac{\ddot{a}(t,x)\cos(\omega y/i) + \ddot{b}(t,x)\sin(\omega y/i)}{w/i} \tag{3.132}$$

Substituting φ_t,φ_{xx} and φ_{yy} in the defining equation $\varphi_t - \varphi_{xx} - \varphi_{yy} + 2f_y = 0$ leads to

$$\begin{aligned} f_y &= \frac{1}{2}\left(\frac{\ddot{a}(t,x)\phi cos(\omega y/i) + \ddot{b}(t,x)sin(\omega y/i)}{\omega/i}\right) \\ &\quad -\frac{1}{2}\left(\frac{\dot{a}(t,x)\phi cos(\omega y/i) + \dot{b}(t,y)sin(\omega y/i)}{w/i}\right), \end{aligned} \tag{3.133}$$

Integrating (3.129) and (3.133) yields to

$$\begin{aligned} f &= \frac{1}{2w^2}\left(\ddot{d}(t,y)cos(\omega x/i) - \ddot{c}(t,y)\phi sin(\omega x/i)\right) \\ &\quad +\frac{1}{2w^2}\left(\dot{c}(t,y)\phi sin(\omega x/i) - \dot{d}(t,y)cos(\omega x/i)\right) \\ &\quad +\frac{1}{2w^2}\left(\ddot{b}(t,x)cos(\omega y/i) - \ddot{a}(t,x)\phi\phi sin(\omega y/i)\right) \\ &\quad +\frac{1}{2w^2}\left(\dot{a}(t,x)\phi cos(\omega y/i) - \dot{b}(t,x)cos(\omega y/i)\right) + \frac{K_0}{2}, \end{aligned} \tag{3.134}$$

Differentiating (3.129)and (3.133) with x and y respectively we obtained

$$\begin{aligned} f_{xx} &= \frac{1}{2}\phi sin(\omega x/i)\left(\dot{c}(t,y)-\ddot{c}(t,y)\right) \\ &\quad +\frac{1}{2}cos(\omega x/i)\left(\ddot{d}(t,y)-\dot{d}(t,y)\right), \end{aligned} \tag{3.135}$$

$$\begin{aligned} f_{yy} &= \frac{1}{2}\phi sin(\omega y/i)\left(\dot{a}(t,x)-\ddot{a}(t,x)\right) \\ &\quad +\frac{1}{2}cos(\omega y/i)\left(\ddot{b}(t,x)-\dot{b}(t,x)\right), \end{aligned} \tag{3.136}$$

Differentiating (3.134) with respect to t leads to

$$\begin{aligned} f_{t} &= \frac{1}{2w^2}\phi sin(\omega x/i)\left(\ddot{c}(t,y)-\dddot{c}(t,y)\right) \\ &\quad +\frac{1}{2w^2}cos(\omega x/i)\left(\dddot{d}(t,y)-\ddot{d}(t,y)\right) \\ &\quad +\frac{1}{2w^2}\phi cos(\omega y/i)\left(\ddot{a}(t,x)-\dddot{a}(t,x)\right) \\ &\quad +\frac{1}{2w^2}cos(\omega y/i)\left(\dddot{b}(t,x)-\ddot{b}(t,x)\right), \end{aligned} \tag{3.137}$$

Substituting f_t, f_{xx} and f_{yy} in the defining equation $f_t = f_{xx} + f_{xx}$ and separating we obtain the following equations

$$\frac{\dddot{c}+\ddot{c}}{\ddot{c}+\dot{c}} = w^2 \tag{3.138}$$

$$\frac{\dddot{d}-\ddot{d}}{\ddot{d}-\dot{d}} = w^2 \tag{3.139}$$

$$\frac{\dddot{a}+\ddot{a}}{\ddot{a}+\dot{a}} = w^2 \tag{3.140}$$

$$\frac{\dddot{b}-\ddot{b}}{\ddot{b}-\dot{b}} = w^2 \tag{3.141}$$

Solving the above equations we obtain

$$c = \frac{A_0 e^{w^2 t}}{w^2(w^2+1)} + A_1 + A_2 e^{t} \tag{3.142}$$

$$a = \frac{C_0 e^{w^2 t}}{w^2(w^2+1)} + C_1 + C_2 e^{t} \tag{3.143}$$

$$b = \frac{D_0 e^{w^2 t}}{w^2(w^2+1)} + D_1 + D_2 e^{-t} \tag{3.144}$$

$$d = \frac{B_0 e^{w^2 t}}{w^2(w^2+1)} + B_1 + B_2 e^{-t} \tag{3.145}$$

3.3.1 Infinitesimals

$$
\begin{aligned}
T &= -\frac{2}{\omega}\left(\frac{C_0 e^{\omega^2 t}}{(w^2+1)} + C_2 e^t\right)\phi sin(\omega y/i) \\
&\quad +\frac{2}{\omega}\left(\frac{D_0}{\omega^2-1} - D_2 e^{-t}\right) cos(\omega y/i) + R_0, \qquad (3.146)
\end{aligned}
$$

$$
\begin{aligned}
\xi &= \frac{i\phi}{\omega}\left(\frac{A_0 e^{\omega^2 t}}{w^2(\omega^2+1)}\right) cos(\omega x/i) \\
&\quad +\frac{i\phi}{\omega}\left(A_1 + A_2 e^t\right) cos(\omega x/i) \\
&\quad +\frac{i}{w}\left(\frac{B_0 e^{\omega^2 t}}{\omega^2(w^2+1)}\right) sin(\omega x/i) \\
&\quad +\frac{i}{w}\left(B_1 + B_2 e^{-t}\right) sin(\omega x/i) \qquad (3.147)
\end{aligned}
$$

$$
\begin{aligned}
\varphi &= \frac{i\phi}{\omega}\left(\frac{C_0 e^{\omega^2 t}}{w^2(w^2+1)} cos(\omega y/i)\right) \\
&\quad +\frac{i\phi}{\omega}\left(C_1 + C_2 e^t\right) cos(\omega y/i) \\
&\quad +\frac{i}{\omega}\left(\frac{D_0 e^{\omega^2 t}}{w^2(\omega^2-1)}\right) sin(\omega y/i) \\
&\quad +\frac{i}{\omega}\left(D_1 + D_2 e^{-t}\right) sin(\omega y/i) \qquad (3.148)
\end{aligned}
$$

