NON-EUCLIDEAN GEOMETRY AND ITS POSSIBLE ROLE IN THE SECONDARY SCHOOL MATHEMATICS SYLLABUS

by

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SUMMARY

There are numerous problems associated with the teaching of Euclidean geometry at secondary schools today. Students do not see the necessity of proving results which have been obtained intuitively. They do not comprehend that the validity of a deduction is independent of the 'truth' of the initial assumptions. They do not realise that they cannot reason from diagrams, because these may be misleading or inaccurate. Most importantly, they do not understand that Euclidean geometry is a particular interpretation of physical space and that there are alternative, equally valid interpretations.

A possible means of addressing the above problems is the introduction of non-Euclidean geometry at school level. It is imperative to identify those students who have the pre-requisite knowledge and skills. A number of interesting teaching strategies, such as debates, discussions, investigations, and oral and written presentations, can be used to introduce and develop the content matter.

KEY TERMS

Euclidean geometry; parallel postulate; non-Euclidean geometry; mathematical-historical aspects; hyperbolic geometry; consistency; Poincaré model; philosophical implications; senior secondary mathematics syllabus; teaching-learning problems in geometry; misconceptions about mathematics; Van Hiele model of development in geometry; mathematical competency of teachers; pre-assessment strategies; teaching-learning strategies; evaluation strategies

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CHAPTER 1

INTRODUCTION

> The character of necessity ascribed to the truths of mathematics and even the peculiar certainty attributed to them is an illusion.
>
> **John Stuart Mill**

1.1 INTRODUCTORY REMARKS

On being asked what the sum of the angles in a triangle is, the majority of people, if not all of those being asked, would answer without any hesitation that it is 180°. So firmly entrenched is this theorem in Euclidean geometry in the minds of people, that it becomes quite irrelevant whether they are able to substantiate it with a valid deductive argument. After all, it is in agreement with commonsense and experience as is evident in surveying, architecture and engineering . Thus, for the majority of people, this theorem in Euclidean geometry which states that the sum of the angles in a triangle equals 180° is as incontestable as the fact that the sun rises in the east and sets in the west.

The origin of the above-mentioned mindset can be traced to the manner in which Euclidean geometry is encountered at school. Although most mathematics textbooks in current use in schools formally introduce Euclidean geometry as a mathematical system in which the theorems are the logical consequences of a certain set of definitions and axioms, in reality, very little time is spent in the classroom on ensuring that students get a good grasp of Euclidean geometry as a particular interpretation of physical space, and that alternative sets of definitions and axioms can give rise to alternative, equally valid interpretations. A possible reason for this omission on the part of teachers is the fact that evaluation procedures as outlined in the syllabus focus exclusively on the content of Euclidean geometry. Another possible reason is that many teachers may never have had the opportunity to gain such insights into Euclidean geometry so that they are hardly in a position to facilitate the acquisition of these insights by their students. Thus, students memorise theorem upon theorem, and solve rider upon rider without ever gaining an overview of the mathematical system which they are supposed to be mastering. Studying Euclidean geometry at school can therefore be likened to having all the pieces of a jigsaw puzzle without having the picture which gives an indication of how these pieces have to be fitted together.

It is to be expected then, that a person, upon discovering that the sum of the angles of a triangle may be less than, more than or equal to 180°, depending on the premises chosen at the outset, will require time and effort to grapple with these new ideas. He is compelled to critically re-examine the basis of a belief which, up to this point, has been perceived by him as being unquestionable. And in the process he will liberate himself from the mental chains which the mathematical world had managed to liberate itself from only after 2000 years of intense efforts.

A possible didactic solution to the problem of Euclidean geometry not being taught and studied in its proper context in schools, thereby resulting in students remaining closed to the possibility of alternative geometrical perspectives, is the introduction of an elementary course in non-Euclidean geometry in the Std 10 Higher Grade syllabus and/or the Additional Mathematics syllabus . The term 'non-Euclidean geometry' refers to any geometry which is based on some, and the negation of other postulates of Euclidean geometry. This study investigates the mathematical-historical aspects of non-Euclidean geometry and its possible role in the afore-mentioned syllabus.

1.2 ASPECTS THAT PROMPTED THE STUDY

On a general note, Morris Kline (1963:553), the noted historian of mathematics, contends that non-Euclidean geometry is one of the concepts which have revolutionised the way we think about our world and our place in it. However, whereas the influence of other revolutionary concepts have been acknowledged, non-

Euclidean geometry is relatively unknown. As an illustration, students become acquainted with the theory of evolution of Darwin and the heliocentric theory of Copernicus at school, but seldom, if ever, are they made aware of the momentous discoveries of Gauss, Lobachevsky and Bolyai. It is by time that these names become as familiar to students as the names Newton and Einstein.

On a mathematical note, the excessive emphasis on the memorisation of facts and the mastery of skills which is evident from both internal and external examination papers, raises doubts as to whether students are acquiring mathematical knowledge which is lasting and which impacts on the way they think and approach problem situations. For example, few students can recall intricate facts such as the cosine rule and procedures such as 'completing the square' once they have left school. The importance that is given to facts and skills in the syllabus also evokes concern that the attainment of higher levels of cognition such as analysis, synthesis and evaluation are not being encouraged at school level. It is no coincidence that students generally lack the ability to think independently, creatively and critically. The fact that negative attitudes towards mathematics, which could possibly hamper achievement, are commonplace amongst students, and that few of them actually pursue careers in mathematics, brings into dispute the fact that the true nature of mathematics and mathematical activity are filtering through to students. If mathematical activity entails following repetitive procedures for the purpose of arriving at the right answer, then students can surely not be blamed for avoiding careers in mathematics. The strict separation in the syllabus of the mathematical contents from the historical circumstances surrounding their discovery, creating the impression that mathematics developed independently of the spirit of the times, causes uncertainty as to whether students are acquiring an appreciation for the history of mathematics. Often, attempts by teachers to incorporate aspects of the history of mathematics into their lessons are met with apathy and even irritation on the part of students.

On a purely geometrical note, the fact that many of the results in Euclidean geometry currently being studied at school are regarded as facts which are so 'obvious' that they need not be proven, advances the question as to whether these results actually promote an understanding of deductive arguments as necessary means of verifying knowledge which has been obtained intuitively. Most teachers have at some stage in their careers encountered students who refuse to see the point in proving a result which they already know to be true. Although the current trend to concretise the learning of geometry with the aid of blocks, tangrams, geoboards etc. is commendable, it creates scepticism as to whether sufficient efforts are being made at school level to facilitate an understanding of a geometry as an abstract system. The designers of the current in-service courses for teachers have not given serious attention to this matter, neither are there indications that they will do so in the near future. The common practices of students to reason in 'circles', to base their arguments on diagrams which may be inaccurate or misleading, and to ignore alternative, equally valid proofs, cast doubt over the fact that studying Euclidean geometry helps students to gain a conceptual understanding of proofs.

The seriousness of the afore-mentioned concerns points not only to a need to improve the teaching of Euclidean geometry, but also to a need to search for alternative content matter in geometry. This study is essentially an investigation of such content matter and its presentation in the classroom with the aim of compensating for the deficiencies of current content matter and teaching practices in geometry.

1.3 FORMAL STATEMENT OF THE RESEARCH PROBLEM

This study gives an exposition of the mathematical-historical aspects of non-Euclidean geometry with a view towards demonstrating that its inclusion in the Std 10 Higher Grade syllabus and/or the Additional Mathematics syllabus is a possible solution to the problem of putting Euclidean geometry in its proper perspective as the geometry of flat surfaces which will not be valid when enormous distances on earth or in space are considered. This study also makes practical suggestions as to the extent of the content to be covered and ways of making it interesting and meaningful for the students.

The following discrepancies will be pursued :

1.3.1 whether the elementary content matter of non-Euclidean geometry is suitable for study by those students whom teachers have identified as having the prerequisite knowledge and skills

1.3.2 whether the study of non-Euclidean geometry promotes the advancement from the 4th to the 5th Van Hiele level of development of geometric thought

1.3.3 whether the study of non-Euclidean geometry can help students to reconceive their perceptions with respect to the following issues commented on by Borasi in Cooney and Hirsh (1990:176):

(a) the nature of mathematical activity

(b) the scope of mathematical activity

(c) the nature of mathematics

(d) the origin of mathematics

1.3.4 whether the study of non-Euclidean geometry can be related to occurrences in the real world

1.3.5 whether the study of non-Euclidean geometry at school level will encourage those teachers, who are not sufficiently mathematically competent, to take steps towards improving their situation in the interest of the mathematics education of their students

1.4 ANALYSIS OF THE PROBLEM

To provide further insight into the problem under discussion, and to enable the recommendations being made in this study to be given fair consideration, it is essential to give a brief sketch of the main subproblems. These will naturally be examined in far greater detail in the main body of this study.

1.4.1 'Geometry' versus 'Euclidean geometry'

Although the geometry syllabus makes specific reference to Euclidean geometry, the majority of teachers casually use the term 'geometry' to imply 'Euclidean geometry', thereby making it difficult for their students to even contemplate the existence of alternative geometries such as Lobachevskian geometry or Riemannian geometry. In the instances that the term 'Euclidean geometry' is used, it is seldom used for the purpose of re-inforcing a proper understanding of Euclidean geometry. Thus, few students realise that the geometry which they are studying is based not on 'self-evident truth', but on the assumptions of the Greek geometer, Euclid. Other assumptions, some possibly contradicting those of Euclid, can, on condition that they are consistent, form the basis of other geometries which will not be Euclidean.

The issue is complicated further when students encounter analytical geometry, because they gain the impression that they are studying an alternative geometry. Usiskin in Lindquist (1987:23) discusses this problem at length. Regrettably, there are even teachers who hold this misconception. Teaching non-Euclidean geometry at school will necessitate the proper usage of the term 'Euclidean geometry', and will make clear the distinction between an approach to and the nature of a geometry.

1.4.2 Introducing non-Euclidean geometry at the appropriate stage

Ideally, all students should have their visions broadened by studying non-Euclidean geometry. However, the conceptual leaps and the technicalities involved in studying non-Euclidean geometry make it obvious that it is not suitable for study by all students and at any stage in the development of geometric thought.

The ability to understand non-Euclidean geometry is identified in the Van Hiele model of development in geometry as the highest level of development. According to the model, progress between the levels is sequential. A mismatch between the level of the content and the level of the student can hamper progress to the next level (Crowley in

Lindquist 1987:4). These factors imply that it is imperative that teachers identify those students who are ready for the study of non-Euclidean geometry. A Van Hiele based test, such as the one which is detailed in this study, would help to accomplish this.

1.4.3 Choosing suitable content matter

Because the idea of teaching non-Euclidean geometry at school level is very likely to be met with opposition, careful consideration has to go into choosing suitable content matter. From a perusal of some of the works on non-Euclidean geometry, it becomes clear that the field is vast and that the approaches are varied. For example, Bonola (1955) uses a trigonometric approach as was originally done by Bolyai and Lobachevsky, whereas Greenberg (1974) bases his approach on Hilbert's axiomatisation of Euclidean geometry.

In order to choose content matter which can be studied meaningfully by students, the following points have to be borne in mind :

(a) the aim of teaching non-Euclidean geometry at school is not to make students experts on the subject, but to provide them with another 'way of seeing'

(b) the content matter has to be introduced in such a way so that it can be built onto the students' existing knowledge as naturally as possible

(c) the content matter has to reflect the restraining effect of the spirit of the times on the discovery of non-Euclidean geometry and the intellectual courage of its discoverers

(d) the content matter has to reflect the impact of non-Euclidean geometry on human thinking

In the light of the above-mentioned points, the approaches of Trudeau (1987), Bonola (1955), Gray (1979) and Greenberg (1974) have been found to be appropriate for school level.

1.4.4 Devising strategies for effective teaching

Euclidean geometry was the first branch of mathematics to be logically organised. Niven in Lindquist (1987:38) remarks that this fact is partly responsible for the dull and uninspiring methods that have become a feature of the teaching of Euclidean geometry. It is important that, for non-Euclidean geometry to fulfil its function at school level, it be taught in a way that will help students to make it their own.

According to the constructivist theory of Piaget, students do not passively receive knowledge through sensory experiences. Rather, they actively construct knowledge on the basis of their existing knowledge and experiences. The traditional role of teachers as custodians of knowledge therefore has to change to that of facilitators of learning.

This theory applied to the teaching of non-Euclidean geometry implies that teachers should create a learning environment in which autonomous thinking and critical reflection, the very activities which led to the discovery of non-Euclidean geometry, are given priority. Teachers have to listen carefully to their students' reasoning, especially when they take their first steps in this strange new world. By posing probing questions and encouraging lively discussions, they can help their students to realise that Euclidean geometry is a habit of thought which can be broken with some effort.

The teaching strategy which is best suited for putting the afore-mentioned ideas into practice is the discovery strategy. However, this strategy cannot just be used indiscriminately, because repeated failure at making a discovery may discourage future attempts by students. A good variety of strategies will thus have to be looked into.

1.4.5 Teaching non-Euclidean geometry as a means of encouraging a greater degree of mathematical competency amongst teachers

From in-service workshops it is evident that not all teachers are competent to teach even the mathematics prescribed in the current syllabus. How will these teachers possibly then be capable of teaching a topic which is beyond the scope of the current syllabus? Usiskin in Lindquist (1987:20) points out that many teachers "may never have encountered a non-Euclidean geometry". It is obvious that teachers cannot teach topics which they have very little knowledge of with great success. Having to teach non-Euclidean geometry will mean that teachers will have to familiarise themselves with the subject if they have not already done so. In the process, they will benefit in both a personal and professional capacity. The same can be said about any other topic in mathematics which enables teachers to put the syllabus contents into better perspective for the students.

From the above remarks it becomes clear that both didactic and mathematical content aspects need to be given attention in pre-service and in-service courses for teachers.

1.5 AIMS OF THE STUDY

The primary aims of this study are :

1.5.1 to contextualise the discovery of non-Euclidean geometry by briefly sketching the foundations of Euclidean geometry with special emphasis on the logical flaws in the *Elements*

1.5.2 to illustrate the intensity of the attempts "to vindicate Euclid from all defects" by giving an account of some research on the parallel postulate

1.5.3 to emphasise the magnitude of the discovery of non-Euclidean geometry by discussing some of its philosophical implications

1.5.4 to show that the study of non-Euclidean geometry is suitable for school level by detailing the elementary ideas associated with it

1.5.5 to motivate the study of non-Euclidean geometry at school level by speculating on the general idea that it will broaden the vision of students

1.5.6 to show that non-Euclidean geometry need not be taught in the tedious way that Euclidean geometry has traditionally been taught by proposing some interesting strategies for teaching it

1.6 METHOD OF RESEARCH

A literature study will be undertaken. Analysis of the foundations of Euclidean geometry, critical commentary on the *Elements*, and the status of Euclidean geometry up to the early 19th century will be examined. The research on the parallel postulate will be traced, and different viewpoints on the consequent development of non-Euclidean geometry will be considered. Various approaches to the study of non-Euclidean geometry will also be studied. In addition, current problems associated with the teaching and learning of geometry at secondary school level, as well as models which could possibly be adapted to teaching non-Euclidean geometry, will be perused.

1.7 PROGRAMME OF STUDY

In Chapter 2, the foundations of Euclidean geometry are briefly reviewed. The viewpoint that the postulates of Euclid are 'self-evident truths' is critically examined. A discussion of the existence of logical flaws in the *Elements* and the uneasiness which it evoked in mathematicians, concludes this chapter.

Chapter 3 gives an account of some of the most intense efforts to re-inforce the foundations of Euclidean geometry. The general scepticism which the parallel postulate had created, resulted in research spanning 2000 years. Of these, the work of the Jesuit priest, Saccheri, is undoubtedly the most important - so determined was Saccheri to "vindicate Euclid from all defects", that he did not realise that he had stumbled upon non-Euclidean geometry in the process.

In Chapter 4, the three main characters associated with the discovery of non-Euclidean geometry, namely Gauss, Lobachevsky and Bolyai are discussed. The hyperbolic postulate is stated. Some of the elementary theorems of hyperbolic geometry, the non-Euclidean geometry which was first developed and which can be most easily understood, are stated and proven to show how they differ from well-known Euclidean ones.

Chapter 5 raises the issue about the consistency of hyperbolic geometry. The consistency of hyperbolic geometry is shown to hinge on the consistency of Euclidean geometry. One of the models for hyperbolic geometry in Euclidean geometry, namely the Poincaré model, is described in detail. This chapter also emphasises the fact that proof of the consistency of hyperbolic geometry put it on par with Euclidean geometry from a purely logical standpoint.

In Chapter 6, some philosophical implications of the discovery of non-Euclidean geometry are examined. The realisation dawned on mathematicians that no specific geometry can be identified as the correct interpretation of physical space - a geometry is chosen on the basis of its appropriateness under certain circumstances. For example, Euclidean geometry can be used when building a bridge, but a non-Euclidean geometry is necessary when investigating the effects of gravity on objects in space.

In Chapter 7, a detailed motivation for the study of non-Euclidean geometry at school level is presented. This motivation is presented in the light of current problems which are being experienced in the teaching and learning of geometry, such as the general inability of students to devise logically valid proofs.

In Chapter 8, strategies for verifying that students have the prerequisite knowledge and skills for studying non-Euclidean geometry are proposed. These strategies are in accordance with the Van Hiele model. Strategies for introducing and developing the content matter in such a way as to give students the opportunity to construct their own knowledge are discussed at length. The importance of continually evaluating students' assimilation of the strange new ideas which they are confronted with, becomes evident from this chapter.

Chapter 9 concludes this study with a summary, conclusions and recommendations for the future.

CHAPTER 2

THE ELEMENTS OF EUCLID

> We have learned from the very pioneers of this science not to have any regard to mere plausible imaginings when it is a question of the reasonings to be included in our general doctrine.
>
> Proclus

2.1 INTRODUCTION

The history of non-Euclidean geometry is one of the most protracted, ironic and undoubtedly one of the most important in the history of mathematics. It can be traced back to the compilation of the *Elements* by Euclid. On discovery of logical deficiencies in the *Elements*, mathematicians became determined to compensate for these and consequently embarked on research in Euclidean geometry. Ironically, this research culminated in the discovery of non-Euclidean geometry and not in the elimination of the perceived flaws in the *Elements*.

In this chapter, the compilation of the *Elements* and the high regard with which it was held, are discussed with the aim of showing how difficult it was for contradictory ideas to gain widespread acceptance. Details are also provided on the logical flaws in the *Elements* which caused such great concern amongst mathematicians.

2.2 THE EGYPTIAN INFLUENCE ON ANCIENT GREEK GEOMETRY

The ancient Egyptians became very skilled at the task of laying out the boundary lines of properties after the annual flooding of the fertile areas around the Nile. Based on observation and experiment, they concluded that a straight line is the shortest distance between two points, that the sum of the angles in a triangle equals two right angles, that the ratio of the circumference of a circle to its diameter is a constant, and a number of other results which are familiar to us through our study of geometry at high school. The Greeks became acquainted with these empirical principles and gave the name 'geometry', meaning 'earth measurement', to this science. The Greeks pursued geometry not only for its practical utility, but also for its intellectual appeal. Meschkowski (1965:6) relates the anecdote that Euclid, on being questioned about the usefulness of mathematics, called to a slave and said: "Give that man a few pieces of gold. He studies in order to make a profit." (Perhaps there is some comfort in this incident for those who are currently resisting the implementation of a new mathematics syllabus in South African schools in which the emphasis falls on 'relevance' and 'applicability'.) Unlike the Egyptians and the Babylonians, the Greeks did not rely on inductive arguments on which to base their conclusions, but wanted to prove their results deductively.

2.3 THE COMPILATION OF THE ELEMENTS

Around 300 B.C. the Greek geometer, Euclid, undertook a systematic compilation of the mathematical discoveries of his predecessors into thirteen books which he called the *Elements*. The *Elements* is regarded as the most durable and influential book in the history of modern mathematics. According to Meschkowski (1965:5), the fact that Greek mathematics had produced a work of such rigour around 300 B.C. is evidence that mathematical thought had progressed way beyond the empirical methods which were characteristic of the mathematics practised by ancient civilisations. Until the 19th century the *Elements* was not only valued for its contribution to geometry in particular, but its method, the axiomatic method, served as a model for scientific thought (Barker 1964:16).