$$
\begin{aligned}
f &= -\frac{1}{2}\left(\frac{A_0e^{\omega^2t}}{w^2+1}\phi sin(\omega x/i)\right)-\frac{1}{2}\left(\frac{A_2e^{t}}{\omega^2}\phi sin(\omega x/i)\right)\\
&+\frac{1}{2}\left(\frac{B_0e^{\omega^2t}}{\omega^2-1}cos(\omega x/i)\right)+\frac{1}{2}\left(\frac{B_2e^{-t}}{\omega^2}cos(\omega x/i)\right)\\
&+\frac{1}{2}\left(\frac{A_0e^{\omega^2t}}{\omega^2(\omega^2+1)}\phi sin(\omega x/i)\right)+\frac{1}{2}\left(\frac{A_2e^{t}}{w^2}\phi sin(\omega x/i)\right)\\
&-\frac{1}{2}\left(\frac{B_0}{\omega^2(\omega^2-1)}cos(\omega x/i)\right)+\left(\frac{B_2e^{-t}}{\omega^2}cos(\omega x/i)\right)\\
&-\frac{1}{2}\left(\frac{C_0e^{\omega^2t}}{\omega^2+1}\phi sin(\omega y/i)\right)-\left(\frac{C_2e^{t}}{\omega^2}\phi sin(\omega y/i)\right)\\
&-\frac{1}{2}\left(\frac{C_2e^{2}}{\omega^2}\phi sin(\omega y/i)\right)+\frac{1}{2}\left(\frac{D_0e^{w^2t}}{\omega^2-1}cos(\omega y/i)\right)\\
&+\left(\frac{D_2e^{-t}}{\omega^2}cos(\omega y/i)\right)+\frac{1}{2}\left(\frac{C_0e^{\omega^2t}}{w^2(w^2+1)}cos(\omega y/i)\right)\\
&+\left(\frac{C_2e^{t}}{w^2}cos(\omega y/i)\right)-\frac{1}{2}\left(\frac{D_0e^{\omega^2t}}{w^2(w^2-1)}cos(\omega y/i)\right)\\
&+\left(\frac{D_2e^{-t}}{\omega^2}cos(\omega y/i)\right)+\frac{K_0}{2}, \qquad (3.149)
\end{aligned}
$$

3.3.2 The symmetries

According to (3.146), the infinitesimals: (3.147), (3.148) and (3.149), lead to the generators

$$
\begin{aligned}
X_1 &= \left(\frac{i\phi e^{\omega^2t}}{\omega^3(\omega^2+1)}cos(\omega x/i)\right)\frac{\partial}{\partial x}\\
&+\frac{e^{\omega^2t}\phi sin(\omega x/i)}{2(w^2+1)}\left(\frac{1-w^2}{\omega^2}\right)u\frac{\partial}{\partial u} \qquad (3.150)
\end{aligned}
$$

$$
X_2 = i\phi cos(\omega x/i)\frac{\partial}{\partial x}, \qquad (3.151)
$$

$$
X_3 = \frac{i\phi}{w}cos(\omega x/i)\frac{\partial}{\partial x}, \qquad (3.152)
$$

$$
\begin{aligned}
X_4 &= \frac{ie^{\omega^2 t}}{w\omega^3(\omega^2+1)} sin(\omega x/i)\frac{\partial}{\partial x} \\
&\quad + \frac{e^{\omega^2 t} cos(\omega x/i)}{\omega^2 - 1}\left(1 - \frac{1}{\omega^2}\right) u\frac{\partial}{\partial u}
\end{aligned} \tag{3.153}
$$

$$
X_5 = \frac{i}{\omega} sin(\omega x/i)\frac{\partial}{\partial x}, \tag{3.154}
$$

$$
\begin{aligned}
X_6 &= \frac{ie^{-t}}{w} sin(\omega x/i)\frac{\partial}{\partial x} \\
&\quad + \frac{e^{-t} cos(\omega x/i)}{\omega^2}\left(1 + \frac{1}{\omega^2 - 1}\right) u\frac{\partial}{\partial u},
\end{aligned} \tag{3.155}
$$

$$
\begin{aligned}
X_7 &= \frac{-2e^{\omega^2 t}\phi sin(\omega y/i)}{\omega(w^2+1)}\frac{\partial}{\partial t} \\
&\quad + \frac{i\phi e^{\omega^2 t}}{\omega^3(\omega^2+1)} cos(\omega y/i)\frac{\partial}{\partial y} \\
&\quad - \frac{e^{\omega^2 t}}{\omega^2+1}\phi sin(\omega y/i) u\frac{\partial}{\partial u} \\
&\quad + \frac{e^{w^2 t}}{\omega^2(w^2+1)} cos(\omega y/i) u\frac{\partial}{\partial u}
\end{aligned} \tag{3.156}
$$

$$
X_8 = \frac{i\phi}{\omega} cos(\omega y/i)\frac{\partial}{\partial y}, \tag{3.157}
$$

$$
\begin{aligned}
X_9 &= \frac{i\phi e^t}{w} cos(\omega y/i)\frac{\partial}{\partial y} \\
&\quad - \frac{2\phi e^t}{w} sin(\omega x/i)\frac{\partial}{\partial t} \\
&\quad - \frac{e^t\phi}{\omega^2} sin(\omega y/i) u\frac{\partial}{\partial u} \\
&\quad + \frac{e^t}{w^2} cos(\omega y/i) u\frac{\partial}{\partial u}
\end{aligned} \tag{3.158}
$$

$$
\begin{aligned}
X_{10} &= \frac{2e^{\omega^2 t}}{w^2 - 1} cos(\omega y/i)\frac{\partial}{\partial t} \\
&\quad + \frac{ie^{\omega^2 t}}{\omega^3(w^2-1)} sin(\omega y/i)\frac{\partial}{\partial y} \\
&\quad + \frac{e^{\omega^2 t}}{2(w^2-1)} cos(\omega y/i) u\frac{\partial}{\partial u} \\
&\quad - \frac{e^{\omega^2 t}}{2\omega^2(\omega^2-1)} cos(\omega y/i) u\frac{\partial}{\partial u}
\end{aligned} \tag{3.159}
$$