Although there are no existing copies of the original work by Euclid, his writings have been reconstructed from the numerous commentaries, reviews and remarks by other writers. Amongst these is a version with textual changes and additions by Theon of Alexandria in the 4th century A.D. Early in the 19th century a Greek manuscript discovered in the Vatican Library was found by internal evidence to precede the version of Theon (Reid 1963:19). Proclus in his Commentary supports the viewpoint that the *Elements* was compiled to serve as a textbook for students rather than a reference for professional mathematicians. The *Eléments de géométrie* of Legendre, which was an attempt to simplify and rearrange the propositions in the *Elements* to facilitate more effective teaching and learning, served as a prototype for the high school versions most widely used in this century. In fact, so successful was Legendre's version in creating popular interest in geometry, that its publication has been hailed by Howard Eves (1981:72) as a "pedagogically great moment in mathematics".

From the outset the *Elements* was recognised as a masterpiece. The form of presentation was not originial. The logical arrangement of definitions, axioms, theorems and proofs stem from earlier Greek writers. The mathematical contents was not original. The theory of proportion which enabled the Greeks to work with commensurable magnitudes i.e. magnitudes which can be expressed as a ratio of whole numbers, as well as incommensurable magnitudes, was the work of Eudoxus. Traditionally, Euclid has been credited only with the proof of the theorem of Pythagoras, although he may well have had to revise earlier proofs or devise new ones to comply with his new arrangement of the theorems. Even the title was not original, as so many earlier works had been called the 'Elements' (Reid 1963:17). However, the particular choice of the axioms, the definitions, the arrangement of the theorems and the rigour of the proofs are Euclid's, and are the hallmarks of his genius.

The thirteen books of the *Elements* are arranged according to subject matter: books I - IV deal with plane geometry, book V with the theory of proportion, books VI - X with the theory of numbers and books XI - XIII with solid geometry. Book V, which constitutes the theory of proportion, is regarded by many historians of mathematics as the most outstanding work in the *Elements*. It was this theory as expounded by Eudoxus which enabled Greek mathematics to progress again after the setbacks which it had experienced due to the discovery of irrational numbers (Reid 1963:21). Each of the thirteen books in the *Elements* begins with a list of definitions of the terms that will be used in it. The definitions provided intuitive explanations of the concepts and thus attempted to ensure clarity and uniformity in interpretation amongst the readers. Stephen Barker (1964:21) puts forward yet another purpose of the definitions. According to Barker, the definitions helped to prevent fallacious arguments in the proofs, because the inclusion of new, undefined terms in the theorems implies the inclusion of unstated premises in the arguments, so that the conclusions follow logically from more premises that have been accounted for.

2.4 CRITICAL COMMENTARY ON THE ELEMENTS

The following are some of the definitions that appear at the beginning of Book I numbered according to Sir Thomas Heath's version of the *Elements*:

1. A point is that which has no part.
2. A line is a breadthless length.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.
23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

In modern times, the definitions of Euclid have been criticised for their vagueness and their logical futility. For example, the concept of the straight line is expressed in terms of the concept "lies evenly between its points" which is by no means unambiguous. Even the seemingly unobjectionable definition of the straight line as the shortest distance between two points introduces the concept of distance into the definition which is equally complex to comprehend. Thus, any attempt to define all the terms to be used in geometry will result either in a vicious circle or in an infinite regress. Unlike Euclid, modern mathematicians have realised that any mathematical system should have undefined terms which, together with purposefully chosen assumptions relating to these terms, form the building blocks of the system.

Euclid next states five common notions and five postulates. The common notions are general statements about magnitudes and as such are applicable to all sciences, whereas the postulates are statements about the possibility of constructions and are applicable specifically to geometry. The following are the common notions of Euclid:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Euclid then postulates the following:

1. That a straight line can be drawn from any point to any point.
2. That a finite straight line can be produced continuously in a straight line.
3. That a circle can be described with any centre and radius.
4. That all right angles are equal to one another.
5. That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles.

On considering the above postulates, it becomes clear that Euclid's approach to geometry differs from the empirical approach of the Egyptians. In the first three postulates Euclid is not referring to actual problems of land surveying because he realises that these constructions may not always be possible on earth due to obstacles such as mountain chains, seas and rivers. To Euclid, it was theoretically possible to do these constructions. Euclidean geometry thus comprised propositions about ideal figures in an ideal plane and it can be argued that the Euclidean plane is therefore just as much a figment of the imagination as the surfaces to which the non-Euclidean geometries are applicable. The fourth postulate may seem to be stating the obvious. However, because Euclid defined a right angle as an angle which is equal to its supplement, it could not be deduced by logic alone that all right angles are equal. Since it was to be required in subsequent proofs, it had to be explicitly stated as a postulate. The fifth postulate is clearly more complicated than the previous ones. It is more verbose, less self-evident and hence it lacks the compelling equality of the other postulates. Kline (1972:87) praises Euclid's ingenious formulation of the fifth postulate. Whilst Euclid realised that a postulate about what happens at infinity is bound to be less plausible because it transcends man's limited experiences, he knew that the parallel postulate was indispensable. He thus formulated his postulate so as to involve conditions under which two lines will meet at a finitely distant point, rather than conditions under which they will not meet along their entire lengths. As the fifth postulate played a central role in the discovery of non-Euclidean geometry, more will be said about it in subsequent chapters.

According to Aristotle, it was not necessary for the truth of the postulates to be established, but their truth would be tested by whether the results deduced from them agreed with reality. Kline (1972:59) presumes that Euclid agreed with Aristotle on the truth status of the postulates. However, it is evident from the history of mathematics prior to the discovery of non-Euclidean geometry that the postulates were not recognised for what they in fact were - carefully chosen assumptions - but were presumed to be so 'self-evident' that their lack of proof did not diminish their significance. They could therefore be used to prove a number of theorems, many of them being deep propositions of Euclidean geometry.

Here is Euclid's *Proposition I, Book I* according to Barker (1964:22):

On a given finite straight line to construct an equilateral triangle

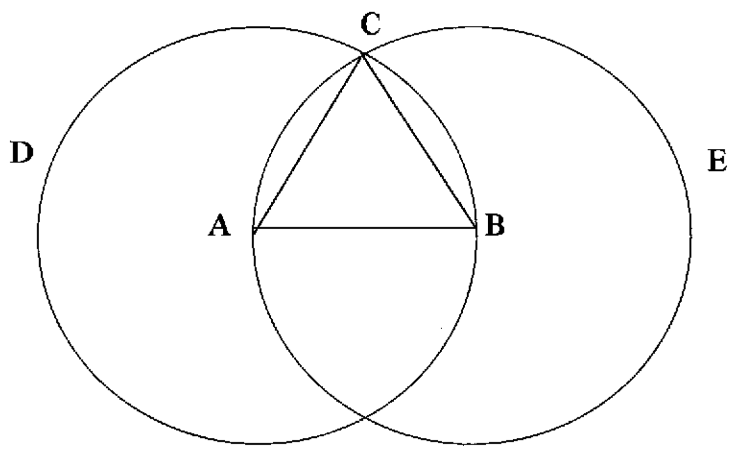


Figure 2.1

Let AB be the finite straight line. (See figure 2.1.) With centre A and radius AB let the circle BCD be drawn (Postulate 3). With centre B and radius BA let the circle ACE be drawn (Postulate 3). From a point C in which the circles cut one another to the points A and B, let the straight lines CA and CB be drawn (Postulate 1). Since the point A is the centre of the circle CDB, AC is equal to AB (Definition 15). Since the point B is the centre of circle CAE, BC is equal to BA (Definition 15). Since CA and CB are each equal to AB, they are equal to each other (Common Notion 1). Therefore the triangle ABC is equilateral and it has been constructed on the given finite straight line AB.

The above theorem illustrates the method that Euclid used to prove his theorems - he attempts to justify each step by recalling the definitions, the common notions or the postulates. It also illustrates the incompleteness of Euclid's system of common notions and postulates. Euclid assumes the existence of the point C, a point of intersection of circles ACE and DCB. Although it is intuitively obvious from the diagram that such a point C exists, diagrams cannot be used to justify existence if the objective is the complete axiomatisation of geometry. Non-Euclidean geometry, unlike Euclidean geometry, did not have the advantage of reasoning from diagrams and since its consistency was also in doubt, it was given a much more rigorous treatment. In order to fill this gap in the proof of *Proposition I*, the following postulate can be used (Moise 1989:242):

The Two-Circle Postulate: If the line segment joining the centres of two circles is less than the sum of their respective radii, then the two circles will intersect in two points. (See figure 2.2.)

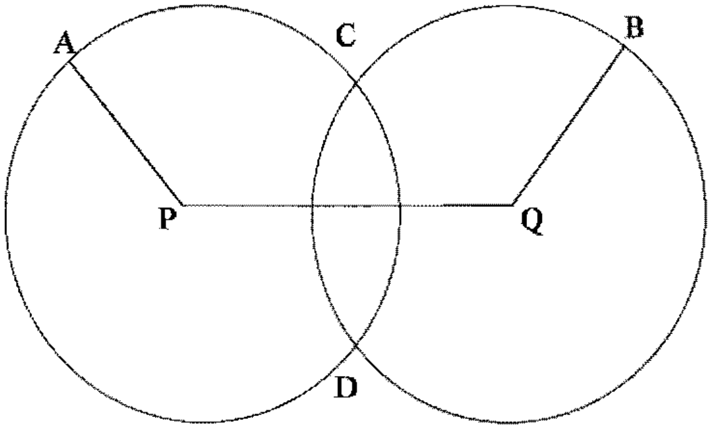


Figure 2.2

According to Meschkowski (1965:7), Euclid tacitly assumes the *Crossbar Theorem* in his proofs: If through the vertex A of a triangle a straight line is drawn which runs along inside the triangle, then this line will intersect the opposite side at a point between its endpoints. (See figure 2.3.)

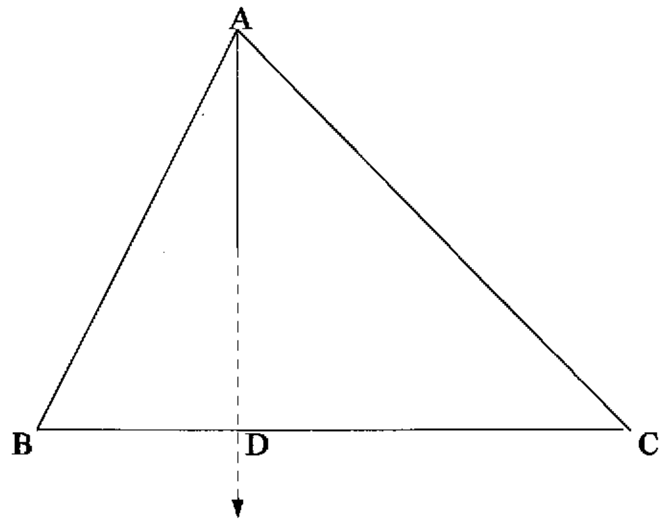


Figure 2.3

Although the diagram in figure 2.3 makes this assertion intuitively obvious, there is no logical basis for rejecting a situation as in figure 2.4:

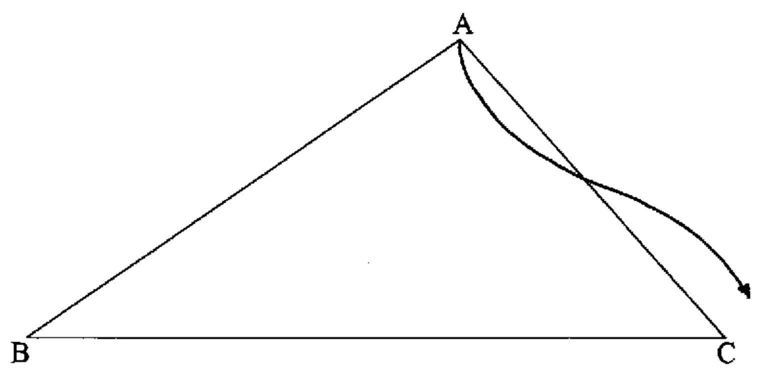


Figure 2.4

To remedy this oversight on the part of Euclid, the *Plane-Separation Postulate* can be used (Moise 1989:74): Given a line and a plane containing it, then the set of all points of the plane that do not lie on the line is the union of two disjoint sets such that (1) each of the sets is convex, and (2) the line segment joining any two points, one belonging to each set, intersects this line. In Moise (1989) this postulate is used to prove a number of incidence theorems, amongst them being the *Postulate of Pasch*: If a line intersects a side of a triangle at a point between the endpoints of that side, then it intersects either of the other two sides. (See figure 2.5.)

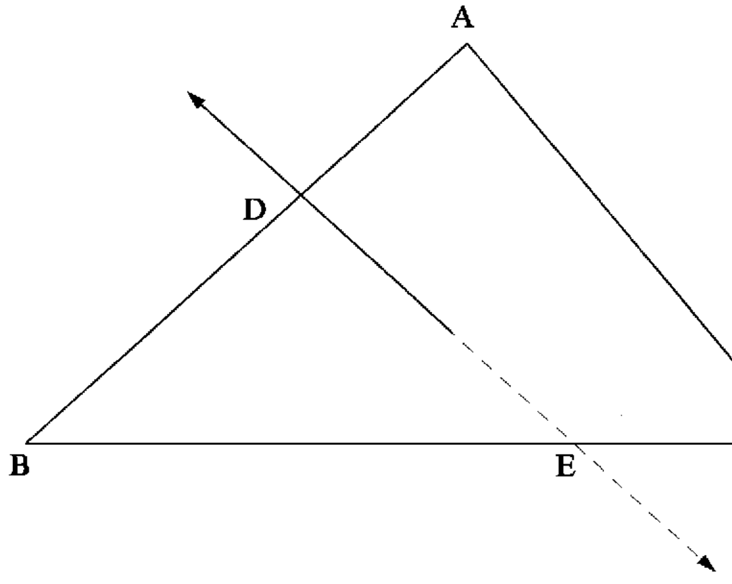


Figure 2.5

These incidence theorems in turn lead to the proof of the *Crossbar Theorem*.

2.5 CONCLUSION

From the previous discussions it is clear that there are logical defects in the *Elements*. Kline (1972:88) claims that there are also defects in the actual proofs and in the reference to theorems of a general nature which are proven only for special cases. It therefore seems that the claim by modern writers that the *Elements* is overrated is justified. However, we have to bear in mind that the achievements of the past cannot be judged according to the standards of rigour of the present. For two thousand years no critical re-examination of the *Elements* was deemed necessary, and the puzzle which the fifth postulate presented mathematicians with remained unsolved as will be illustrated in the next chapter. Perhaps Euclid had the insight which countless mathematicians between his time and the time of Gauss lacked: he knew that the fifth postulate was unprovable from the other postulates, common notions and definitions.

CHAPTER 3

THE RESEARCH ON THE PARALLEL POSTULATE

> It is inconceivable that this enevitable darkness, this eternal eclipse, this blot on geometry was allowed.
>
> Wolfgang Bolyai

3.1 INTRODUCTION

In the previous chapter, it was emphasised that the Greeks revered Euclidean geometry because they believed its foundation, the definitions, the common notions and the postulates, constituted self-evident truths which agreed with their limited experience. The theorems which were logical consequences of the postulates, were therefore also truths in the ordinary sense, and so the whole of Euclidean geometry came to be regarded as the true formulation of the properties of physical space (Reid 1963:148).

The status of Euclidean geometry remained unchallenged until about 1800. In fact, Morris Kline (1972:862) records that there were many attempts to build arithmetic, algebra and analysis on Euclidean geometry so that their truth could also be assured. But how could we be sure that Euclidean geometry was indeed the truth about physical space? The philosopher, Immanuel Kant, attempted to answer this question in his *Critique of Pure Reason* (1781). According to Kant, our minds possess certain modes of space and time which he called intuitions, in terms of which we organise and process our experiences of the external world. Because our minds compel us to interpret these experiences in only one way, certain principles about physical space are prior to experience. Kant maintained that these principles, called *a priori* synthetic truths, were those of Euclidean geometry, and on these grounds he argued that physical space must necessarily be Euclidean (Kline 1980:77).

However, the fifth postulate, the so-called parallel postulate which states that if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles, caused great concern amongst mathematicians. It lacked the terseness and thus the compelling nature of the other postulates, yet denying it would appear to be a violation of our 'better judgement'.

In this chapter, the approaches of some of the leading researchers to the problem of the parallel postulate are detailed. Their noble aim was to strengthen the foundations of Euclidean geometry.

3.2 THREE APPROACHES TO THE PROBLEM OF THE PARALLEL POSTULATE

Euclid also proves the following theorems relating to parallel lines:
Proposition 30, Book 1: Straight lines which are parallel to the same straight line are parallel to each other.

Proposition 31, Book 1: Through a given point, one and only one straight line can be drawn which will be parallel to a given straight line.

Proposition 33, Book 1: The straight lines joining the extremities of two equal and parallel straight lines are equal and parallel.

From these theorems we can deduce the equidistant property of parallel lines, the angle sum property of triangles and the properties of similar figures.

There is evidence that Euclid himself was not entirely satisfied with his version of the parallel postulate because he postponed its use until *Proposition 29,* and he devised more complicated proofs than would have been necessary had he used it as was the case with *Proposition 31, Book 1.*

Since antiquity, three approaches have been made to the problem of the parallel postulate. The first approach was to attempt to prove the postulate using the common notions and the other postulates, thereby giving it the status of a theorem in Euclidean geometry. This seemed a feasible idea, because its converse *Proposition 17, Book I* was a theorem in Euclidean geometry.

Proposition 17, Book I: If two intersecting lines are met by a third straight line, the sum of the interior angles which the third line makes with the two intersecting lines is less than two right angles.

The second approach was to reformulate the postulate or the definition of parallels into less objectionable statements. The third and most radical approach was to investigate the nature of geometries that would result if the postulate were negated.

We shall now consider some of the most important attempts on the parallel postulate.

3.3 PTOLEMY'S ARGUMENT BASED ON THE ANGLE SUM OF PAIRS OF INTERIOR ANGLES

The first known attempt to prove the parallel postulate was made by Ptolemy (2 A.D.).

He reasoned as follows (Bonola 1955:4):

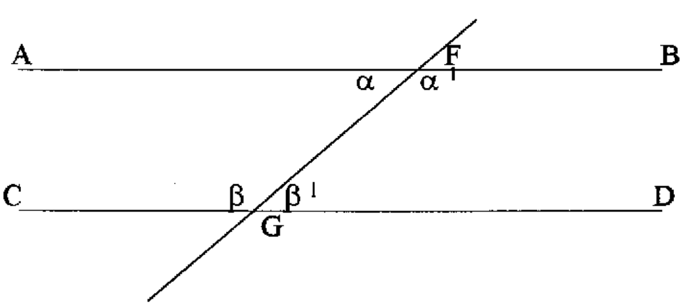


Figure 3.1

Suppose AB and CD are two parallel straight lines and FG a transversal. (See figure 3.1.) Suppose α, β are the two interior angles to the left of FG and α^1, β^1 the two interior angles to the right. Then $\alpha + \beta$ will be either greater than, equal to or less than two right angles. It is assumed that if any one of these cases holds for one pair of parallels, then it holds for every other pair. Suppose $\alpha + \beta >$ two right angles. Since this case holds for FA parallel to GC, it also holds for FB parallel to GD. Hence $\alpha^1 + \beta^1 >$ two right angles. But then $\alpha + \beta + \alpha^1 + \beta^1 >$ four right angles, which is absurd. Similarly $\alpha + \beta$ cannot be less than two right angles. Therefore we must have $\alpha + \beta =$ two right angles.

Euclid's parallel postulate can easily be deduced from Ptolemy's assertion above that if two parallel lines are cut by a transversal, then each pair of interior angles on the same side of the transversal has an angle sum of two right angles. However, this assertion has been shown to be logically equivalent to the parallel postulate (Greenberg 1974:108). Ptolemy was therefore guilty of circular reasoning and his 'proof' did not achieve what he had set out to achieve.