$$X_{11} = \frac{i}{\omega} sin(\omega y/i) \frac{\partial}{\partial y} \tag{3.160}$$

$$\begin{aligned} X_{12} \quad = \quad & -2e^{-t} cos(\omega y/i) \frac{\partial}{\partial t} \\ & + \frac{i}{\omega} e^{-t} sin(\omega y/i) \frac{\partial}{\partial y} \\ & + \frac{3}{2} \frac{e^{-t}}{\omega^2} cos(\omega y/i) u \frac{\partial}{\partial u} \end{aligned} \tag{3.161}$$

$$X_{13} = \frac{\partial}{\partial t}, \tag{3.162}$$

$$X_{14} = u \frac{\partial}{\partial u}. \tag{3.163}$$

The last defining equation leads to an infinite symmetry generator.

$$X_{\infty} = g \frac{\partial}{\partial u}. \tag{3.164}$$

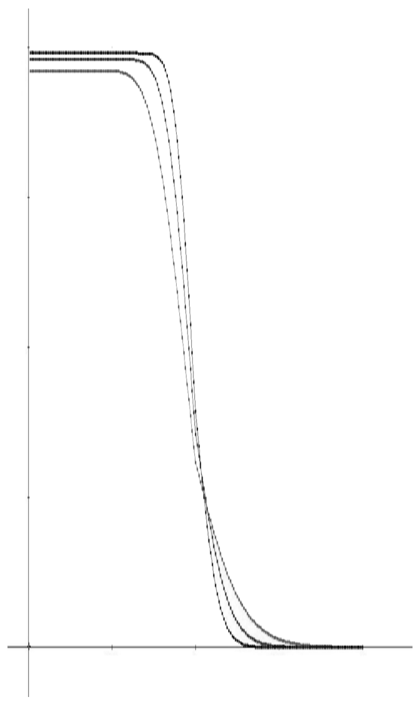

Figure 3.2: Plot of the solution obtained by Fassari and Rinaldi [6] for equation (2.1), similar to the one in Figure 3.1.

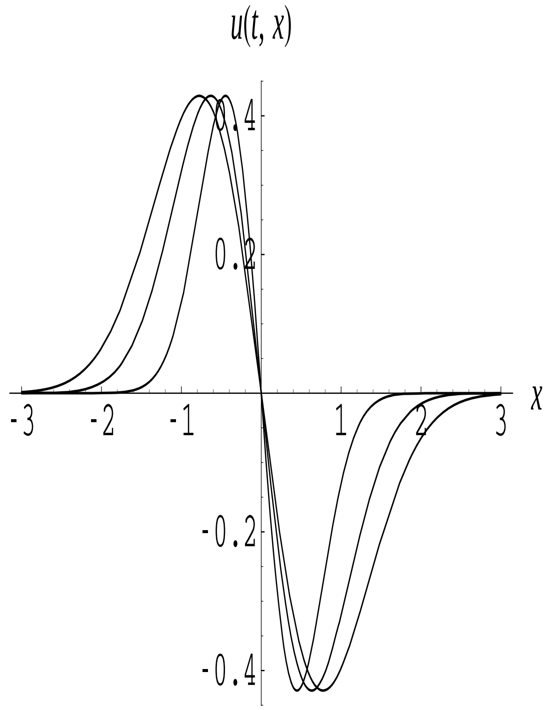

Figure 3.3: Plot of the solution in (3.74) for equation (2.1).

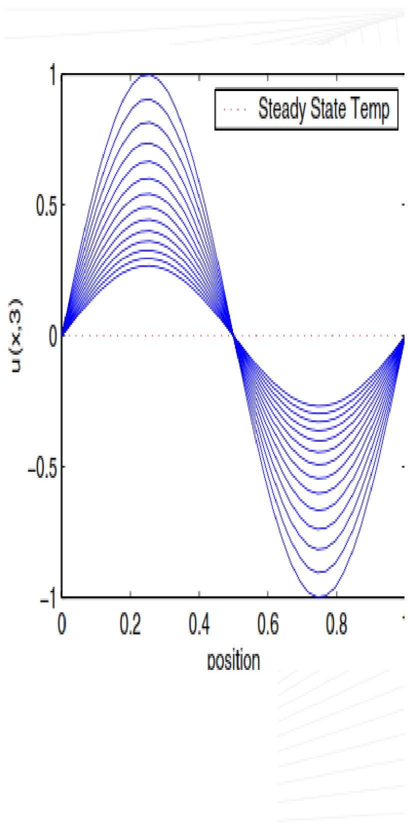

Figure 3.4: Plot of the solution obtained by Richards and Abrahamsen [26] for equation (2.1), similar to the one in Figure 3.3.

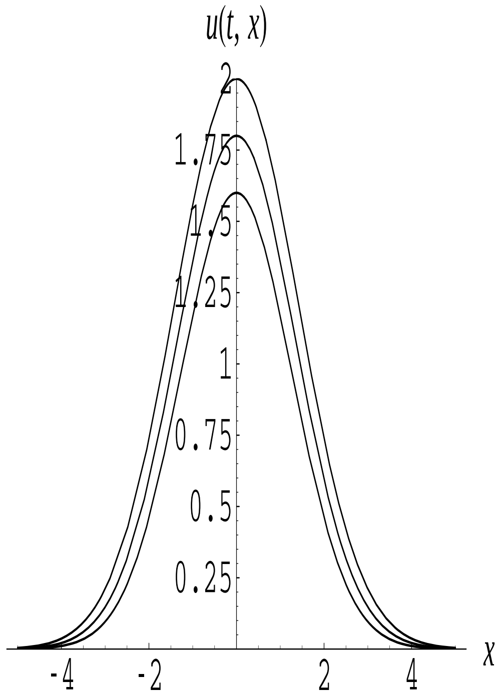

Figure 3.5: Plot of the solution in (3.104) for equation (2.1).

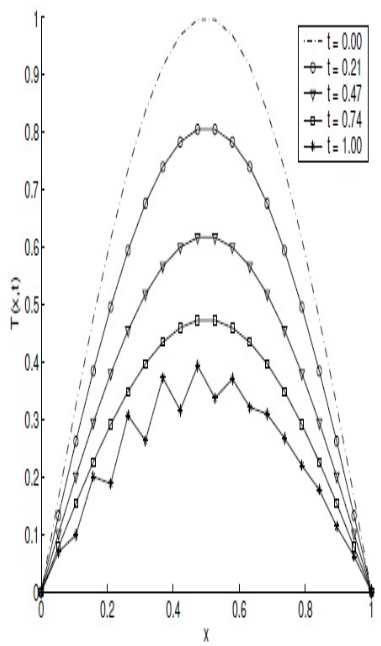

Figure 3.6: Plot of the solution by Gerald Recktenwald, similar to the one in Figure 3.5.

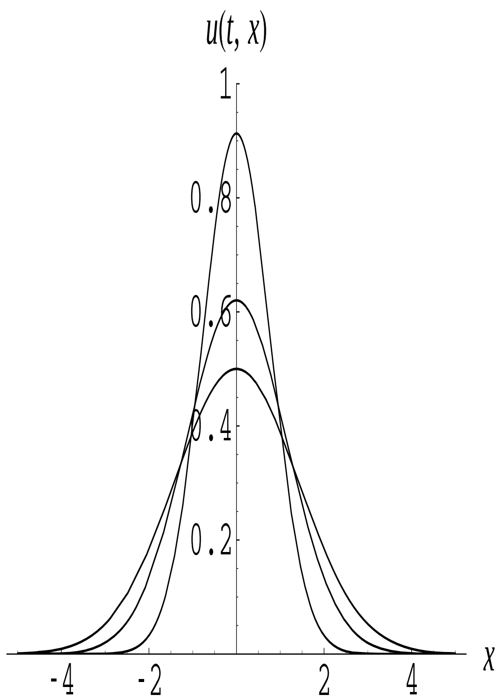

Figure 3.7: Plot of the solution in (3.105) for equation (2.1).

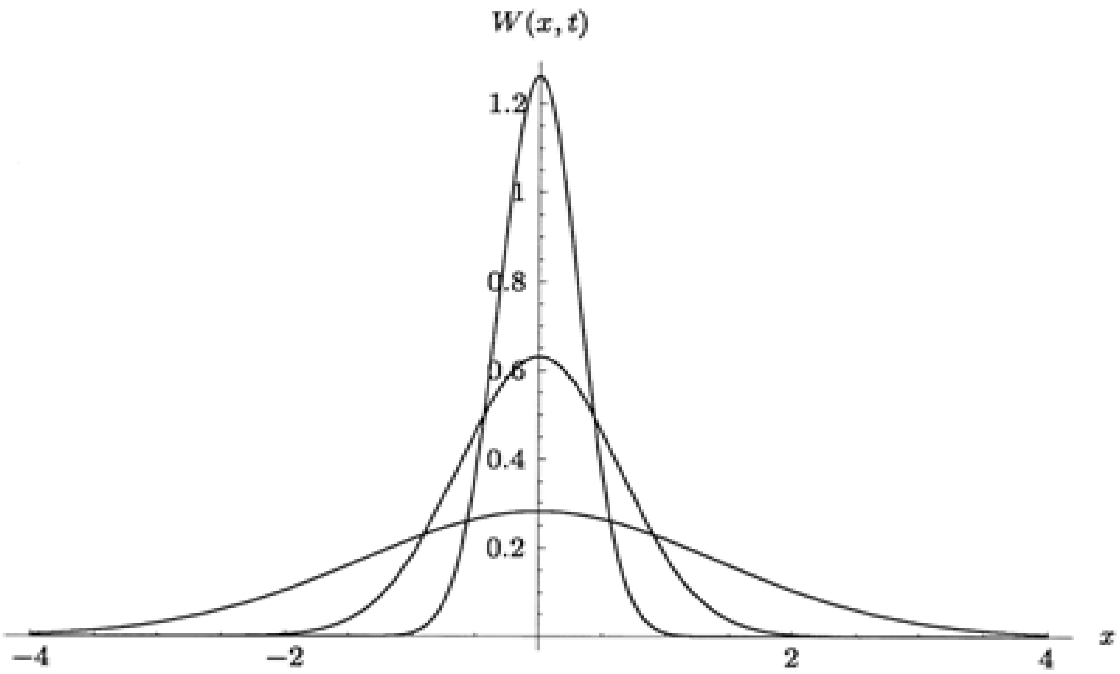

Figure 3.8: Plot of the solution obtained by Balluffi, Allen and Carter [2] for equation (2.1), similar to the one in Figure 3.7.

Chapter 4

Applications

4.1 Applications: *Heat conduction in thin plates.*

As mentioned in the Introduction, there are many methods used in practice to solve (2.1), an equation that finds application in a number of different situations. The backward heat equation

$$u_{xx} = -u_t, \tag{4.1}$$

too, does arise in practice. Unfortunately, without analytical solutions, one could end up applying one of the two equations to a situation to which it does not apply.

For example, in a study on heat conduction in thin plates, Hancork [10] deduced solutions for (2.1) presented in Figure 4.1. These we unpack in Figures 4.2, 4.3 and 4.4 using (3.112). Unfortunately, practical results indicate it is (4.1) which is applicable to this situation. This we deduce from the fact that impractical singularities arise when u is plotted against t when (3.112) is used, but disappear when this expression assumes the form

$$u = F_1 + Ae^{-\frac{x^2}{2(x^2+t^2)}}, \tag{4.2}$$

satisfying both (4.1) and empirical results, plotted in Figure 4.5. These are clearly in the family of the form

$$u = f(t,x)e^{\pm\frac{x^2}{2(x^2+t^2)}} \tag{4.3}$$

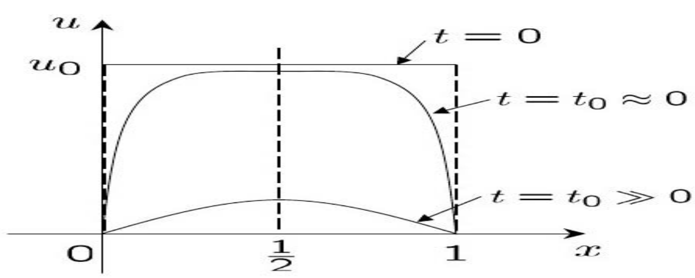

Figure 4.1: Plot of the solution obtained by Hancork [10] for equation (2.1) for cases $t = 0$, $t \approx 0$ and $t >> 0$, all stacked onto the same sketch.