3.4 PROCLUS' APPROACH USING AN ASSERTION ABOUT DISTANCES BETWEEN LINES

The commentator, Proclus (410 - 485 A.D.), explicitly stated his objection to the parallel postulate as follows (Kline 1972:863): "This ought even to be struck out of the postulates altogether; for it is a theorem involving many difficulties... " Proclus further states that although two lines will tend towards each other on the side of the transversal where the sum of the interior angles is less than two right angles, it is plausible but not obvious that they will eventually meet. This conjecture remains to be proven. In support of his argument he gives the example of the hyperbola which tends to but never meets its asymptotes. (See figure 3.2.)

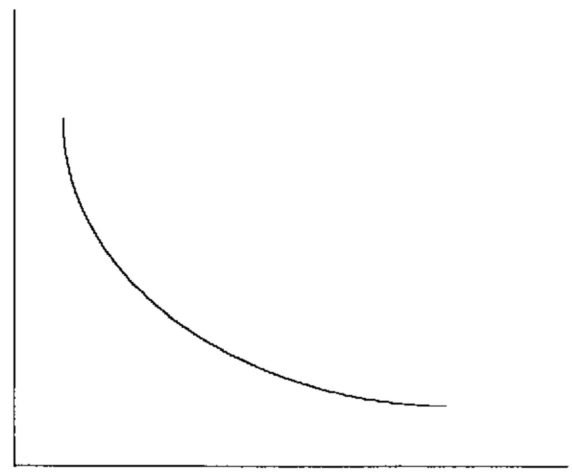


Figure 3.2

Proclus bases his 'proof' of the parallel postulate on the following assumption (Kline 1972:864) : If from one point two straight lines forming an angle be produced indefinitely, then the successive distances between these straight lines will ultimately exceed a finite magnitude. (See figure 3.3.)

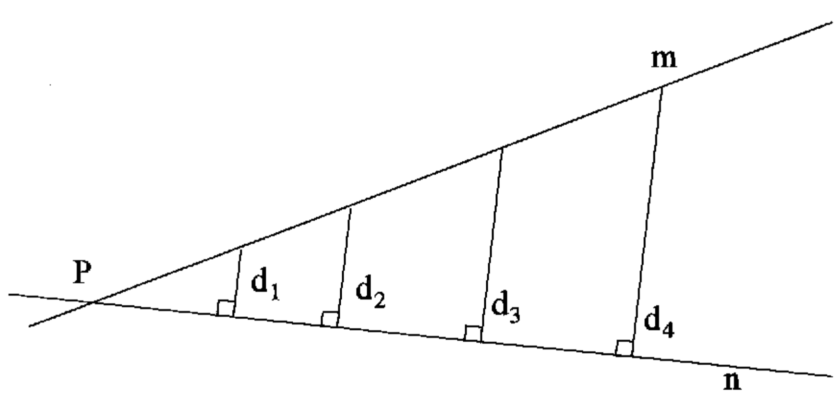


Figure 3.3

Proclus then continues to prove the following lemma:

If a straight line meets one of two parallel lines, then it will also meet the other.

His proof goes as follows (Bonola 1955:5):

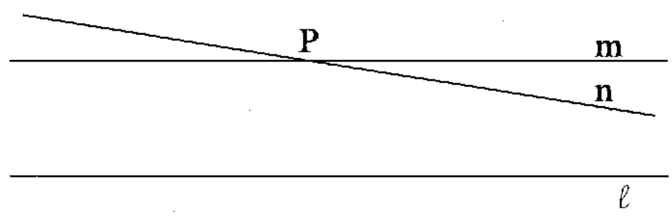


Figure 3.4

Suppose m and ℓ are parallel lines and n a line intersecting m at P. (See figure 3.4.) Then the distance from m to n increases without limit as we move to the right. But since the distance between m and ℓ remains finite, n must necessarily meet ℓ.

The flaw in Proclus' argument lies in his assumption that the distance between two intersecting lines increases without limit while the distance between two parallel lines remains finite. In fact, his lemma is logically equivalent to the parallel postulate (Greenberg 1974:108), and so his efforts amounted to a reformulation rather than a proof of the parallel postulate.

3.5 A PARADOX EVIDENCING THE DISCUSSION OF THE PARALLEL POSTULATE AMONGST THE ANCIENT GREEKS

An interesting paradoxical argument that was well known to the Greeks asserted the following : Two lines which are cut by a third do not meet one another, even when the sum of the interior angles on the same side is less than two right angles.

The following is Gray's version of this argument (1979:38) :

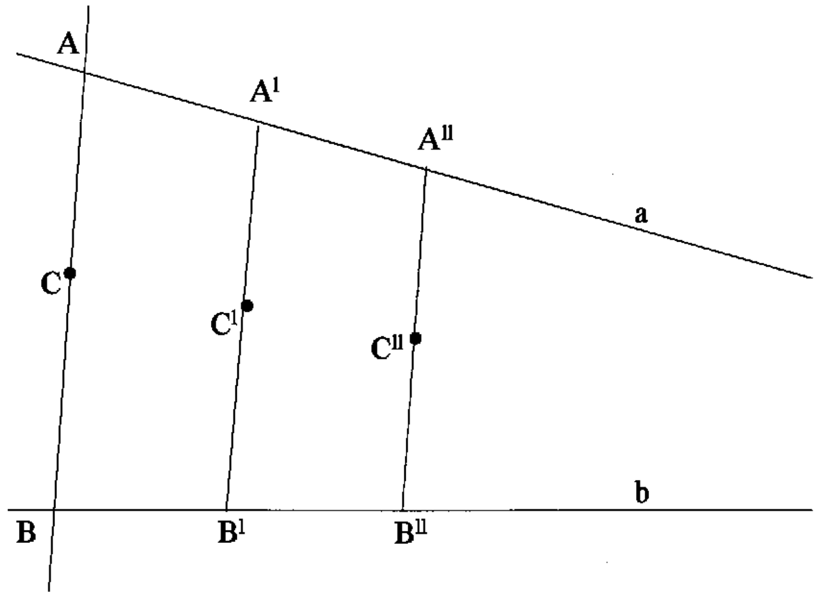


Figure 3.5

Suppose a and b are two lines cut by a third at A and B respectively. (See figure 3.5.) Suppose C is the midpoint of AB. On the side of the transversal on which the sum of the interior angles is less than two right angles, locate the points A^1 and B^1 on a and b respectively, such that AA^1 and BB^1 each equals AC. Then a and b cannot meet between A and A^1 or B and B^1, for then AB would be greater than the sum of the other two sides of the triangle thus formed, contradicting *Proposition 20, Book I.* Draw A^1B^1. Suppose C^1 is the midpoint of A^1B^1. Locate the points A^{11} and B^{11} on a and b respectively, such that A^1A^{11} and B^1B^{11} each equals A^1C^1. As before, a and b cannot meet between A^1 and A^{11} or B^1 and B^{11}. Since this procedure can be continued indefinitely, conclude that a and b will never meet.

The fallacy in the previous argument lies in the fact that the lengths of the line segments AA^1, A^1A^{11}... form a convergent sequence, the limit of which is zero. Although Proclus did not succeed in refuting this argument, his comments relating to it are noteworthy. He states that if the sum of the interior angles is below a certain limiting value, then the lines may still meet. However, any sum above this value will result in the two lines not meeting. By inference, there are a number of lines lying between the two limiting lines which do not meet any given line (Gray 1979:39). (See figure 3.6.).

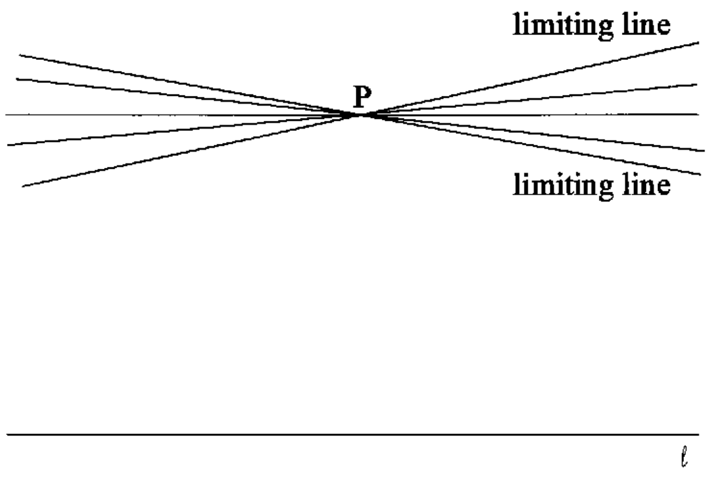


Figure 3.6

3.6 WALLIS' PROPOSAL OF A SUBSTITUTE POSTULATE RELATING TO SIMILAR TRIANGLES

The Englishman, John Wallis (1616 - 1703), gave an exposition on the parallel postulate at Oxford on the evening of 11 July 1663. He had been inspired by the work on the parallel postulate by the Arabian mathematician Nasir - Eddin (1201 - 1274). Wallis based his proof on the following postulate : To every figure there exists a similar figure of arbitrary magnitude.

His argument is as follows (Gray 1979:54):

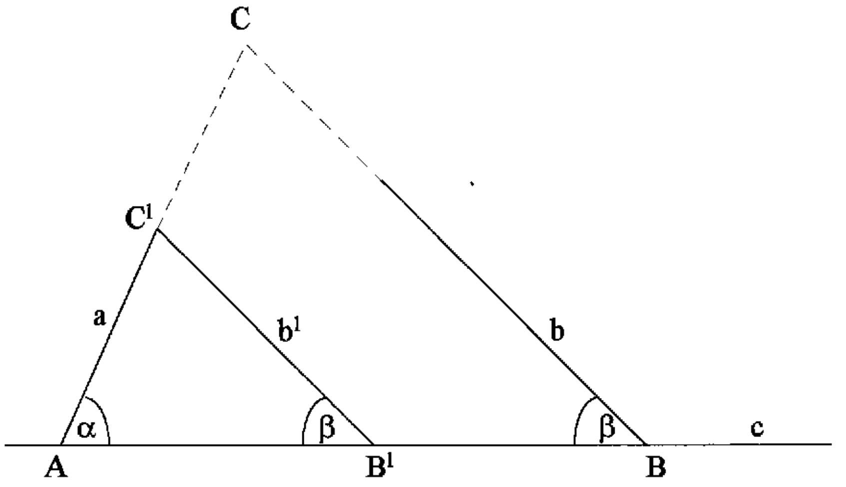


Figure 3.7

Suppose a and b are two lines cut by a third line c at the points A and B respectively. (See figure 3.7.) Suppose the interior angles α and β on the same side of c are such that $\alpha + \beta <$ two right angles. Move the line b along c in the direction of A so that the interior angle β remains constant. Moving b sufficiently close to A will result in a position b^1 of b such that b^1 intersects a. Suppose B^1 and C^1 are the points of intersection of b^1 with c and a respectively. Then AB^1C^1 is a triangle with the angles at A and B^1 equal to α and β respectively. By Wallis' postulate, there exists a triangle ABC similar to triangle AB^1C^1 having AB as one of its sides. Then C is the required point of intersection of the lines a and b.

The above appears to be a valid proof of the parallel postulate if we are prepared to accept the inclusion of Wallis' postulate. Wallis tried to justify its use by asserting that it is a generalisation of Euclid's third postulate which states that it is possible to construct a circle with any given centre and radius. However, Wallis' postulate is no more self-evident than Euclid's parallel postulate, and is in fact logically equivalent to it (Greenberg 1974:132). Thus, in a geometry in which the parallel postulate does not hold, there do not exist similar, non-congruent figures - the arbitrary enlargement or reduction of a figure will necessarily result in its being distorted.

3.7 SACCHERI'S ENCOUNTER WITH A STRANGE NEW WORLD

The most significant attempt on the parallel postulate during the 18th century was made by Girolamo Saccheri (1667 - 1733), a Jesuit priest and professor of mathematics at the University of Pavia. His main interest was in logic, particularly the method of proof by using *reductio ad absurdum*, and in 1697 published the *Logica demonstrativa*. Saccheri was encouraged by the Jesuit father, Tommaso Ceva, to study the *Elements*. He became acquainted with the work of Nasir - Eddin and Wallis, and in 1733 presented his own approach to the parallel postulate in a book titled *Euclides ab Omni Naevo Vindicatus* (Euclid freed of all blemish). Saccheri's novel idea was to apply the *reductio ad absurdum* method to the parallel postulate. He started by assuming the negation of the parallel postulate and in conjunction with the other postulates and the common notions, he hoped to deduce a contradiction. This being the case, his original premise had to be false, and so the parallel postulate would be established.

A summary of Saccheri's approach as it is presented in the translation by George Halsted (1986) with simplifications by Bonola (1955) and Gray (1979) will now be given.

The fundamental figure which Saccheri uses in his discussion is a quadrilateral ABCD with AD equal to BC and the angles at A and B each equal to a right angle. (See figure 3.8.)

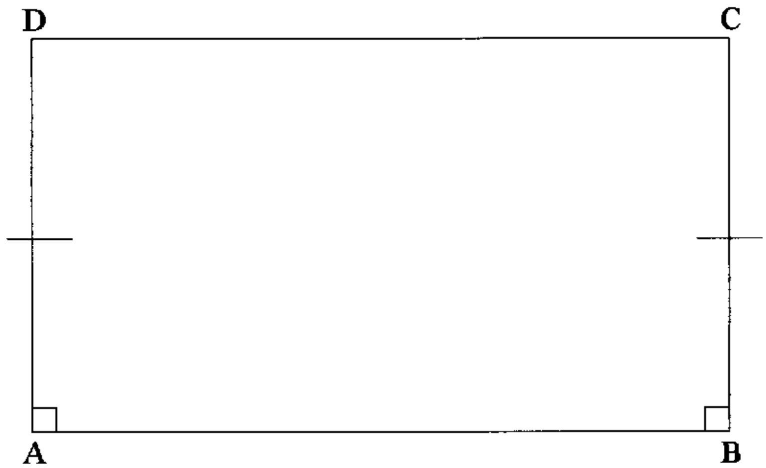


Figure 3.8

By drawing the diagonals AC and BD, and considering the pairs of triangles DAB and CBA, and ADC and BCD, he concludes that ∠ADC equals ∠BCD. Now the parallel postulate is equivalent to the assertion that ∠ADC and ∠BCD are both right angles. Thus, the assumptions that both of these angles are acute, or that both of them are obtuse, are implicit negations of the parallel postulate. Saccheri names these three cases:

(1) Hypothesis of the Right Angle (HRA): ∠ADC and ∠BCD are both right angles

(2) Hypothesis of the Obtuse Angle (HOA): ∠ADC and ∠BCD are both obtuse angles

(3) Hypothesis of the Acute Angle (HAA): ∠ADC and ∠BCD are both acute angles

Saccheri proves the following preliminary result:

(1) On HRA : AB = DC

(2) On HOA : AB > DC

(3) On HAA : AB < DC

Case (1) is clear. For the remaining cases, suppose O and O^1 are the midpoints of AB and DC respectively. (See figure 3.9.) Draw OO^1.

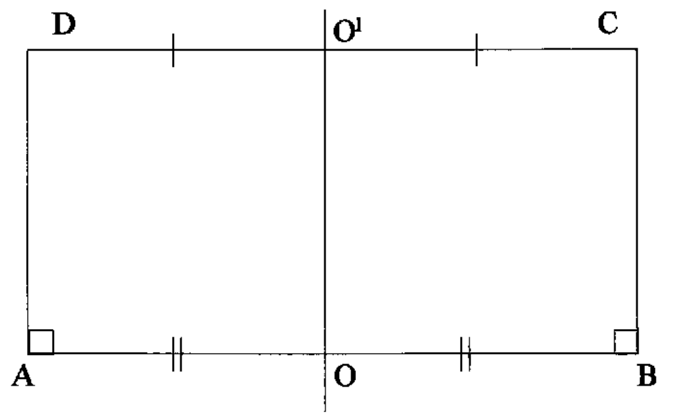


Figure 3.9

By drawing DO, AO^1, OC and O^1B, and considering the resulting pairs of triangles, it can easily be shown that OO^1 is the common perpendicular to AB and DC. Consider quadrilateral OO^1DA and suppose that HOA holds. Then by contradiction, we can prove that AO > DO^1 and so AB > DC. By a similar argument, it follows that on HAA, AB < DC.

Saccheri then goes on to prove that if one of the three hypotheses holds for at least one Saccheri quadrilateral, then it holds for every other Saccheri quadrilateral in space. Next he proves that:

(1) On HRA, the sum of the angles in a triangle equals two right angles

(2) On HOA , the sum of the angles in a triangle is greater than two right angles

(3) On HAA, the sum of the angles in a triangle is less than two right angles

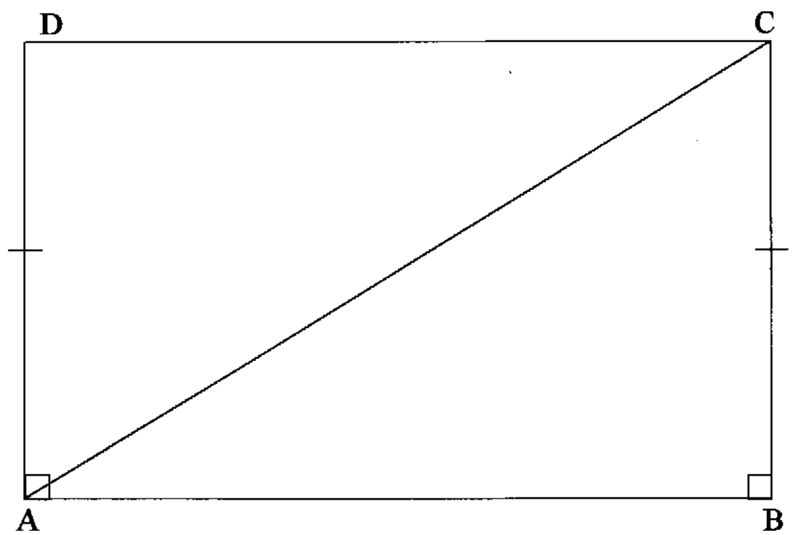


Figure 3.10

His proof of the preceding three statements is as follows:

Suppose ABC is a triangle with a right angle at B. (See figure 3.10.) Draw AD perpendicular to AB with AD equal in length to BC. Draw DC to obtain the quadrilateral ABCD. On HRA, the triangles ABC and CDA are congruent, so that $\angle$BAC equals $\angle$DCA. Then $\angle ABC + \angle ACB + \angle BAC = \angle ABC + \angle ACB + \angle DCA =$ two right angles. On HOA, AB > DC and so $\angle ACB > \angle DAC$, giving $\angle ABC + \angle ACB + \angle BAC >$ two right angles. A similar argument holds on HAA. To prove the result for a general triangle, drop a perpendicular from any vertex to obtain two right-angled triangles.

Saccheri then refutes the HOA as follows:

Suppose ABC is a triangle with a right angle at B, and suppose M is the midpoint of AC. (See figure 3.11.) Draw MN perpendicular to AB with N on AB.

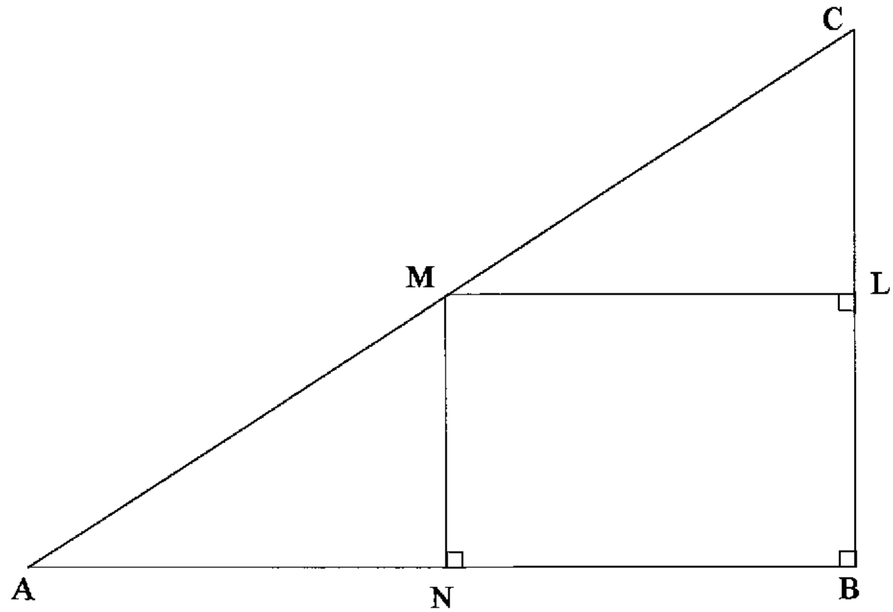


Figure 3.11

On HOA, the sum of the angles of quadrilateral NBCM is greater than four right angles, from which it follows that $\angle AMN < \angle MCB$. Now draw ML perpendicular to CB with ℓ on CB. Since the right-angled triangles ANM and MLC have equal hypotenuses, and since $\angle AMN < \angle MCB$, we have AN < ML. On HOA, $\angle NML >$ a right angle, and so NB > ML. Thus AN < NB. This result implies that if equal intervals are taken along one arm of an acute angle, then they project vertically onto increasing intervals on the other arm. (See figure 3.12.)