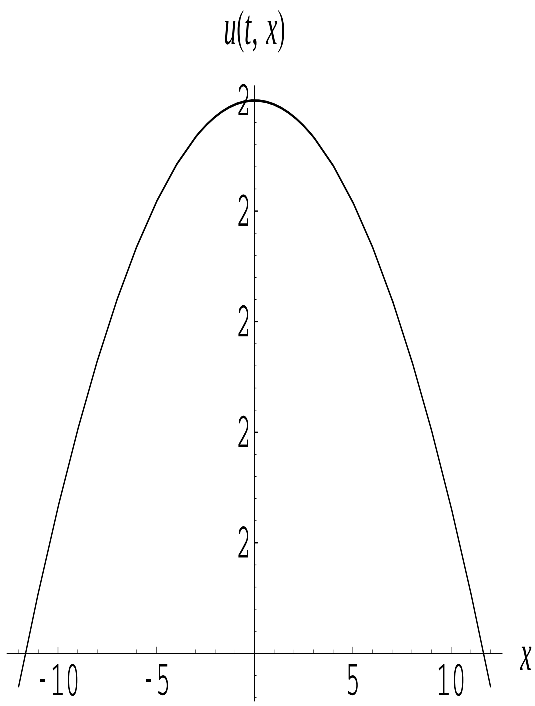

Figure 4.2: Plot of the solution in (3.112) for equation (2.1), similar to the one in Figure 4.1 for $t = t_0 >> 0$.

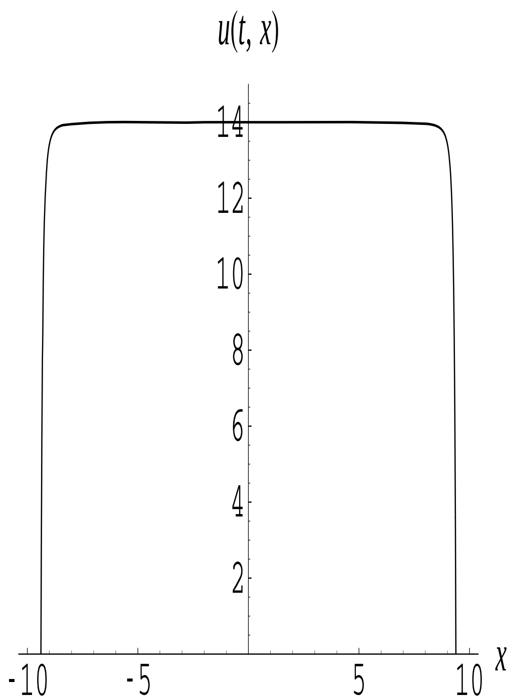

Figure 4.3: Plot of the solution in (3.112) for equation (2.1), similar to the one in Figure 4.1 for $t = t_0 \approx 0$.

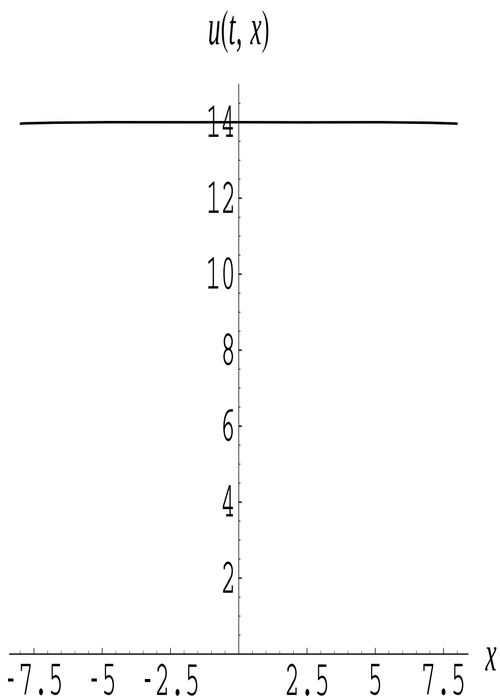

Figure 4.4: Plot of the solution in (3.112) for equation (2.1), similar to the one in Figure 4.1 for $t = t_0 = 0$.

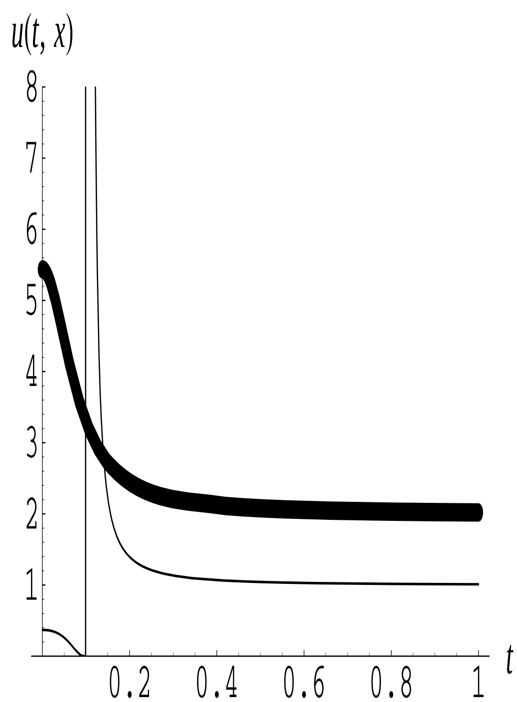

Figure 4.5: The solid curve is from (4.2) for the backward heat equation (4.1), while the other curves are from(3.112) for the heat equation (2.1).

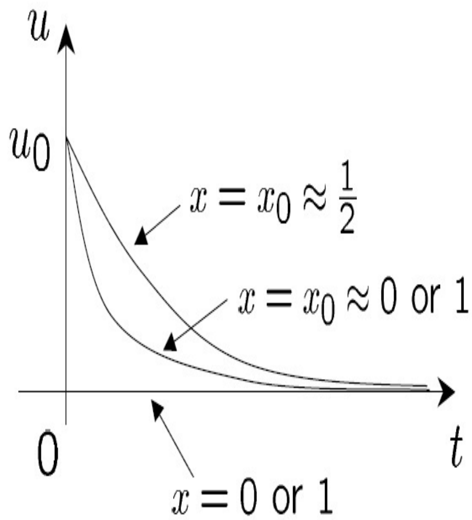

Figure 4.6: The temperature against time curve obtained by Hancork, valid for $t \geq 1/\pi^2$, comparable to the solution in Figure 4.5.

Conclusion

In this study, new symmetries were obtained for the heat equation, and two were used to determine group invariant solutions. It was shown they do not only lead to solutions possible through old symmetries, but also to new solutions, including the ones possible through other methods.

APPENDIX A: Generalising Euler's formulas for solving second order ordinary differential equations

It is well-known that Lie's group theoretical methods seek to reduce procedures for solving differential equations of any challenging form to simple ones that may also have the form

$$a_0\ddot{y} + b_0\dot{y} + c_0 y \;=\; 0, \tag{4.4}$$

for $y = y(x)$, with parameters a_0, b_0 and c_0. It is also accepted that Euler's formulas are suitable for solving such equations. They are:

$$y = \begin{cases} e^{-\frac{b_0}{2a_0}x}\left(Ae^{-\tilde{\omega}x} + Be^{\tilde{\omega}x}\right), & b_0^2 > 4a_0c_0, \\ A + Bx, & b_0^2 = 4a_0c_0, \\ e^{-\frac{b_0}{2a_0}x}[A\cos(\tilde{\omega}x)] & \\ +Be^{-\frac{b_0}{2a_0}x}[\sin(\tilde{\omega}x)], & b_0^2 < 4a_0c_0 \end{cases} \tag{4.5}$$

where $\tilde{\omega} = \sqrt{b_0^2 - 4a_0c_0}/(2a_0)$.

But there is a problem with this system: It does not reduce to $y = A + Bx$ when $b_0 = c_0 = 0$. This is because Euler did not solve the equation to get the formulas.

There has never been a need to do so, primarily because the formulas have been very successful in applications, and they still are.

The need for an exact solution here, is driven by the desire to understand solutions for equation (2.1) through symmetry methods. It is impossible through Euler's formulas to get such exact formula, first let

$$y = \beta z,$$

with $\beta = \beta(x)$ and $z = z(x)$, so that

$$\dot{y} = \dot{\beta} z + \beta \dot{z},$$

and

$$\ddot{y} = \ddot{\beta} z + 2\dot{\beta}\dot{z} + \beta \ddot{z}.$$

These transform (4.4) into

$$a_0\left(\ddot{\beta} z + 2\dot{\beta}\dot{z} + \beta\ddot{z}\right) + b_0\left(\dot{\beta} z + \beta\dot{z}\right) + c_0\beta z \;=\; 0.$$

That is,

$$a_0\beta\ddot{z} + \left(2a_0\dot{\beta} + b_0\beta\right)\dot{z} + \left(a_0\ddot{\beta} + b_0\dot{\beta} + c_0\beta\right) z \;=\; 0. \tag{4.6}$$

Choosing β to satisfy $2a_0\dot{\beta} + b_0\beta = 0$ simplifies equation (4.6). That is,

$$\beta = C_{00} e^{\frac{-b_0}{2a_0}x},$$

for some constant C_{00}. Equation (4.6) assumes the form

$$\ddot{z} \;=\; -\frac{a_0\ddot{\beta} + b_0\dot{\beta} + c_0\beta}{a_0\beta} z.$$

That is,

$$\ddot{z} \;=\; \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right) z.$$

But $\ddot{z}$ can be written as $\dot{z}d\dot{z}/dx$. Therefore,

$$\dot{z}\frac{d\dot{z}}{dz} \;=\; \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right) z,$$

or

$$\dot{z}d\dot{z} \;=\; \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right) z dz.$$

That is,

$$\frac{\dot{z}^2}{2} = \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right)\frac{z^2}{2} + C_{01},$$

for some constant C_{01}. That is,

$$\dot{z} = \sqrt{\left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right)\frac{z^2}{2} + C_{01}},$$

or

$$\frac{dz}{\sqrt{\left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)z^2 + 2C_{01}}} = dx.$$

That is,

$$\frac{dz}{\sqrt{A_{00}^2 - z^2}} = \sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\ dx,$$

with $A_{00}^2 = 2C_{01}/\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}$. Hence,

$$z = \frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}} \times \sin\left(\sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\ x + C_{02}\right), \tag{4.7}$$

for some constant C_{02}. That is,

$$y = C_{00}e^{\frac{-b_0}{2a_0}x}\ \frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}} \times \sin\left(\sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\ x + C_{02}\right). \tag{4.8}$$

Letting

$$\bar{\omega} = \sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}$$

we have

$y = C_{00}e^{\frac{-b_0}{2a_0}x}\ \frac{2C_{01}}{\bar{\omega}}\ \sin\left(\bar{\omega}\ x + C_{02}\right),$

or

$y = C_{00}e^{\frac{-b_0}{2a_0}x}\ 2C_{01}\ [\frac{\sin(C_{02})}{\bar{\omega}}\cos\left(\bar{\omega}\ x\right) + \cos\left(C_{02}\right)\frac{\sin(\bar{\omega}\ x)}{\bar{\omega}}].$

A reduction to the trivial case $\ddot{y} = 0$ requires that $\sin(C_{02}) = C_{03}\sin(\bar{\omega})$ and $\cos(C_{02}) = C_{04}\cos(\bar{\omega})$. That is, $C_{03}^2 + C_{04}^2 = 1$. Hence,