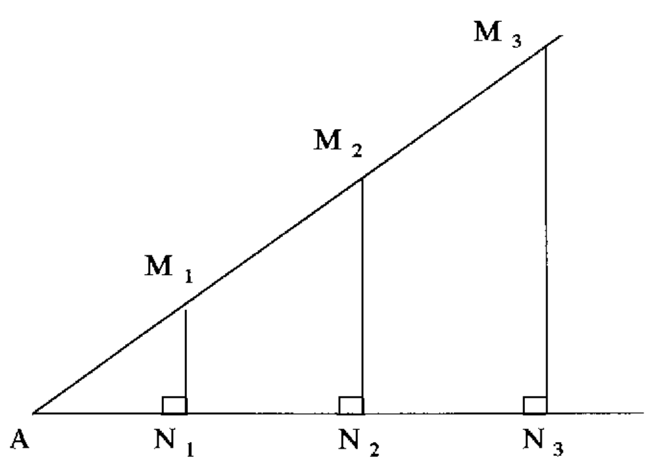


Figure 3.12

Suppose a and b are two lines cut by a third line c at A and B. (See figure 3.13.) Suppose α and β are the interior angles at A and B respectively with $\alpha + \beta <$ two right angles. Then one of these has to be acute, say α. Draw BC perpendicular to a with C on a. On HOA, the sum of the angles of triangle ABC is greater than two right angles. Since $\alpha + \beta <$ two right angles, the angle that BC makes with b is acute.

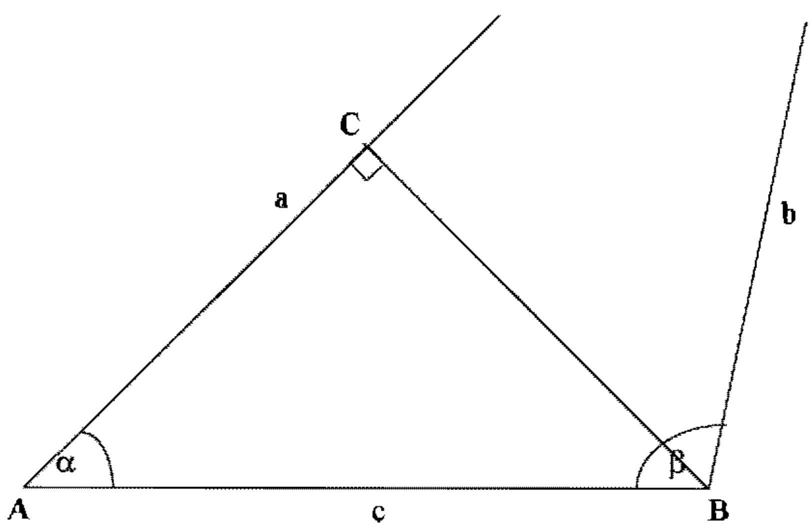


Figure 3.13

Suppose M_1 is a point on b. (See figure 3.14.) Draw M_1N_1 perpendicular to BC with N_1 on BC. Choose M_2 on b such that $BM_2 = 2BM_1$. Draw M_2N_2 perpendicular to BC with N_2 on BC. By our previous result, $BN_1 < N_1N_2$ and so $BN_2 > 2BN_1$. We repeat this process indefinitely choosing M_k on b such that $BM_{k-1} = M_{k-1}M_k$. Draw M_kN_k perpendicular to BC with N_k on BC. Then $BN_k > 2^{k-1}BN_1$. Since BC is finite, by choosing n sufficiently large we can obtain the point N_n such that $BN_n > 2^{n-1}BN_1 >$ BC. Thus C is a point on side BN_n of the right-angled triangle BN_nM_n, and so a meets BM_n i.e. a meets b. This implies that HRA holds, and so HOA must be false.

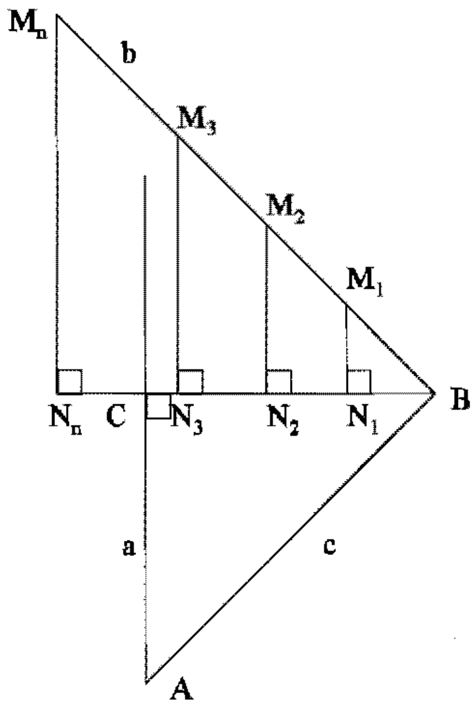


Figure 3.14

Saccheri thus proclaims : "The HOA is absolutely false because it destroys itself" (Saccheri 1733:165).

We note that in the previous argument Saccheri assumes that straight lines can be extended indefinitely, otherwise the existence of the points M_n and N_n cannot be guaranteed.

On HAA, Saccheri deduces that, given any line, there can be drawn a perpendicular to it and a line making an acute angle with it, which do not intersect each other.

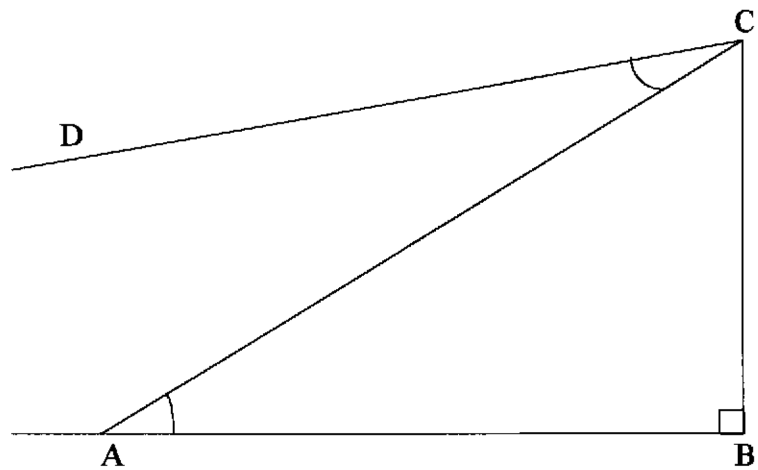


Figure 3.15

To draw these lines, suppose ABC is a triangle with a right angle at B. (See figure 3.15.) At C, draw CD such that ∠DCA equals ∠CAB. Then by the alternate interior angle theorem, DC and AB do not intersect. On the HAA, ∠CAB +∠ACB < a right angle, so that ∠DCB = ∠DCA + ∠ACB < a right angle, and hence the result follows.

With regard to two lines which do not meet, Saccheri examines the conditions under which they will have a common perpendicular.

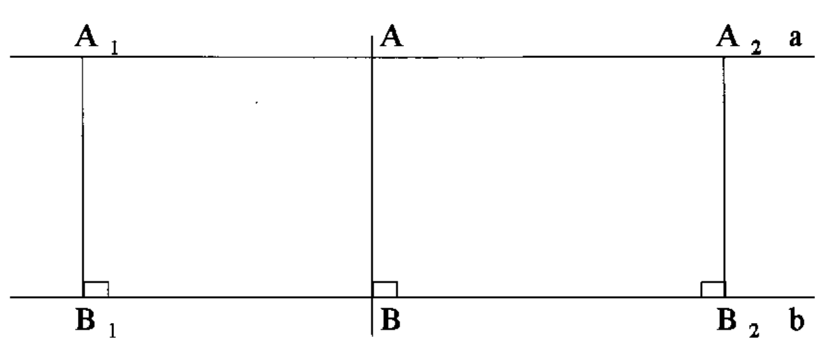


Figure 3.16

Suppose a and b are two lines which do not meet. (See figure 3.16.) Suppose A_1 and A_2 are points on a. Draw A_1B_1 and A_2B_2 perpendicular to b with B_1 and B_2 on b. On the HAA, the angles $\angle B_1A_1A_2$ and $\angle B_2A_2A_1$ of quadrilateral $A_1B_1B_2A_2$ can either be (1) one right and one acute, or (2) both acute, or (3) one acute and one obtuse. In the first case, it is obvious that a and b have a common perpendicular. In the second case, the existence of a common perpendicular is proven by a continuity argument : by moving the line segment A_1B_1 towards A_2B_2 whilst keeping it perpendicular to b, $\angle B_1A_1A_2$ changes from being an acute angle to being an obtuse angle and there is therefore an intermediate position AB at which it is a right angle. In the third case, there is no common perpendicular between B_1 and B_2, nor will there ever be one if $\angle B_2A_2A_1$ is obtuse while $\angle B_1A_1A_2$ is acute or vice versa.

From the above argument, there exist parallel lines which do not have a common perpendicular. Saccheri proves that such lines are asymptotic. :

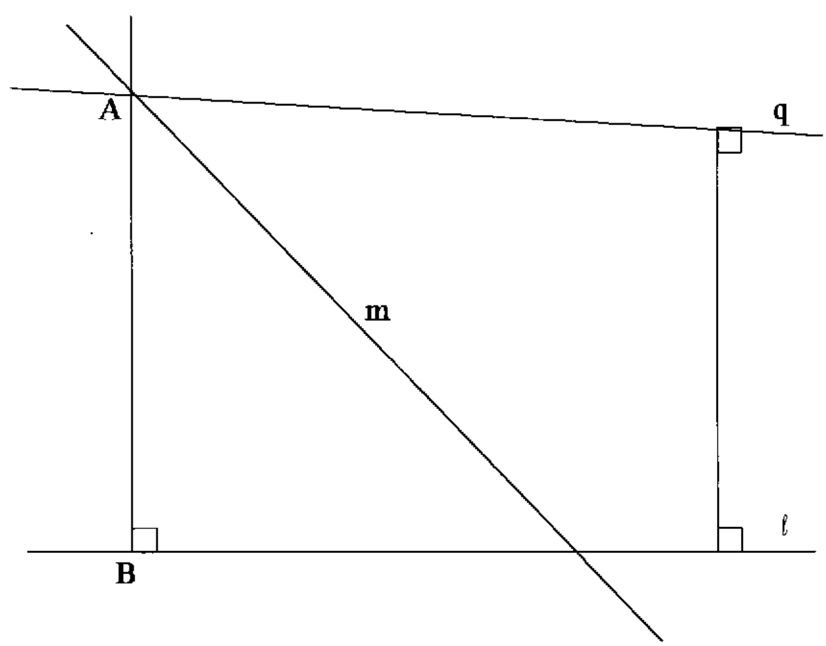


Figure 3.17

Suppose A is a point not on a line ℓ. (See figure 3.17.) Then of the pencil of lines through A, there are those that meet ℓ, and of those that do not meet ℓ, some have a common perpendicular with ℓ. Suppose AB is perpendicular to ℓ with B on ℓ and m is any other line through A meeting ℓ. (See figure 3.18.)

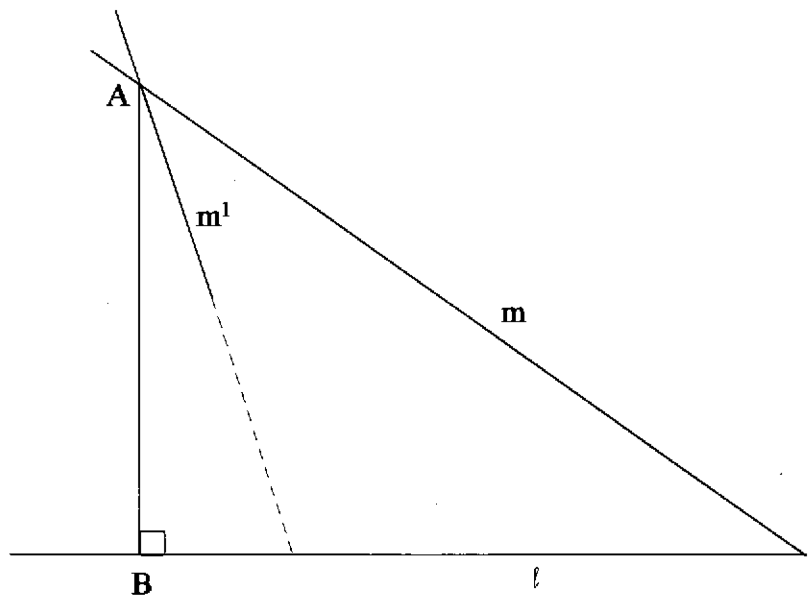


Figure 3.18

Then by the *Crossbar Theorem,* any line m^1 through A which makes a smaller angle with AB than m will also meet ℓ. As these lines move progressively away from AB, no last such line will be encountered. Thus, there is an upper limit α for the angle which any such line makes with AB. Suppose this limiting position of m is m_α. (See figure 3.19.)

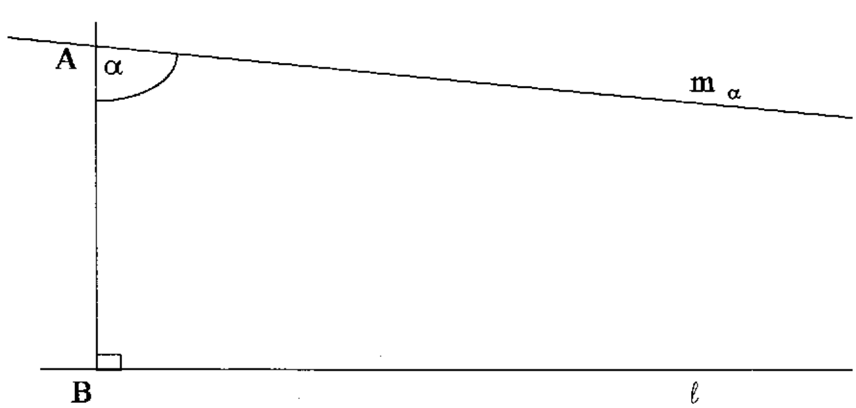


Figure 3.19

Now suppose that q is a line through A parallel to ℓ having a common perpendicular A_2B_2 with ℓ with A_2 and B_2 on q and ℓ respectively. (See figure 3.20.) Suppose q^1 is any other line through A which makes a larger acute angle with AB than q. Extend B_2A_2 to meet q^1 at A_2^1. By the exterior angle theorem, $\angle B_2A_2^1A$ is acute, and from what has been shown earlier, q^1 and ℓ have a common perpendicular between B and B_2.

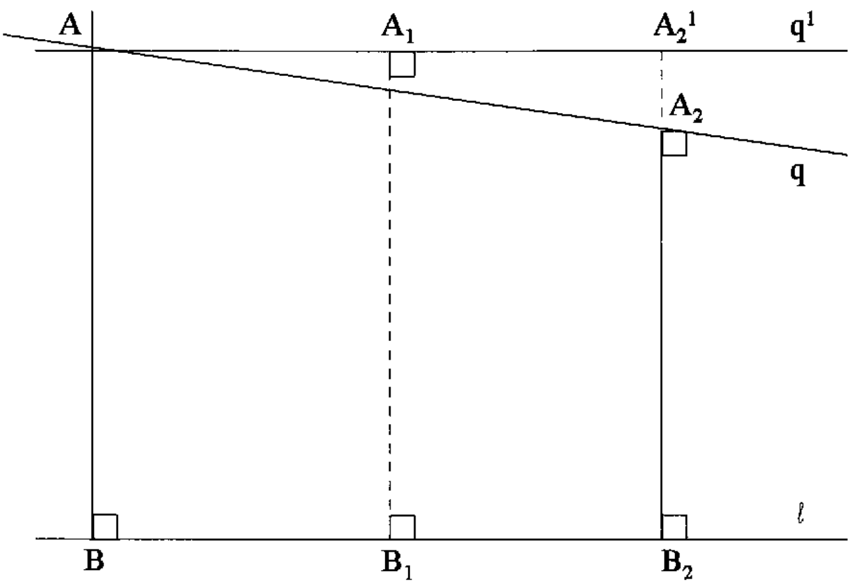


Figure 3.20

As those lines having a common perpendicular with AB move towards AB, no last such line will be encountered. Thus, there is a lower limit β for the angle which any such line makes with AB. Suppose this limiting position of q is q_β.(See figure 3.21.)

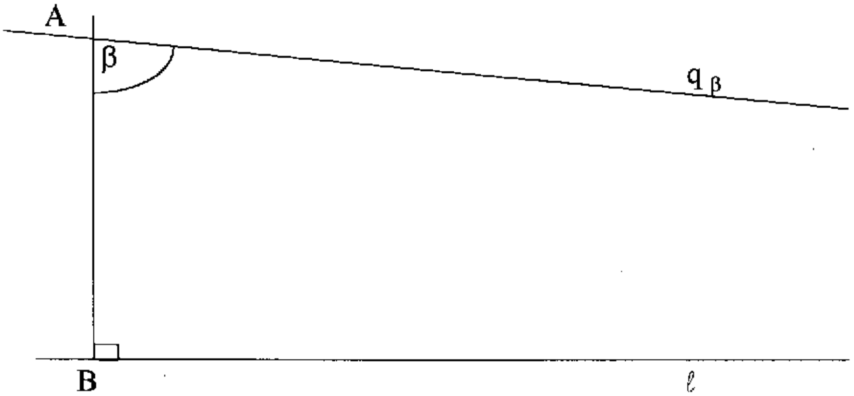


Figure 3.21

Saccheri then shows that the lines m_α and q_β coincide. This line has the following properties :

(1) it does not meet ℓ since it coincides with m_α

(2) it does not have a common perpendicular with ℓ since it coincides with q_β

(3) it is asymptotic to l

Thus, on HAA, there exists in the pencil of lines through A two lines a and b, one asymptotic to ℓ to the left and the other asymptotic to ℓ to the right, which divide the pencil in two parts. The first part consists all the lines which meet ℓ, and the second consists of those which have a common perpendicular with ℓ. (See figure 3.22.)

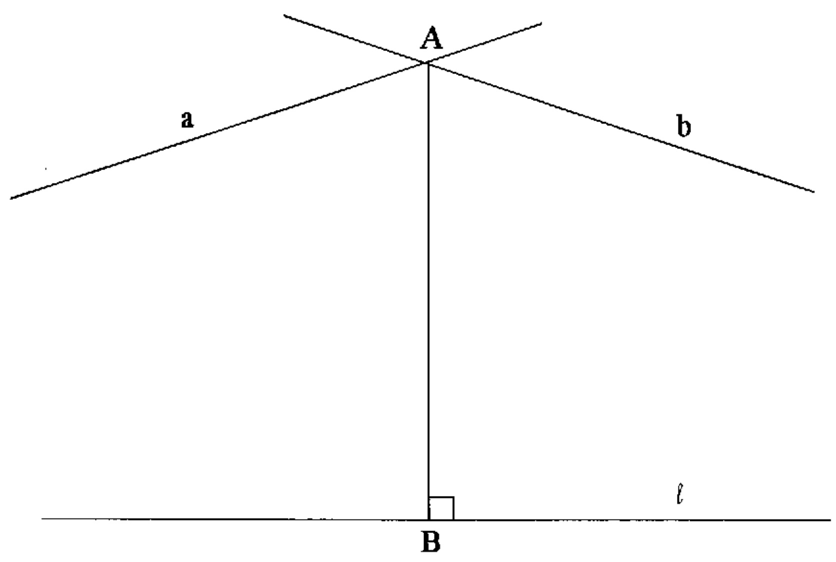


Figure 3.22

According to Saccheri, the lines a and ℓ have a common perpendicular at the point of infinity. This observation misleads him into believing that he has refuted the HAA and he thus concludes: "The HAA is absolutely false, because it is repugnant to the nature of the straight line" (Saccheri 1733:173). In reality however, Saccheri had not discovered a contradiction on the HAA. He had in fact discovered non-Euclidean geometry, but because he was intent on establishing the parallel postulate, he was unable to make an objective evaluation of his discovery. For this reason the work of Saccheri is one of the greatest ironies in the history of non-Euclidean geometry, and possibly in the history of mathematics.