$y = C_{00} e^{\frac{-b_0}{2a_0} x}\ 2C_{01}\ [\frac{C_{03} \sin(\bar{\omega})}{\bar{\omega}} \cos(\bar{\omega}\ x) + C_{04} \cos(\bar{\omega}) \frac{\sin(\bar{\omega}\ x)}{\bar{\omega}}],$

or simply

$$\begin{aligned} y &= C_{00} e^{\frac{-b_0}{2a_0} x}\ 2C_{01}\ \frac{C_{03} \sin(\bar{\omega}) \cos(\bar{\omega}\ x)}{\bar{\omega}} \\ &\quad + C_{00} e^{\frac{-b_0}{2a_0} x}\ 2C_{01}\ \frac{C_{04} \sin(\bar{\omega}\ x)}{\bar{\omega}}. \end{aligned} \tag{4.9}$$

It is very vital to indicate that if the parameters $\bar{\omega}$ in the denominator and $\sin(\bar{\omega})$ are absorbed into the coefficients C_{01} and C_{03}, then formula (4.9) would reduce to one of Euler's formulas. But the consequences would be fatal, as formula (4.9) would not reduce to $y = A + Bx$ when $b_0 = c_0 = 0$, that is, when $\bar{\omega} = 0$.

Unfortunately, this result cannot be found in any university textbook.

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Appendices

Appendix A: Manale's formulas and the infinitesimal ω

It is well-known that Lie's group theoretical methods seek to reduce procedures for solving differential equations of any challenging form to simple ones that may also have the form

$$a_0\ddot{y} + b_0\dot{y} + c_0 y \;=\; 0, \tag{4.10}$$

for $y = y(x)$, with parameters a_0, b_0 and c_0. It is also that accepted Euler's formulas are suitable for solving such equations. They are:

$$y = \begin{cases} e^{-\frac{b_0}{2a_0}x}\left(Ae^{-\tilde{\omega}x} + Be^{\tilde{\omega}x}\right), & b_0^2 > 4a_0c_0, \\ A + Bx, & b_0^2 = 4a_0c_0, \\ e^{-\frac{b_0}{2a_0}x}[A\cos(\tilde{\omega}x)] & \\ +Be^{-\frac{b_0}{2a_0}x}[\sin(\tilde{\omega}x)], & b_0^2 < 4a_0c_0 \end{cases} \tag{4.11}$$

where $\tilde{\omega} = \sqrt{b_0^2 - 4a_0c_0}/(2a_0)$.

But there is a problem with this system: It does not reduce to $y = A + Bx$ when $b_0 = c_0 = 0$. This is because Euler did not solve the equation to get the formulas. There has never been a need to do so, primarily because the formulas have been very successful in applications, and they still are.

The need for an exact solution here, is driven by the desire understand solutions for equation (4.10) through symmetry methods. It is impossible through Euler's formulas. To get such exact formula, first let

$$y = \beta z,$$

with $\beta = \beta(x)$ and $z = z(x)$, so that

$$\dot{y} = \dot{\beta} z + \beta \dot{z},$$

and

$$\ddot{y} = \ddot{\beta} z + 2\dot{\beta}\dot{z} + \beta \ddot{z}.$$

These transform (4.10) into

$$a_0 \left(\ddot{\beta} z + 2\dot{\beta}\dot{z} + \beta \ddot{z} \right) + b_0 \left(\dot{\beta} z + \beta \dot{z} \right) + c_0 \beta z \; = \; 0.$$

That is,

$$a_0 \beta \ddot{z} + \left(2a_0 \dot{\beta} + b_0 \beta \right) \dot{z} + \left(a_0 \ddot{\beta} + b_0 \dot{\beta} + c_0 \beta \right) z \; = \; 0. \tag{4.12}$$

Choosing β to satisfy $2a_0\dot{\beta} + b_0\beta = 0$ simplifies equation (4.12). That is,

$$\beta = C_{00} e^{\frac{-b_0}{2a_0} x},$$

for some constant C_{00}. Equation (4.12) assumes the form

$$\ddot{z} \; = \; -\frac{a_0\ddot{\beta} + b_0\dot{\beta} + c_0\beta}{a_0\beta} z.$$

That is,

$$\ddot{z} \; = \; \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2} \right) z.$$

But $\ddot{z}$ can be written as $\dot{z}dz/dx$. Therefore,

$$\dot{z}\frac{d\dot{z}}{dz} \; = \; \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2} \right) z,$$

or

$$\dot{z}d\dot{z} \; = \; \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2} \right) zdz.$$

That is,

$$\frac{\dot{z}^2}{2} \; = \; \left(\frac{b_0^2 - 4a_0c_0}{4a_0^2} \right) \frac{z^2}{2} + C_{01},$$

for some constant C_{01}. That is,

$$\dot{z} \ = \ \sqrt{\left(\frac{b_0^2 - 4a_0c_0}{4a_0^2}\right)\frac{z^2}{2} + C_{01}},$$

or

$$\frac{dz}{\sqrt{\left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)z^2 + 2C_{01}}} = dx.$$

That is,

$$\frac{dz}{\sqrt{A_{00}^2 - z^2}} = \sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\ dx,$$

with $A_{00}^2 = 2C_{01}/\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}$. Hence,

$$z = \frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}}\ \sin\left(\sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\ x + C_{02}\right),$$

for some constant C_{02}. That is,

$$y = C_{00}e^{\frac{-b_0}{2a_0}x}\ \frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}}\ \sin\left(\sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}\ x + C_{02}\right).$$

Letting

$$\bar{\omega} = \sqrt{-\frac{b_0^2 - 4a_0c_0}{4a_0^2}}$$

we have

$$y = C_{00}e^{\frac{-b_0}{2a_0}x}\ \frac{2C_{01}}{\bar{\omega}}\ \sin\left(\bar{\omega}\ x + C_{02}\right),$$

or

$$y = C_{00}e^{\frac{-b_0}{2a_0}x}2C_{01}[\frac{\sin\left(C_{02}\right)}{\bar{\omega}}\cos\left(\bar{\omega}x\right) + \cos\left(C_{02}\right)\frac{\sin\left(\bar{\omega}\ x\right)}{\bar{\omega}}]$$