3.8 LAMBERT'S DISCOVERY OF ABSOLUTE LENGTHS ON HIS HYPOTHESIS OF THE OBTUSE ANGLE

Saccheri's most distinguished immediate successor was the Swiss mathematician, Johann Heinrich Lambert (1728 - 1777). Lambert became acquainted with the investigation of Saccheri through the doctoral work of George S. Klügel (1739 - 1812). In 1766 he gave his own account of the parallel postulate in a book titled *Die Theorie der Parallellinien.*

Lambert conducts his investigation with a quadrilateral having three right angles. (See figure 3.23.)

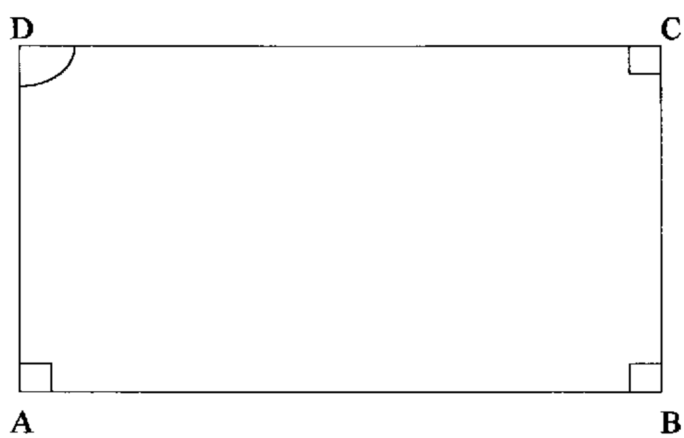


Figure 3.23

He formulates three hypotheses about the nature of the fourth angle :

(1) HRA: the fourth angle is a right angle

(2) HOA: the fourth angle is obtuse

(3) HAA: the fourth angle is acute

The HRA being equivalent to the parallel postulate, he sets to work on the HOA as follows (Bonola 1955:45) :

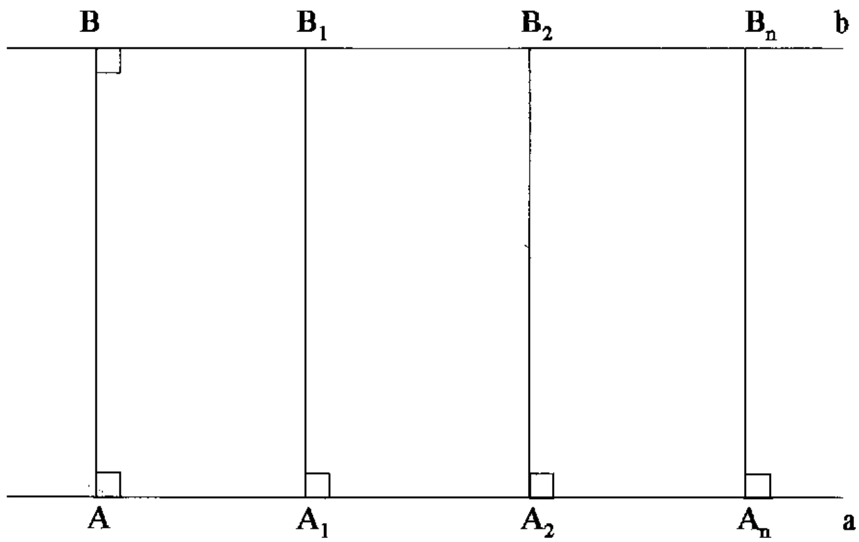


Figure 3.24

Suppose a and b are two lines perpendicular to a third line AB. (See figure 3.24.) From points B_1, B_2..... B_n taken in succession on the line b, draw B_1A_1, B_2A_2..... B_nA_n perpendicular to the line a. According to Lambert, these perpendiculars become progressively smaller as we move to the right, and the difference between any one of these and its successor becomes progressively larger. Thus, $BA - B_nA_n >$ $n\,(BA - B_1A_1)$. Since $n(BA - B_1A_1)$ can be made indefinitely large by choosing n appropriately, but $BA - B_nA_n$ is always smaller than BA, a contradiction arises. Therefore, the HOA is false.

On HAA, Lambert proves that the perpendiculars BA, B_1A_1 ...... B_nA_n, as well as the difference between each of these and its predecessor, become progressively larger.

Since no contradiction arose from this result, he was unable to refute the HAA. On the HAA, he discovered that the defect of a triangle i.e. the difference between the sum of its angles and two right angles, is proportional to its area. He also makes a very important observation regarding the lengths of line segments. Whereas the measurement of angles is absolute i.e. angle sizes are fixed irrespective of how they are constructed in Euclidean geometry, the length of a line segment varies according to the unit of length. On HAA, every line segment can be uniquely associated with some angle, so that we also have an absolute measure for lengths. For example, on any line segment we can construct an equilateral triangle. By Wallis' postulate, no triangle can be similar to it without being congruent to it as well, so that a side of this equilateral triangle can be identified with one of its angles.

On HOA, Lambert found that lengths are also absolute, and that the excess of the sum of the angles of a triangle above two right angles is proportional to its area. On a sphere of radius r, the area of a triangle with angles α, β, σ is $r^2(\alpha + \beta + \sigma - \pi)$, and so obtaining a formula like $r^2[\pi - (\alpha + \beta + \sigma)]$ - which equals $(ir)^2(\alpha + \beta + \sigma - \pi)$ - he remarks: "For this I should almost conclude that the third hypothesis would occur in the case of the imaginary sphere" (Gray 1979:67).

Lambert did not come to a definite conclusion regarding his investigations on HAA. Howard Eves (1981:70) suggests that the indecisiveness that came across in his work held him from publishing it. In 1786 it was published posthumously by Johann Bernoulli III.

3.9 LEGENDRE'S POPULARISATION OF THE PROBLEM OF THE PARALLEL POSTULATE

The French mathematician, Adrien - Marie Legendre (1752 - 1833), made a number of attempts in the classical tradition at proving the parallel postulate. These appeared over a number of years in the various editions of his *Eléments de géométrie* (1794 - 1823).

In one of his attempts he shows that the sum of the angles in a triangle cannot exceed two right angles, thereby refuting Saccheri's HOA. His proof is as follows (Bonola 1955:55) :

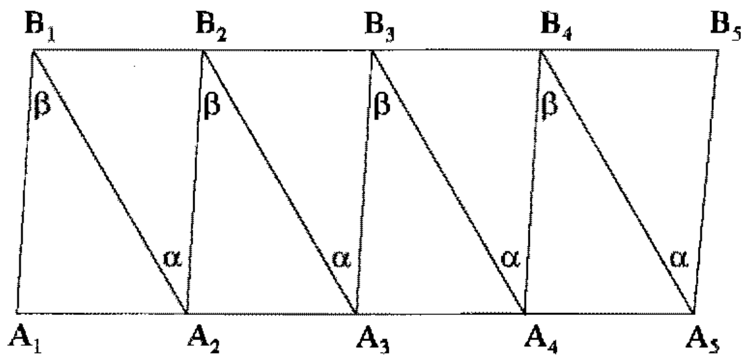


Figure 3.25

Suppose A_1A_2, A_2A_3, A_3A_4 .... A_{nn+1} are n equal segments on a line.(See figure 3.25.) Construct n congruent triangles $B_1A_1A_2$, $B_2A_2A_3$... $B_nA_nA_{n+1}$ on the same side of the line. Join B_1, B_2... B_n to obtain a second set of congruent triangles $B_1A_2B_2$, $B_2A_3B_3$ ... $B_{n-1}A_nB_n$. Complete the triangle $B_nA_{n+1}B_{n+1}$ so that it is congruent to the triangles in the latter group. Suppose $\angle A_1B_1A_2$ and $\angle B_1A_2B_2$ equal β and α respectively. Suppose $\beta > \alpha$. Then in triangles $A_1B_1A_2$ and $B_1A_2B_2$, $A_1A_2 > B_1B_2$ since A_1B_1 equals A_2B_2 and A_2B_1 is a common side. Since $A_1B_1B_2$... $B_{n+1}A_{n+1}$ is longer than the line segment A_1A_{n+1}, it follows that $A_1B_1 + nB_1B_2 + B_{n+1}A_{n+1} > nA_1A_2$. Therefore $2A_1B_1 > n(A_1A_2 - B_1B_2)$. But since n can be made as large as required, a contradiction arises. Thus, $\beta \leq \alpha$ and so the sum of the angles in triangle $A_1B_1A_2$ is less than or equal to two right angles.

In another attempt Legendre reasons from the following premise: Through a point within an angle, we can always draw a straight line which will intersect both arms of the angle.

His 'proof' goes as follows (Gray 1979:71):

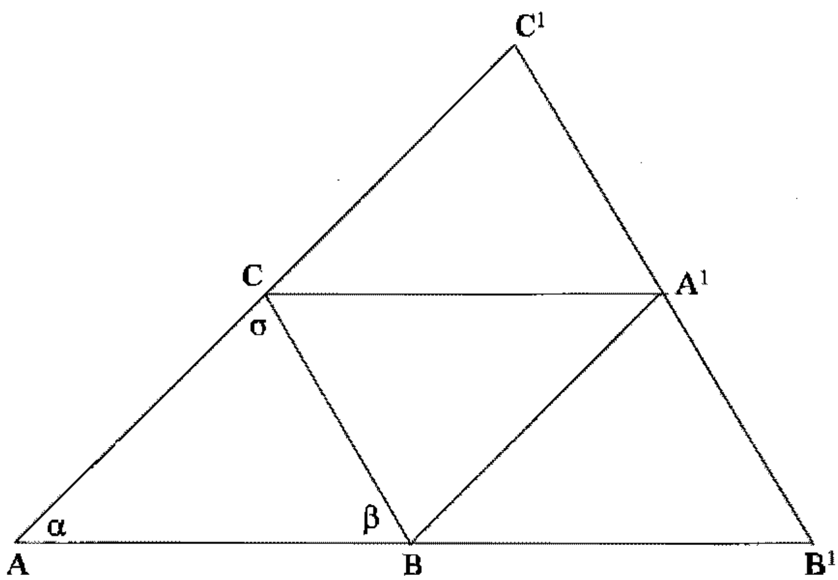


Figure 3.26

Suppose the angles at A, B and C in triangle ABC are equal to α, β and σ respectively, and $\alpha + \beta + \sigma <$ two right angles. (See figure 3.26.) Suppose the defect equals δ. Locate the point A^1 symmetrical to A with respect to BC. By Legendre's premise, a line can be drawn through A^1 meeting AB produced and AC produced at B^1 and C^1 respectively. Draw A^1B and A^1C. Since the defect of triangle AB^1C^1 is the sum of the defects of its component triangles, this defect has to be greater than or equal to 2δ. Continuing in this way, we eventually obtain a triangle having a defect greater than or equal to $2^n\delta$. For sufficiently large n, the defect would be greater than two right angles, which is absurd. So $\delta = o$ and $\alpha + \beta + \sigma$ equals two right angles, which establishes the parallel postulate.

The previous 'proof' of Legendre is flawed in that his premise is an equivalent statement to the parallel postulate. So Legendre did not succeed in extending the frontiers of research on the parallel postulate. However, the elegance and simplicity which characterised his presentations made the discussion widely accessible.

3.10 WOLFGANG BOLYAI'S DISILLUSIONMENT WITH HIS MANY FAILED ATTEMPTS

The Hungarian geometer, Wolfgang Bolyai (1775 - 1856), had a great interest in the parallel postulate from the time that he was a student at Göttingen (1796 - 1799). He substituted the parallel postulate with a number of other equally dubious hypotheses. The most acclaimed of these is the following: Through any three points not a line, it is possible to draw a circle. He deduces the parallel postulate from this hypothesis as follows (Greenberg 1974:135):

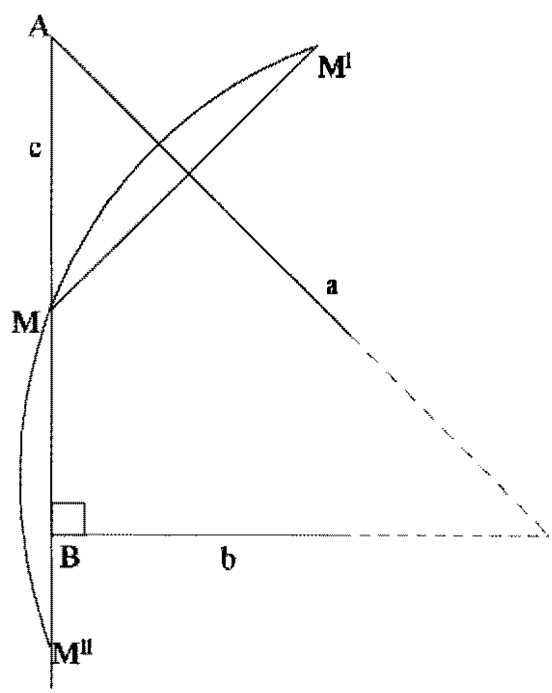


Figure 3.27

Suppose a and b are two lines cut by a third line c at A and B respectively. (See figure 3.27.) Suppose that the angle which a makes with c is acute, and that b is perpendicular to c. Suppose M is a point on AB between A and B. Locate the points M^1 and M^{11} symmetrical to M with respect to a and b respectively. Then M, M^1 and M^{11} are not collinear. By Bolyai's hypothesis, a circle can be drawn through these three points. Since a and b are the perpendicular bisectors of the chords MM^1 and $M^{11}M$, they must necessarily meet at the centre of the circle.

Wolfgang Bolyai became disheartened by his many failed attempts at settling the question of the parallel postulate. This is evident in the words of advice which he gives to his son János as quoted in Meschkowski (1964:31) :

> "You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone..... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labours; my creations are far better than others and yet I have not achieved complete satisfaction. For here it is true that *si paullum a summo discessit, vergit ad imum*. I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind.

János of course did not heed this advice. It is ironic, yet somehow fitting, that the object of the father's lamentations would later become the cause of his son's acclaim.

3.11 CONCLUSION

Despite the determination of mathematicians to prove the parallel postulate, all their attempts ended in failure. None of the substitute postulates which had been proposed was found to be entirely satisfactory. D' Alembert (1717 - 1783) thus referred to the parallel postulate as the "scandal of geometry". Of the many substitute postulates proposed, the one most frequently used in modern geometry textbooks is the version proposed by the Scottish mathematician, John Playfair (1748 - 1819) in 1795: Through a given point P not on a line ℓ, there exists a unique line parallel to ℓ.

The inability of the most distinguished mathematicians of the time to produce a valid proof of the parallel postulate suggested to Klügel and Abraham G. Kästner (1719 - 1800) amongst others, that perhaps it was unprovable so that it, or an equivalent version of it, had to be accepted without proof. Klügel made the observation that it is experience and sensory perception, rather than logical reasoning, which underlies our bias towards the parallel postulate (Kline 1972:868). In the following chapter, it will be shown that once these suggestions had gained acceptance, the discovery of non-Euclidean geometry became logically inevitable.

CHAPTER 4

THE DISCOVERY OF NON - EUCLIDEAN GEOMETRY

Out of nothing I have created a strange new universe.

János Bolyai

4.1 INTRODUCTION

Because the problem of the parallel postulate remained unresolved despite mathematicians' pre-occupation with it, the idea that it was unprovable from the other definitions, common notions and postulates started gaining momentum. Research therefore became focused on the nature of the geometry that would result if the parallel postulate were negated, but all the other postulates of Euclidean geometry were retained. Hence the discovery of non-Euclidean geometry became imminent.

According to Kline (1972:869), the discovery of non-Euclidean geometry is a vivid illustration of how mathematicians build on the work of their predecessors to obtain new insights. It is therefore not surprising that, when the intellectual climate is right, these insights may occur to several mathematicians working independently of one another.

In this chapter, details will be provided on the insights of those mathematicians who are normally credited with the discovery of non-Euclidean geometry, namely Gauss, Lobachevsky and Bolyai. In addition, some of the elementary theorems in hyperbolic geometry, the particular non-Euclidean geometry discovered by the afore-mentioned mathematicians, will be stated and proven in contrast with some well-known Euclidean ones.

4.2 GAUSS' CONTEMPLATION ON THE TRUE NATURE OF PHYSICAL SPACE

Carl Friedrich Gauss (1777 - 1855) is regarded by certain authors, such as Bonola (1955), as the first to have obtained clear insight into a geometry in which the parallel postulate is denied. Gauss' contributions have become known to us from his letters to fellow researchers, two short reviews in the *Göttingische gelehrten Anzeigen* of 1816 and 1822 and some notes of 1831 found amongst his papers after his death.

Gauss attended the University of Göttingen from 1795 to 1798. He was fully aware of the many flawed attempts at proving the parallel postulate because all these attempts had been studied carefully by his teacher, Kästner. Perhaps Gauss is guilty of exaggeration when he claims in a letter to Schuhmacher that as far back as 1792 he had already become convinced of the independence of the parallel postulate, because there is evidence that Gauss was still trying to prove the parallel postulate from other more plausible assumptions much later (Bühler 1981:100). However, in a letter to Wolfgang Bolyai in 1799, Gauss expresses his uncertainty about the provability of the parallel postulate (Bonola 1955:65):

> As for me, I have already made some progress in my work. However, the path I have chosen does not lead at all to the goal which we seek, and which you assure me you have reached. It seems rather to compel me to doubt the truth of geometry itself. It is true that I have come upon much which by most people would be held to constitute a proof; but in my eyes it holds as good as nothing. For example, if one could show that a rectilinear triangle is possible, whose area would be greater than any given area, then I would be ready to prove the whole of geometry absolutely rigorously. Most people would certainly let this stand as an axiom; but I, no! It would, indeed, be possible that the area might remain below a certain limit, however far apart the three angular points were taken.

He confirms this position in his reviews of 1816 and 1822, and in his letter to Olbers in 1817 in which he writes (Kline 1972:872):

> I am becoming more and more convinced that the necessity of our geometry cannot be proved, at least not by human reason, nor for human reason. Perhaps in another life we

> will be able to obtain insight into the nature of space, which is now unattainable. Until then we must place geometry not in the same class with arithmetic, which is purely *a priori*, but with mechanics.

From about 1813 onwards Gauss directed his efforts towards the development of the fundamental theorems of the new geometry which he first named anti-Euclidean geometry, then astral geometry, and finally non-Euclidean geometry. He outlines some of his new insights in a letter to Taurinus in 1824 (Greenberg 1974:145):

> The assumption that the sum of the three angles is less than two right angles leads to a curious geometry, quite different from ours, but thoroughly consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of a constant, which cannot be designated *a priori*. The greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen infinitely large, the two coincide. The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible.

Gauss was unable to find any contradictions in non-Euclidean geometry. However, there was no guarantee that subsequent investigations would not reveal a contradiction, and so the question of consistency remained open. According to Bühler (1981:100), Gauss was more concerned about the geometry of physical space than the philosophical implications of the independence of the parallel postulate. In an attempt to determine which of the two geometries, Euclidean or non-Euclidean, is the more accurate description of physical space, Gauss measured the sum of the angles of the triangle formed by the Brocken, Hohenhagen and Inselsberg mountains. According to Kline (1972:873), the sum slightly exceeded two right angles. However, because experimental error had to be taken into consideration, no conclusive evidence was obtained from this attempt. Because the defect of a triangle is proportional to its area in non-Euclidean geometry, only a large triangle, such as one formed by three celestial bodies, would show up a significant deviation from two right angles.