A reduction to the trivial case $\ddot{y} = 0$ requires that $\sin(C_{02}) = C_{03}\sin(\bar{\omega})$ and $\cos(C_{02}) = C_{04}\cos(\bar{\omega})$. That is,

$$C_{03}^2 + C_{04}^2 = 1.$$

Hence,

$$\begin{aligned} y = C_{00}e^{\frac{-b_0}{2a_0}x}2C_{01}[\frac{C_{03}\sin\left(\bar{\omega}\right)}{\bar{\omega}}\cos\left(\bar{\omega}\ x\right) \\ + C_{04}\cos\left(\bar{\omega}\right)\frac{\sin\left(\bar{\omega}\ x\right)}{\bar{\omega}}] \end{aligned} \tag{4.13}$$

or simply

$$
\begin{aligned}
y &= C_{00} e^{\frac{-b_0}{2a_0} x} 2C_{01} \frac{C_{03} \sin(\bar{\omega}) \cos(\bar{\omega} x)}{\bar{\omega}} \\
&+ C_{00} e^{\frac{-b_0}{2a_0} x} 2C_{01} \frac{C_{04} \sin(\bar{\omega}\ x)}{\bar{\omega}}.
\end{aligned} \tag{4.14}
$$

It is very vital to indicate that if the parameters $\bar{\omega}$ in the denominator and $\sin(\bar{\omega})$ are absorbed into the coefficients C_{01} and C_{03}, then formula (4.14) would reduce to one of Euler's formulas. But the consequences would be fatal, as formula (4.14) would not reduce to $y = A + Bx$ when $b_0 = c_0 = 0$, that is, when $\bar{\omega} = 0$. Unfortunately, this result cannot be found in any university textbook.

Appendix B: Useful limit results

It is true that

$$\lim_{\mu\to 0}\left\{\frac{\sin\left(\frac{\mu t}{i}\right)}{\mu}\right\}=\frac{t}{i}. \tag{4.15}$$

This can be written in the form

$$\lim_{\mu\to 0}\left\{\frac{\sin\left(\frac{\mu x}{i}\right)}{\mu}-\frac{t}{i}\right\}=0,$$

or

$$\lim_{\mu\to 0}\left\{\frac{\sin\left(\frac{\mu t}{i}\right)}{\mu}-\frac{t}{i}\cos\left(\frac{\mu t}{i}\right)\right\}=0. \tag{4.16}$$

Removing the 'lim' for greater clarity:

$$\frac{\sin\left(\frac{\mu t}{i}\right)}{\mu}=\frac{t}{i}\cos\left(\frac{\mu t}{i}\right).$$

That is,

$$\sin\left(\frac{\mu t}{i}\right)=\frac{t}{i}\mu\cos\left(\frac{\mu t}{i}\right), \tag{4.17}$$

or

$$\cos\left(\frac{\mu t}{i}\right)=\frac{i}{t}\frac{\sin\left(\frac{\mu t}{i}\right)}{\mu}.$$

We then have

$$\frac{\cos\left(\frac{\mu t}{i}\right)}{\mu^q}=\mu\frac{\cos\left(\frac{\mu t}{i}\right)}{\mu^{q+1}}. \tag{4.18}$$

Carrying out the derivative on the right hand side:

$$\frac{\cos\left(\frac{\mu t}{i}\right)}{\mu^q}=\frac{-\mu\left(\frac{t}{i}\right)\sin\left(\frac{\mu t}{i}\right)+\cos\left(\frac{\mu t}{i}\right)}{\mu^{q+1}}. \tag{4.19}$$

Substituting (4.17)

$$\frac{\cos\left(\frac{\mu t}{i}\right)}{\mu^q}=\frac{-\mu^2\left(\frac{t}{i}\right)^2\cos\left(\frac{\mu t}{i}\right)+\cos\left(\frac{\mu t}{i}\right)}{\mu^{q+1}} \tag{4.20}$$

That is,

$$\mu\cos\left(\frac{\mu t}{i}\right)=\mu^2t^2\cos\left(\frac{\mu t}{i}\right)+\cos\left(\frac{\mu t}{i}\right), \tag{4.21}$$

which can be expressed in the form

$$\mu^2\cos\left(\frac{\mu t}{i}\right)-\mu^3t^2\cos\left(\frac{\mu t}{i}\right)=\frac{i}{t}\sin\left(\frac{\mu t}{i}\right). \tag{4.22}$$

Since $\sin\left(\frac{\mu t}{i}\right) = 0$ for μ small, it follows then that

$$\mu^2 \cos\left(\frac{\mu t}{i}\right) = \mu^3 t^2 \cos\left(\frac{\mu t}{i}\right). \tag{4.23}$$

Since $e^{\mu t}$ can be expressed in the form $cos(\mu t/i) + i\sin(\mu t/i)$, then

$$\mu^2 e^{\mu t} = \mu^3 t^2 \cos\left(\frac{\mu t}{i}\right) \tag{4.24}$$

so that

$$\sqrt{\mu} e^{\mu t/4} = \left[\mu^3 \cos\left(\frac{\mu t}{i}\right)\right]^{\frac{1}{4}} \sqrt{t}, \tag{4.25}$$

or

$$\sqrt{\mu} e^{-\mu t/4} = \left[\mu^3 \cos\left(\frac{\mu t}{i}\right)\right]^{\frac{1}{4}} \sqrt{t}, \tag{4.26}$$

Therefore (4.25) and (4.26) can then be written in the form

$$u = \frac{\sqrt{\mu}}{\left[\mu^3 \cos\left(\frac{\mu t}{i}\right)\right]^{\frac{1}{4}} \sqrt{t}} \phi(\eta), \tag{4.27}$$

with $\mu = \omega^4(\omega^2 - 1)$ in the case of (4.24) and $\mu = \omega^4(\omega^2 + 1)$ for (4.25). That is,

$$u = \frac{1}{\sqrt{(\omega^2 - 1)t}\ \omega^2} \phi(\eta) \tag{4.28}$$

for (4.24), and

$$u = \frac{1}{\sqrt{(\omega^2 + 1)t}\ \omega^2} \phi(\eta) \tag{4.29}$$

for (4.26).