Despite his reputation as the greatest mathematician in Europe at the time, Gauss did not have the confidence to publish his findings in non-Euclidean geometry. He wrote to

Bessel in 1829 that he feared the "clamour of the Boeotians", that is, the uproar by those that were not capable of reaching such deep understanding. Another possible reason for Gauss' reluctance to publish could have been that his endeavours in other fields kept him from producing a refined version of his work, so that it did not meet the high standards which he had set for himself.

4.3 BOLYAI'S DISCOVERY OF A STRANGE NEW UNIVERSE

János Bolyai (1802 - 1860) was a Hungarian officer in the Austrian army. His father, Wolfgang, undoubtedly contributed to his pre-occupation with the parallel postulate. Until 1820 he was intent on finding a proof of the parallel postulate by using a similar approach to that which had been used by Saccheri and Lambert. These flawed attempts set his thinking in a new direction, so that by 1823 he was working in earnest on the fundamental ideas in non-Euclidean geometry. In a letter to his father he discloses his resolution to publish a tract on the theory of parallels as soon as he has the opportunity to organise his ideas. His father urged him to publish the proposed tract as soon as possible, and to assist his son toward that end, he offered to include it as an appendix to his book on elementary mathematics. The expansion and arrangement of ideas proceeded more slowly than János had anticipated. Eventually in 1832, *The Science of Absolute Space* appeared as a twenty-six page appendix to the *Tentamen*. Wolfgang sent a copy of his son's work to Gauss who replied as follows (Greenberg 1974:144):

> If I begin with the statement that I dare not praise such a work, you will of course be startled for a moment: but I cannot do otherwise; to praise it would amount to praising myself; for the entire content of the work, the path which your son has taken, the results to which he is led, coincide almost exactly with my own meditations which have occupied my mind for from thirty to thirty-five years. On this account I find myself surprised to the extreme. My intention was, in regard to my own work, of which very little up to the present has been published, not to allow it to become known during my lifetime. Most people have not the insight to understand our conclusions and I have encountered only a few who received with any particular interest what I communicated to them. In order to understand these things, one must first have a keen perception of what is needed, and upon this point the majority are quite confused. On the other hand,

> it was my plan to put all down on paper eventually, so that at least it would not finally perish with me. So I am greatly surprised to be spared this effort, and am overjoyed that it happens to be the son of my old friend who outstrips me in such a remarkable way.

Gauss' reply was a heavy blow to János. He could not come to terms with the fact that others had preceded him in the discovery of non-Euclidean geometry. He went as far as to believe that his father had secretly been passing on his discoveries to Gauss. He became so disillusioned that he never published his research again.

4.4 LOBACHEVSKY'S EXTENSIVE PUBLICATIONS ON NON - EUCLIDEAN GEOMETRY

The Russian mathematician, Nikolai Ivanovich Lobachevsky (1793 - 1856), was the first to actually publish a systematic development of non-Euclidean geometry. Lobachevsky studied mathematics at the University of Kazan under J.M.C Bartels who was closely connected to Gauss. Like Gauss and Bolyai, he began his investigations by attempting to prove the parallel postulate. The futility of these attempts caused him to believe that the problems which had to be resolved were due to reasons other than which they had been attributed to. His earliest paper on non-Euclidean geometry, *On the Principles of Geometry*, was published in the *Kazan Messenger* in 1829. This paper attracted only slight attention in Russia and, because of language and distance barriers, practically no attention elsewhere. He followed this up with numerous other publications: *Imaginary geometry* (1835), *New Principles of Geometry with a Complete Theory of Parallels* (1835 - 1838), *Applications of the Imaginary Geometry to Some Integrals* (1836), *Géométrie Imaginaire* (1837). In the *New Principles of Geometry* he writes (Bonola 1955:92):

> The fruitlessness of the attempts made since Euclid's time, for the space of 2000 years aroused in me the suspicion that the truth, which it was desired to prove, was not contained in the data themselves; that to establish it the aid of experiment would be needed, for example, of astronomical observations, as in the case of other laws of nature.

In 1840 he published a summary of his investigations, *Geometrische Untersuchungen zur Theorie der Parallellinien*, which has become the best known of his works. His last publication, *Pangéométrie*, appeared in 1855 after he had become blind.

Although the discoveries of Lobachevsky were not duly acknowledged during his lifetime, the non-Euclidean geometry which he developed is often referred to as Lobachevskian geometry, and he has been honoured with the title 'the Copernicus of geometry'.

4.5 THE DELAYED ACCEPTANCE OF NON - EUCLIDEAN GEOMETRY

The discoveries of Gauss, Bolyai and Lobachevsky were not widely acclaimed at first. This can perhaps be best explained by George Cantor's law of conservation of ignorance: "A false conclusion, once arrived at and widely accepted, is not easily dislodged, and the less it is understood, the more tenaciously it is held" (Kline 1980:87). The posthumous publication of Gauss' notes and correspondence on non-Euclidean geometry gave prominence to the work of Lobachevsky and Bolyai. Gray (1987:59) claims that it was Gauss' radical ideas on curvature which finally settled the debate about the nature and validity of non-Euclidean geometry. Certain geometers, such as C.L. Gerling (1788 - 1864), J. Hoüel (1823 - 1886), F. Schmidt (1827 - 1901) and E. Beltrami (1835 - 1900), became committed to making the new ideas accessible to the mathematical world. Others, such as B. Riemann (1826 - 1866) and A. Cayley (1821 - 1895), expanded the results and applied them to other branches of mathematics. By 1901 David Hilbert proclaimed non-Euclidean geometry as the most important discovery in geometry in the 19th century (Gray in Phillips 1987:37).

4.6 A DISCUSSION ON SOME OF THE ELEMENTARY THEOREMS IN HYPERBOLIC GEOMETRY

A few elementary theorems of the geometry of Gauss, Lobachevsky and Bolyai will now be examined. This geometry is nowadays referred to as 'hyperbolic geometry', a name suggested by Felix Klein (1849 - 1925) in 1871. Trudeau (1987:159) suggests that the name is derived from the Greek term *hyperbole*, meaning 'excess', since in this geometry the number of parallel lines through a given point to a given line exceeds the number in Euclidean geometry. A more plausible suggestion is that when a plane is placed at a tangent to a point on a surface to which this geometry is applicable, it cuts the rest of the surface in two hyperbolas (Reid 1963:156).

Hyperbolic geometry has as its common notions and postulates all those of Euclidean geometry, except that the parallel postulate, according to the version by Playfair, is replaced by its negation which shall be referred to as the hyperbolic postulate.

Hyperbolic Postulate: There exists a line ℓ and a point P not on ℓ such that there are at least two distinct lines parallel to ℓ passing through P. (See figure 4.1.)

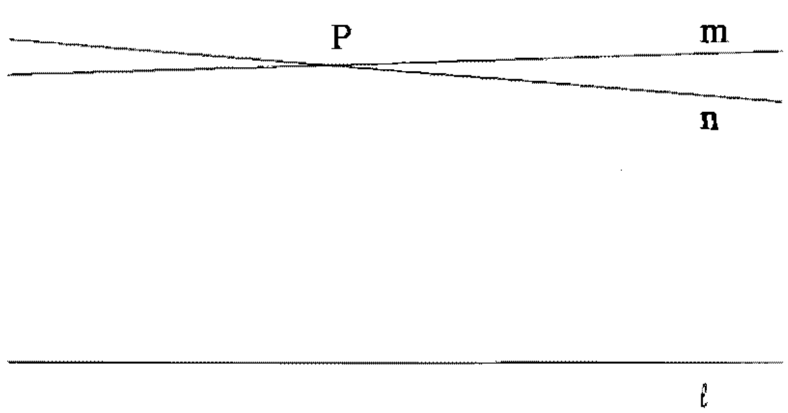


Figure 4.1

All the theorems of Euclidean geometry which do not require the parallel postulate for their proofs are therefore theorems of hyperbolic geometry. (These are the theorems of so-called 'neutral geometry'.) For a discussion on some of the new results in hyperbolic geometry, Greenberg (1974) and Trudeau (1987) have been referred to.

Lemma: In hyperbolic geometry, there exists a triangle with an angle sum of less than two right angles.

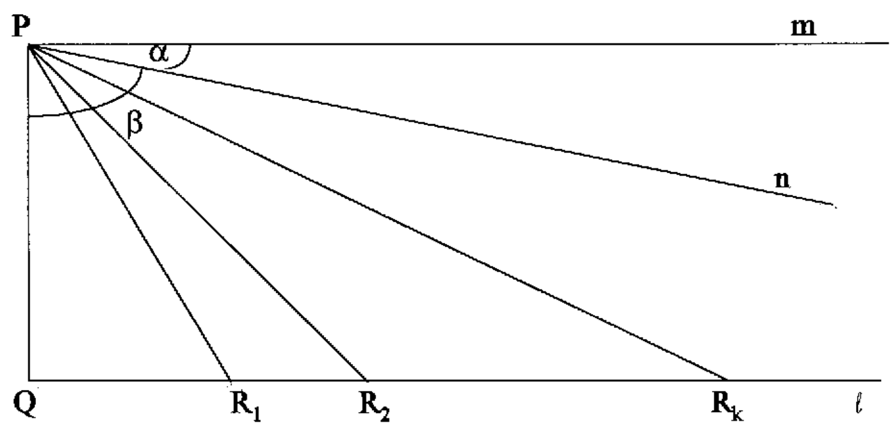


Figure 4.2

By the hyperbolic postulate, there exists a line ℓ and a point P not on ℓ, such that there are at least two distinct lines parallel to ℓ passing through P. (See figure 4.2) Draw PQ perpendicular to ℓ with Q on ℓ. Suppose line m passing through P is perpendicular to PQ. Then from neutral geometry, m is parallel to ℓ. Suppose n is another parallel to ℓ passing through P. Suppose that the angles between m and n, and n and PQ, are α and β respectively. Choose a point R_1 on ℓ as shown, such that PQ equals QR_1. Then by the theorem of Legendre and Saccheri, the sum of the angles of the right-angled isosceles triangle PR_1Q is less than or equal to two right angles. Hence $\angle PR_1Q \leq \frac{1}{2}$ right angle. Choose R_2 on ℓ on the same side of Q as R_1, such that PR_1 equals R_1R_2. By the exterior angle theorem, $\angle PR_2Q \leq \frac{1}{2}^2$ right angle. By choosing k sufficiently large, we obtain a point R_k on ℓ such that PR_{k-1} equals $R_{k-1}R_k$ and $\angle PR_kQ \leq \frac{1}{2}^k$ right angle $< \alpha$. Now PR_k lies between PQ and n, for otherwise n would intersect QR_k by the *Crossbar Theorem*, contradicting the assumption that n is parallel to ℓ. Thus, $\angle R_kPQ < \beta$, and hence $\angle PR_kQ + \angle R_kPQ < \alpha + \beta$ = a right angle, which yields the result.

Now Saccheri and Legendre had shown that if the sum of the angles in a triangle is more than, less than or equal to two right angles in a specific triangle, then it is the case for every other triangle, and so we have the following result:

Theorem: In hyperbolic geometry, the sum of the angles in a triangle is less than two right angles.

The following is an immediate consequence of the above theorem:
Corollary: In hyperbolic geometry, the exterior angle of a triangle is greater than the sum of its remote interior angles.

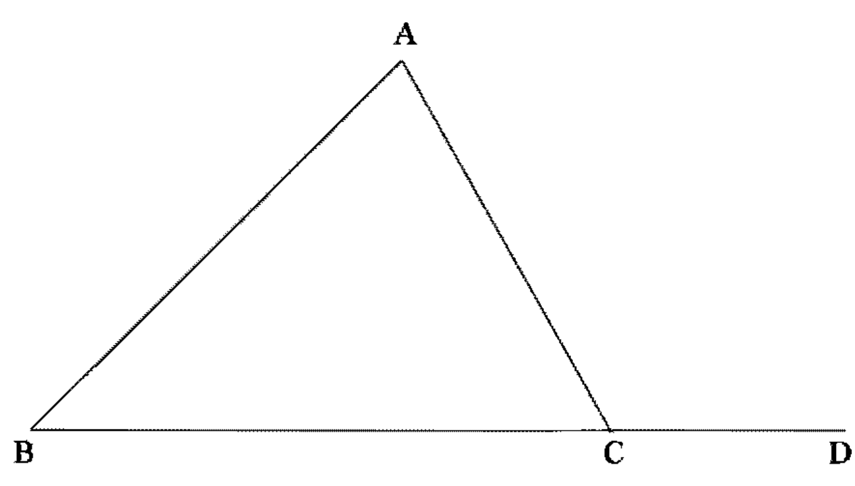


Figure 4.3

Suppose the side BC of triangle ABC is extended to D as shown in figure 4.3. Then by the previous theorem, the sum of $\angle BAC$, $\angle ABC$ and $\angle ACB$ is less than two right angles. But the sum of $\angle ACD$ and $\angle ACB$ equals two right angles, so that $\angle ACD$ is greater than the sum of $\angle BAC$ and $\angle ABC$.

Another consequence is the following:

Corollary: In hyperbolic geometry, the sum of the angles in a convex quadrilateral is less than four right angles.

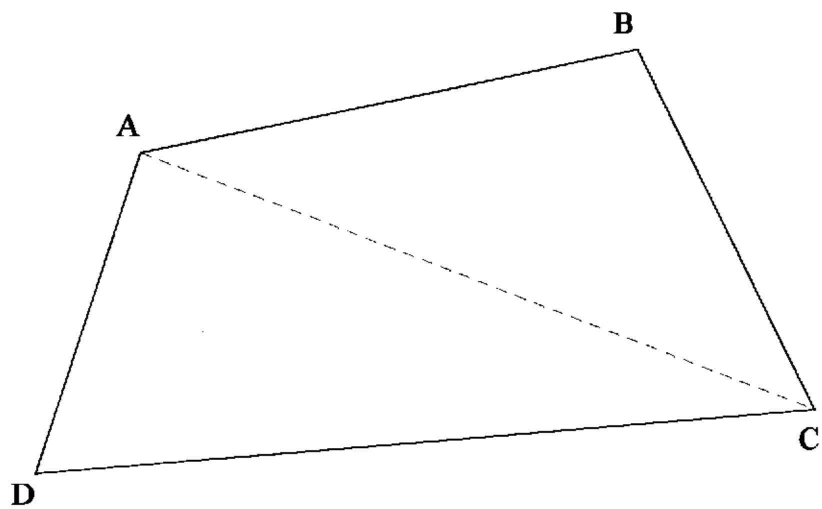


Figure 4.4

Suppose ABCD is a convex quadrilateral. (See figure 4.4.) Draw diagonal AC. Since ABCD is convex, AC lies between AB and AD, and between BC and CD. Hence the sum of the angles of triangles ABC and ACD equals the sum of the angles of quadrilateral ABCD. By the previous theorem, the sum of the angles in each of the triangles is less than two right angles, from which it follows that the sum of the angles in ABCD is less than four right angles.

Since rectangles are special types of convex quadrilaterals, it follows that the sum of their angles should be less than four right angles. But by definition, each of the angles of a rectangle equals a right angle, so that the sum of its angles equals four right angles. Thus we have the following important result in hyperbolic geometry.

Theorem: In hyperbolic geometry, rectangles do not exist.

We recall that Playfair's postulate asserts that for any given line ℓ and any point P not on ℓ, there is a unique line parallel to ℓ through P. The hyperbolic postulate, as the negation of Playfair's postulate, implies that for a certain line ℓ and a certain point P not on ℓ, the parallel to ℓ through P is not unique. The universal hyperbolic theorem ensures that this is the case for an arbitrary line ℓ and an arbitrary point P not on ℓ.

Universal Hyperbolic Theorem: In hyperbolic geometry, for every line ℓ and every point P not on ℓ, there are at least two distinct lines parallel to ℓ passing through P.

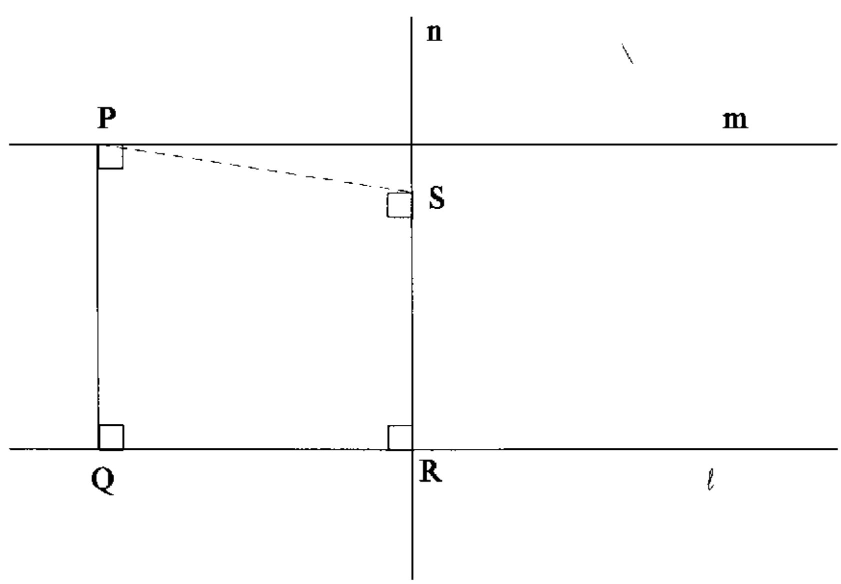


Figure 4.5

Suppose ℓ is any line and P any point not on ℓ. (See figure 4.5.) Draw PQ perpendicular to ℓ with Q on ℓ. Through P, draw m perpendicular to PQ. Suppose R is any other point on ℓ. Draw n perpendicular to ℓ through R. Draw PS perpendicular to n with S on n. By the alternate interior angle theorem, PS is parallel to ℓ. If S were on m, then PQRS would be a rectangle. Since rectangles do not exist in hyperbolic geometry, it follows that S is not on m, so that m and PS are distinct parallels to ℓ through P.

We recall that Wallis' postulate, which asserts the existence of similar, non-congruent triangles, is equivalent to the parallel postulate. Therefore, the negation of the parallel postulate implies the negation of Wallis' postulate, so that we have the following result in hyperbolic geometry:

Theorem: In hyperbolic geometry, if two triangles are similar, then they are congruent.

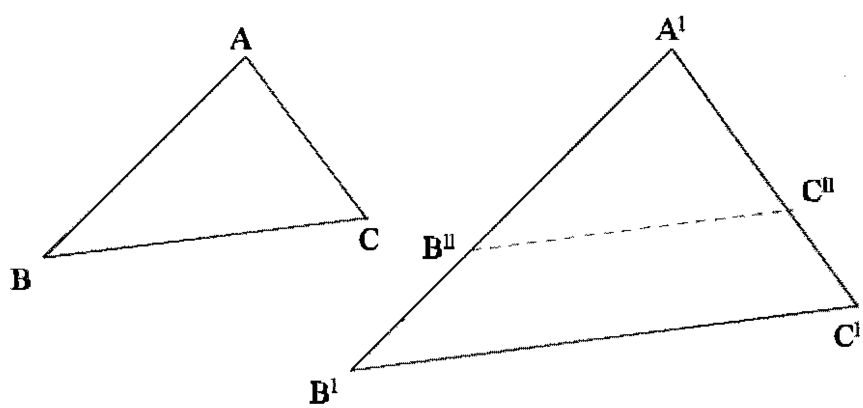


Figure 4.6

Suppose that triangles ABC and $A^1B^1C^1$ are similar but not congruent. (See figure 4.6.) Then no pair of corresponding sides is equal, for then these triangles would be congruent by the angle, side, angle criterion. Thus, at least two sides of one triangle must be greater than the two corresponding sides of the other triangle, say $A^1B^1 > AB$ and $A^1C^1 > AC$. Hence there exists points B^{11} on A^1B^1, and C^{11} on A^1C^1, such that A^1B^{11} equals AB, and A^1C^{11} equals AC. By the side, angle, side criterion, triangles ABC and $A^1B^{11}C^{11}$ are congruent, and hence the pairs of corresponding angles, $\angle ABC$

and $\angle A^1B^{11}C^{11}$, and $\angle ACB$ and $\angle A^1C^{11}B^{11}$, are equal. From similarity however, $\angle ABC$ equals $\angle A^1B^1C^1$, and $\angle ACB$ equals $\angle A^1C^1B^1$, so that $\angle A^1B^{11}C^{11}$ equals $\angle A^1B^1C^1$, and $\angle A^1C^{11}B^{11}$ equals $\angle A^1C^1B^1$. Now $\angle A^1B^{11}C^{11} + \angle B^1B^{11}C^{11} = \angle A^1C^{11}B^{11} + \angle C^1C^{11}B^{11}$ = two right angles. Since $\angle A^1B^{11}C^{11} = \angle A^1B^1C^1$ and $\angle A^1C^{11}B^{11} = \angle A^1C^1B^1$, the sum of the interior angles of quadilateral $B^{11}C^{11}C^1B^1$ equals four right angles, contradicting the result in hyperbolic geometry that the sum of the angles in a convex quadilateral is less than four right angles.

Theorem: In hyperbolic geometry, the theorem of Pythagoras is false.

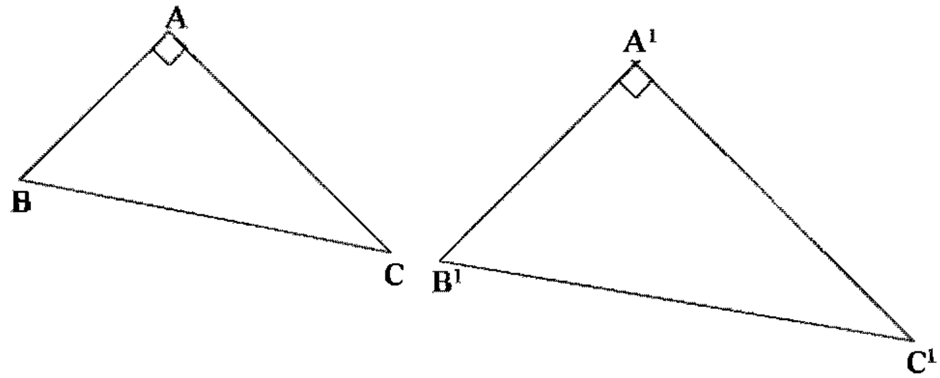


Figure 4.7

Suppose triangles ABC and $A^1B^1C^1$ are such that $\angle BAC$ and $\angle B^1A^1C^1$ are right angles, and A^1B^1 and A^1C^1 equal 2AB and 2AC respectively. (See figure 4.7.) Suppose the theorem of Pythagoras is true. Then $(B^1C^1)^2$ equals $4(AB^2 + AC^2)$, which in turn equals $4BC^2$. Hence B^1C^1 equals 2BC, and thus triangles ABC and $A^1B^1C^1$ are similar. Since similar, non-congruent triangles do not exist in hyperbolic geometry, the theorem of Pythagoras is false.

The fact that two lines are parallel i.e. they do not meet, does not necessarily imply that they are everywhere equidistant from each other. This oversight flawed Proclus' attempted proof of the parallel postulate. Whilst it is certainly true that parallel lines are everywhere equidistant on the assumption of the parallel postulate, they cease to be so on the assumption of the hyperbolic postulate. This is expressed in the following result:

Theorem: In hyperbolic geometry, if ℓ and m are two distinct parallel lines, then there are at most two points on ℓ equidistant from m.

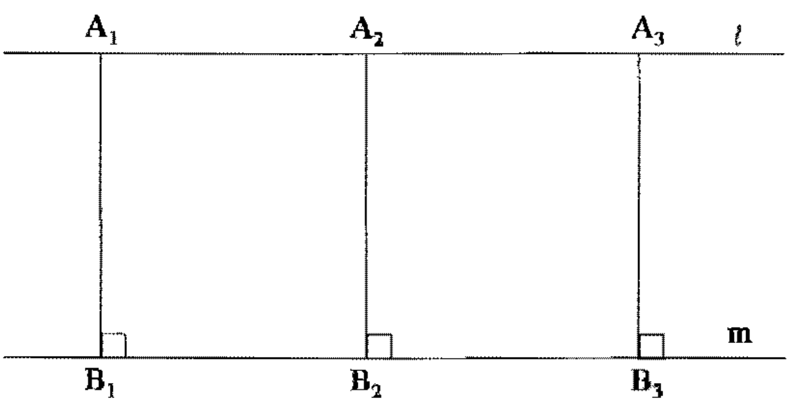


Figure 4.8

Suppose A_1, A_2 and A_3 are three points on ℓ which are equidistant from m. (See figure 4.8.) Then by definition, the perpendiculars to m, A_1B_1, A_2B_2 and A_3B_3 are equal. The quadrilaterals $A_1B_1B_2A_2$, $A_1B_1B_3A_3$ and $A_2B_2B_3A_3$ are Saccheri quadrilaterals, the fundamental figures used in the investigations by Saccheri. Since the two angles in a Saccheri quadrilateral which are not right angles, are equal, it follows that $\angle B_1A_1A_2 = \angle B_2A_2A_1$, $\angle B_1A_1A_3 = \angle B_3A_3A_1$, and $\angle B_2A_2A_3 = \angle B_3A_3A_2$. By transivity, $\angle B_2A_2A_1 = \angle B_2A_2A_3$, and hence each equals a right angle. Hence these Saccheri quadrilaterals are rectangles, but since rectangles do not exist in hyperbolic geometry, the points A_1, A_2 and A_3 cannot be equidistant from m.

The above theorem shows that points on ℓ which are equidistant from m either occur in pairs as in figure 4.9:

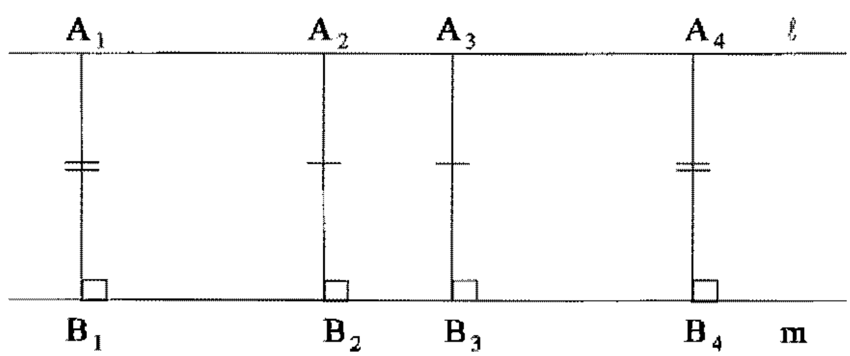


Figure 4.9

or do not occur at all as in figure 4.10:

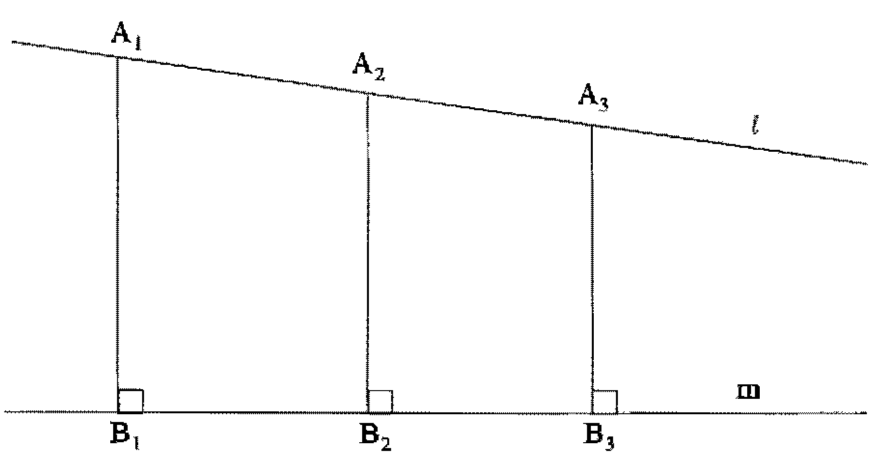


Figure 4.10

Theorem: In hyperbolic geometry, if ℓ and m are parallel lines, and there is a pair of points on ℓ equidistant from m, then m and ℓ have a unique common perpendicular which is shorter than any other line segment between ℓ and m.

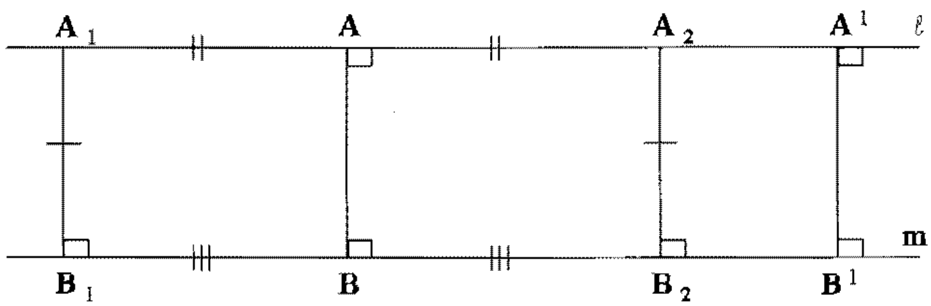


Figure 4.11

Suppose A_1 and A_2 are two points on ℓ which are equidistant from m. (See figure 4.11.) Then by definition, the perpendiculars to m, A_1B_1 and A_2B_2, are equal. Thus $A_1B_1B_2A_2$ is a Saccheri quadrilateral. Suppose A and B are the midpoints of A_1A_2 and B_1B_2 respectively. Since the line segment joining the midpoints of the summit and base (in this case A_1A_2 and B_1B_2 respectively) of a Saccheri quadrilateral is a common perpendicular to these two sides, it follows that AB is a common perpendicular to ℓ and m. Suppose A^1B^1 is another common perpendicular to ℓ and m. Then ABB^1A is a rectangle, which does not exist in hyperbolic geometry. Hence AB is unique.

Suppose A_k and B_k are any two points on ℓ and m respectively, such that A_kB_k is distinct from AB. Then either (1) one of the pairs A_k and A, and B_k and B coincides, or (2) none of the pairs coincide and A_kB_k is perpendicular to m, or (3) none of the pairs coincide and A_kB_k is not perpendicular to m. In case (1), suppose A_k and A coincide. Then $\angle A_kB_kB$ is acute since the sum of the angles of triangle A_kB_kB is less than two right angles. Hence $A_kB < A_kB_k$. In case (2), consider quadrilateral A_kB_kBA. (See figure 4.12.)

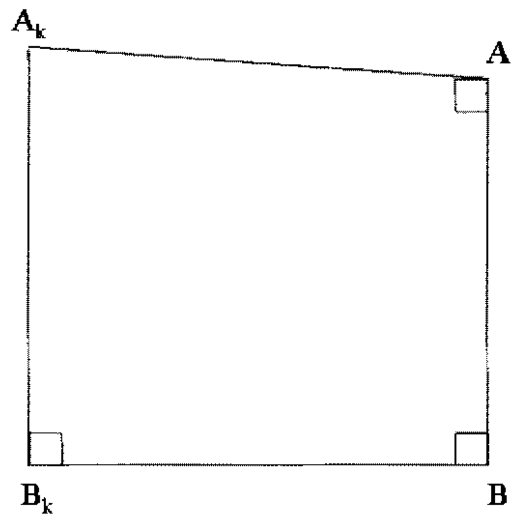


Figure 4.12

Since the sum of the angles in a convex quadrilateral is less than four right angles in hyperbolic geometry, $\angle B_kA_kA$ is acute, and hence $AB < A_kB_k$ from a result by Lambert. In case (3), draw $A_kB_k^1$ perpendicular to m with B_k^1 on m. (See figure 4.13.) As before, $\angle A_kB_kB_k^1$ is acute and hence $A_kB_k^1 < A_kB_k$. From case (2), $AB < A_kB_k^1$ and by transitivity, $AB < A_kB_k$.

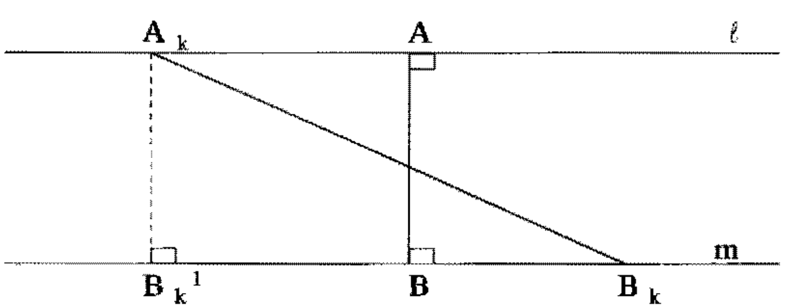


Figure 4.13

We recall that when the equidistance property for parallel lines fails in hyperbolic geometry, there is the possibility of having two parallel lines such that there is no pair of points on the one which is equidistant from the other. An intuitive argument for the existence of such parallel lines goes as follows: Suppose ℓ is a line and P a point not on ℓ. (See figure 4.14.) Draw PQ perpendicular to ℓ with Q on ℓ. Through P, draw m perpendicular to PQ. By the alternate interior angle theorem, m is parallel to ℓ, and by the universal hyperbolic theorem, there exists another line n through P parallel to ℓ.

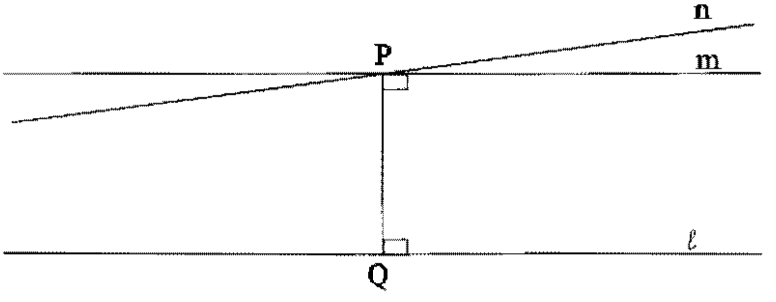


Figure 4.14

Consider the pencil of lines through P. Of these, some, like n, do not meet ℓ, and some, like PQ, meet ℓ. By continuity, there exists a line a, asymptotic to ℓ on the left, and a line b, asymptotic to ℓ on the right, such that a and b separate those lines which meet ℓ from those which do not meet ℓ. (See figure 4.15.) The lines which meet ℓ are those between a and PQ on the left of PQ, and those between b and PQ on the right of PQ.

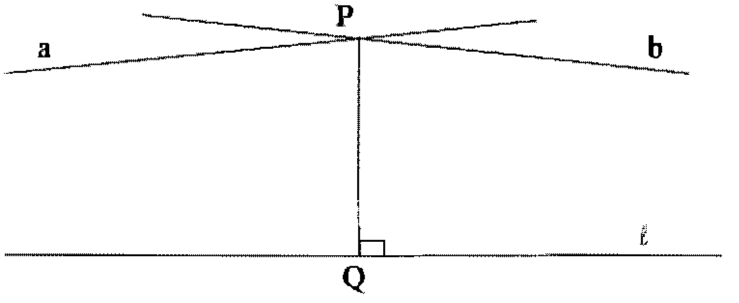


Figure 4.15

This is exactly what Saccheri had discovered in trying to refute the HAA. It is not surprising that we obtain the same results in hyperbolic geometry as Saccheri had obtained on the HAA:

Theorem: The hyperbolic postulate is logically equivalent to the HAA.

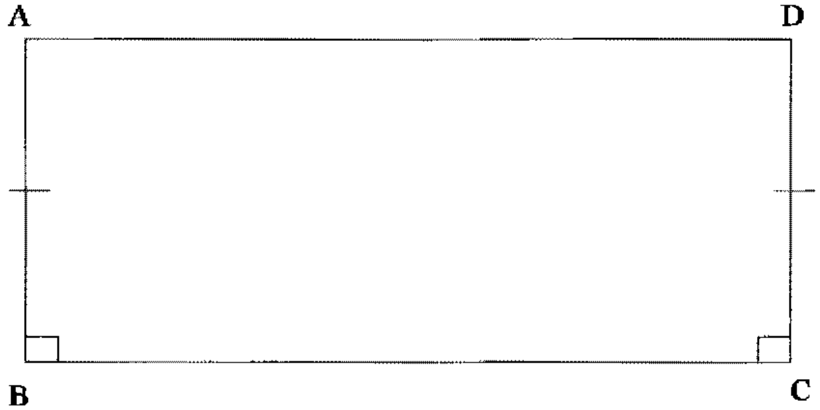


Figure 4.16

Suppose that the hyperbolic postulate holds, and that ABCD is a Saccheri quadrilateral. (See figure 4.16.) Since the sum of the angles in a convex quadrilateral is less than four right angles, it follows that the sum of the angles in ABCD is less than four right angles. Since ∠BAD and ∠CDA are equal, they are each acute. Thus the HAA holds.

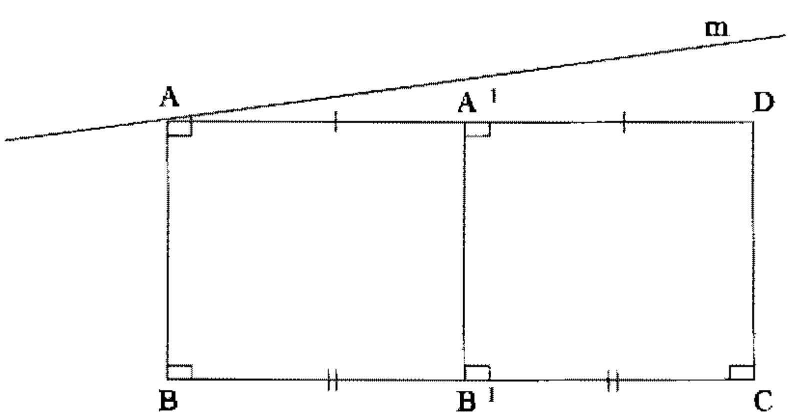


Figure 4.17

Conversely, suppose that the HAA holds. (See figure 4.17.) The line segment joining the midpoints of the summit AD and the base BC is a common perpendicular to these two sides. By the alternate interior angle theorem, the lines AD and BC are parallel. Through A, draw m perpendicular to AB. As before, m is parallel to line BC. Since $\angle$BAD is acute, the lines m and AD are distinct. Hence the hyperbolic postulate holds.

Theorem: The asymptotic parallels to a line through a point make equal acute angles with the perpendicular from the point to the line.

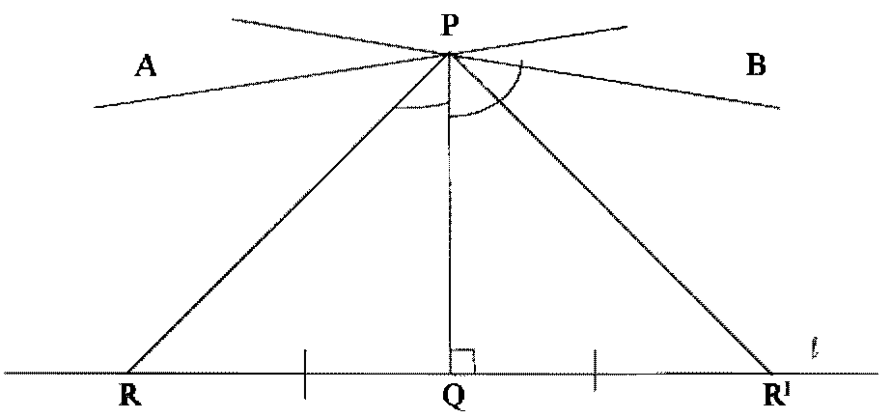


Figure 4.18

Suppose AP and PB are parallels to ℓ through P, asymptotic to ℓ on the left and right respectively. (See figure 4.18.) Draw PQ perpendicular to ℓ with Q on ℓ. Suppose $\angle$APQ and $\angle$BPQ are not equal, say $\angle APQ > \angle BPQ$. Since all the lines through P between AP and PQ on the left of PQ will meet ℓ, there is one such line meeting ℓ at a point R such that $\angle$RPQ equals $\angle$BPQ. Suppose R^1 is the point on ℓ symmetrical to R with respect to PQ. By the side, angle, side criterion, triangles RPQ and R^1PQ are congruent. Hence $\angle$RPQ equals $\angle R^1PQ$, and by transitivity, $\angle R^1PQ$ equals $\angle$BPQ. But PB and PR^1 are distinct and hence there is a contradiction. Now suppose $\angle$APQ is a right angle. (See figure 4.19.) Then APB is a straight line and therefore all the other lines through P will meet ℓ, contradicting the hyperbolic postulate.

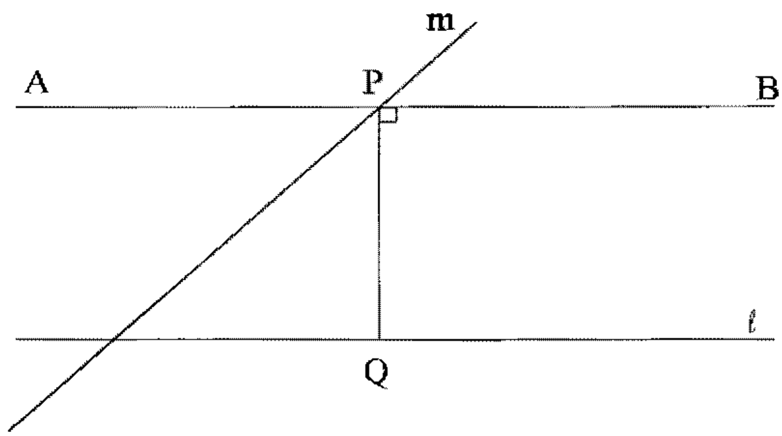


Figure 4.19

Suppose $\angle APQ$ is obtuse. (See figure 4.20.) Then the perpendicular to PQ through P on the left of PQ is between AP and PQ on the left. By the alternate interior angle theorem, this perpendicular will not meet ℓ, contradicting the fact that AP is the left asymptotic parallel to ℓ.

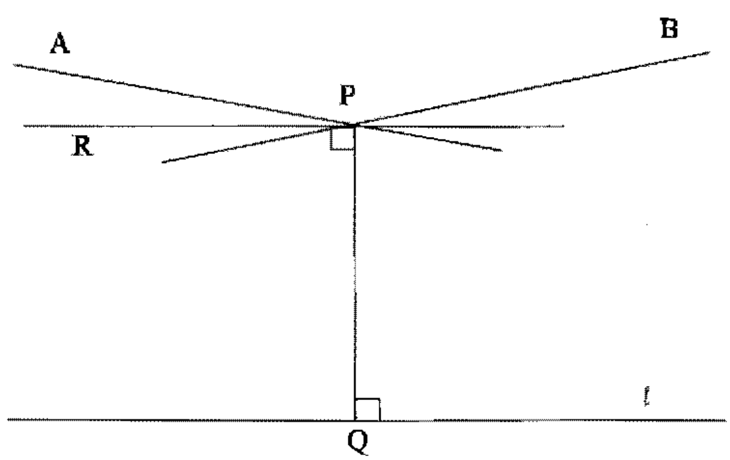
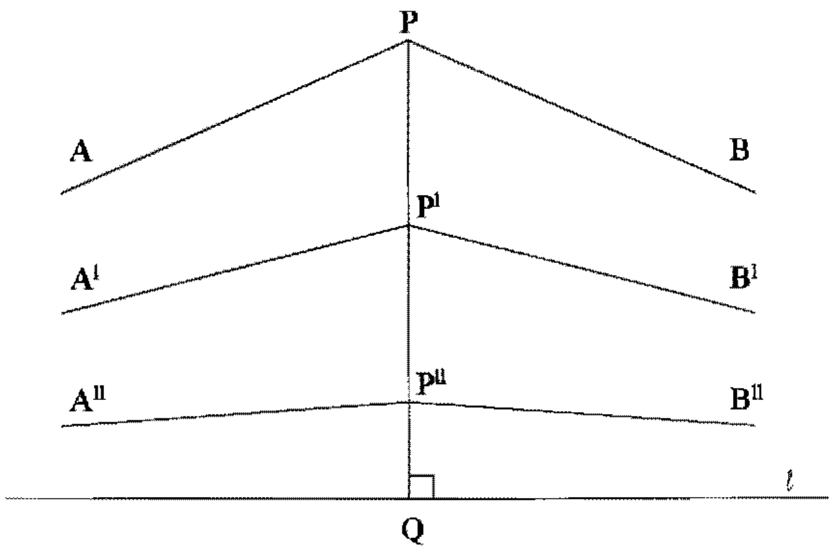


Figure 4.20

Either of the angles $\angle APQ$ or $\angle BPQ$ is called the angle of parallelism at P with respect to ℓ. The size of the angle of parallelism depends on the length of PQ and is denoted by $\pi(PQ)$. As PQ becomes shorter, $\pi(PQ)$ approaches a right angle. (See figure 4.21.)



$\angle APQ < \angle A^1P^1Q < \angle A^{11}P^{11}Q <$ right angle

Figure 4.21

In hyperbolic goemetry, two types of parallels to a line ℓ have thus been identified, namely those that are asymptotic to ℓ, and those that have a common perpendicular with ℓ. A proof by Hilbert verified that these are the only types of parallels.

4.7 CONCLUSION

There are numerous more complex theorems in hyperbolic geometry than those that have been presented in this chapter. However, since this is a very elementary treatment of the subject presented with the purpose of demonstrating its possible application in the secondary school context, these will not be discussed. At face-value, the theorems of hyperbolic geometry seem contrary to what we believe and have experienced to be true. However, when we consider the reasoning by which they were obtained, our feelings of uneasiness subside, and we are undoubtedly enriched by the new perspective which they provide us with.

Despite Morris Kline's claim (1972:878) that the great credit given to Bolyai and Lobachevsky is unjustified because much of the groundwork for their discoveries had already been done, they certainly showed great intellectual and moral courage in publicly advancing ideas that were contrary to the spirit of the times. The philosophy of Kant was no doubt one of the factors which contributed to the delayed acceptance of non-Euclidean geometry. The opponents of non-Euclidean geometry still cherished the hope that it would be shown to be inconsistent. This will be the issue under discussion in the next chapter.

CHAPTER 5

THE CONSISTENCY OF NON-EUCLIDEAN GEOMETRY

It is impossible to establish the logical consistency of any complex deductive system except by assuming principles of reasoning whose own internal consistency is as open to question as that of the system itself.

Kurt Gödel

5.1 INTRODUCTION

After the discoveries of Bolyai, Lobachevsky and Gauss were presented to the mathematical world, the next question to be answered was naturally the question pertaining to the consistency of hyperbolic geometry. If hyperbolic geometry were shown to be inconsistent, then it would be rendered useless and uninteresting as a mathematical system.

In this chapter, the consistency of hyperbolic geometry relative to that of Euclidean geometry will be demonstrated by means of the Poincaré model. Thus, since the consistency of Euclidean geometry was not in doubt, the consistency of hyperbolic geometry was affirmed. Hyperbolic geometry was thereby established as a valid mathematical system, worthy of investigation by all those who are capable of and interested in doing so.

5.2 THE INDIRECT METHOD OF PROVING THE CONSISTENCY OF HYPERBOLIC GEOMETRY

To prove that hyperbolic geometry is inconsistent, two theorems which logically contradict each other would have to be discovered. Saccheri and Lambert had tried and failed in this regard. We recall the words of Gauss: "All my efforts to discover a contradiction, an inconsistency, in this non-Euclidean geometry have been without success...." (Greenberg 1974:181).

The efforts of Gauss and others point towards the consistency, rather than the inconsistency of hyperbolic geometry. But how, if at all, can we establish beyond any doubt that hyperbolic geometry, or any formal axiomatic system for that matter, is consistent ? One possibility is to find an interpretation - a manner of assigning meaning to the undefined terms of the system - under which all the postulates turn out to be true statements. Such an interpretation is called a model of the system. According to Barker (1964:46), a drawback of this approach is that it requires absolute certainty about the truth of the interpreted statements. Another method is to demonstrate the consistency of a system relative to another system about which we are more confident. This is done by finding an interpretation within the latter system in which the postulates of the former system turn out to be true.

In the case of hyperbolic geometry, only relative proofs are known. Since no serious doubt had ever been expressed about the consistency of Euclidean geometry, it seemed reasonable to let the consistency of hyperbolic geometry hinge on the consistency of Euclidean geometry. The consistency of Euclidean geometry would consequently establish the independence of the parallel postulate, for if the parallel postulate were provable from the other postulates of Euclidean geometry, then both the parallel postulate and its negation, the hyperbolic postulate, would he valid in hyperbolic geometry. Thus, hyperbolic geometry would be inconsistent, but since it is consistent relative to Euclidean geometry, no proof of the parallel postulate exists. In retrospect, the futility of the attempts to prove the parallel postulate becomes clear. According to Greenberg (1974:183), it is ironic that a proof of the parallel postulate, which was sought in order to place Euclidean geometry on a more solid foundation, would have rendered it inconsistent.

5.3 THE POINCARÉ MODEL FOR HYPERBOLIC GEOMETRY

Bolyai and Lobachevsky were the first to offer plausible explanations for the consistency of hyperbolic geometry. During the last third of the 19th century a number of models for hyperbolic geometry were constructed. The model that will be discussed is one that was proposed by the French mathematician, physicist and philosopher, Henri Poincaré (1854 - 1912). The following is a summary of the presentations of this model by Moise (1989), Greenberg (1974) and Trudeau (1987).

Suppose δ is a fixed circle in the Euclidean plane. The hyperbolic plane is represented by all those points in the interior of δ i.e. those points whose distance from the centre of δ is less than the radius of δ. A circle σ is said to be orthogonal to δ if the tangents of the two circles are perpendicular to each other at each of their points of intersection. Lines in the Poincaré model are represented either by the intersection of diameters of δ with the interior of δ, or by the intersection of circles orthogonal to δ with the interior of δ. (See figure 5.1.)

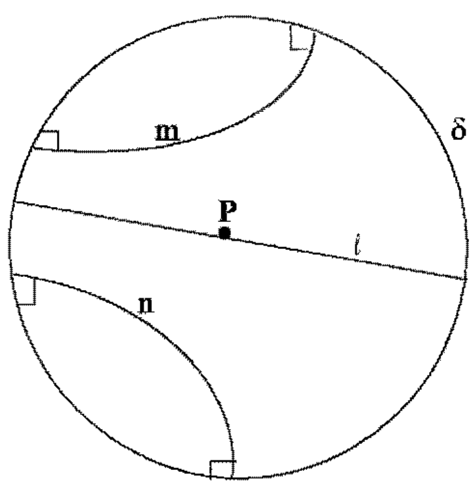


Figure 5.1

In this model, the concepts of incidence and betweenness are defined exactly as in Euclidean geometry. Thus, a point lies on a line if it lies on it in the Euclidean sense. In the case of three points A, B and C on an arc of a circle orthogonal to δ, B lies between A and C if, of the rays emanating from the centre of this circle through each of these points, the one through B lies between the other two. (See figure 5.2.)

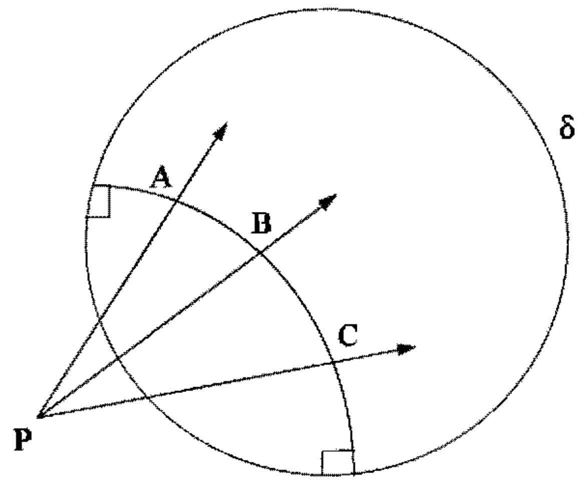


Figure 5.2

We define lengths of segments in the Poincaré model as follows: For any pair of points X, Y in the interior of δ or on δ, let XY denote the usual length of the segment joining X and Y. In figure 5.3, the points W and Z do not lie on the hyperbolic plane, but they are points of the original Euclidean plane, and hence the lengths XW, XZ, YW and YZ are defined. The Poincaré length $\partial(XY)$ is defined by

$$\partial(XY) = \left| \log_e \frac{XW / XZ}{YW / YZ} \right|$$

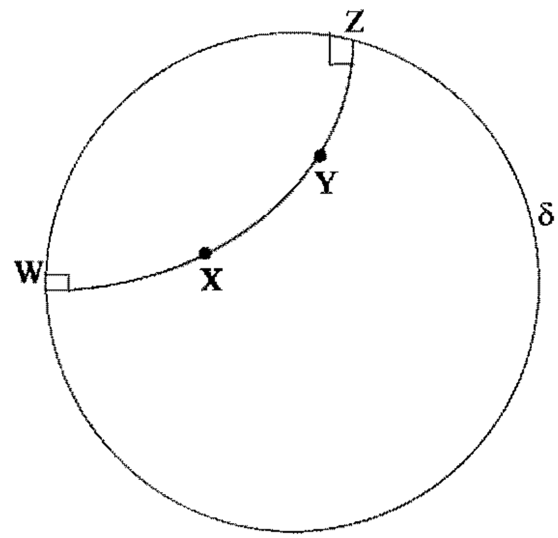


Figure 5.3

From the previous definition, we note that the order of X and Y is irrelevant:

$$\begin{aligned}\partial(YX) &= \left|\log_e \frac{YW / YZ}{XW / XZ}\right| \\ &= \left|\log_e \left(\frac{XW / XZ}{YW / YZ}\right)^{-1}\right| \\ &= \left|-\log_e \frac{XW / XZ}{YW / YZ}\right| \\ &= \left|\log_e \frac{XW / XZ}{YW / YZ}\right| \\ &= \partial(XY)\end{aligned}$$

Thus, in the Poincaré model, the line segments joining points A and B, and C and D are equal in length in the Poincaré sense if $\partial(AB) = \partial(CD)$.

The size of any angle in the case of two arcs is defined to be the size of the angle formed by the two tangent rays at the point of intersection of the two arcs. (See figure 5.4.)

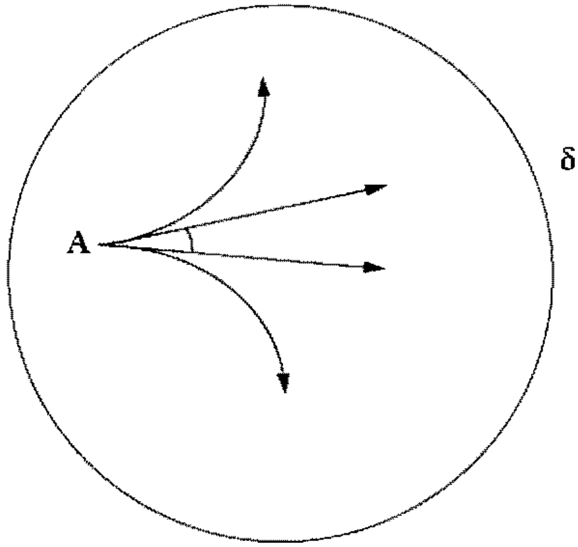


Figure 5.4

With the above assignment of meanings to the undefined terms, all the postulates of neutral geometry hold in the Poincaré model. However, as illustrated in figure 5.5, the parallel postulate does not hold. Through the point P not on ℓ, there are infinitely many lines parallel to ℓ, with a and b the left and right asymptotic parallels to ℓ respectively.

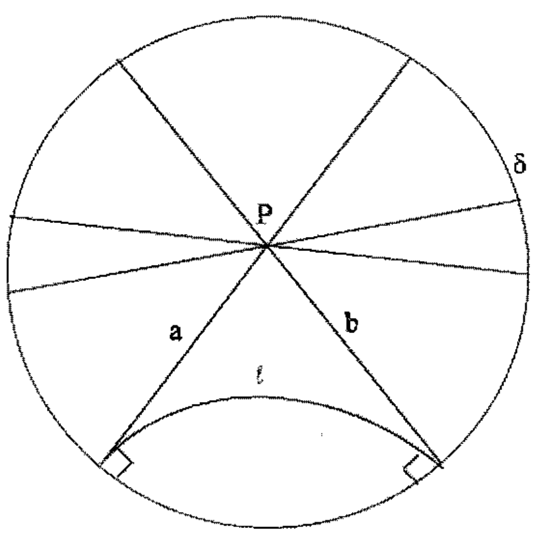


Figure 5.5

Figure 5.6 illustrates that a line m parallel to l but not asymptotic to it, has a common perpendicular with it which is the shortest distance between the two lines.

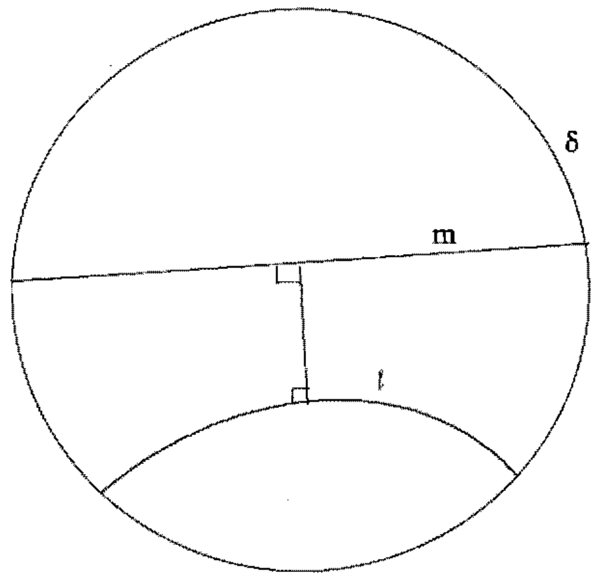


Figure 5.6

The summit angles of a Saccheri quadrilateral are acute and the summit is longer than the base in the Poincaré model. (See figure 5.7.)

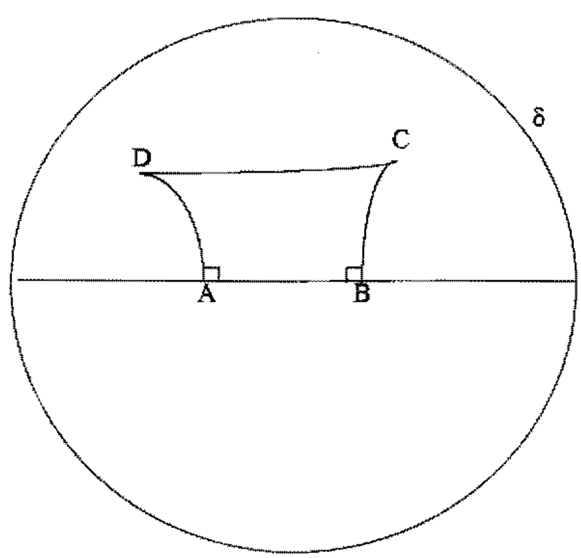


Figure 5.7

Figure 5.8 illustrates that the sum of the angles in a triangle is less than two right angles.

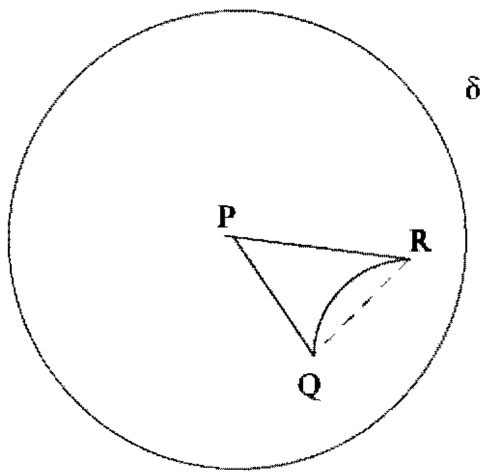


Figure 5.8

5.4 CONCLUSION

The Poincaré model shows that if Euclidean geometry is consistent, then so is hyperbolic geometry. But how, if at all, can we know for sure that Euclidean geometry is consistent ? We can try to mimic the method used to establish the consistency of hyperbolic geometry. However, "to try to establish the consistency of Euclidean geometry relative to some other geometry would not be very helpful, for any other geometry is at least as suspect as regards its consistency as is Euclidean geometry" (Barker 1964:47). Thus, a mathematical system whose consistency was not in doubt, had to be found for this purpose. Analytical geometry was used to construct a model for Euclidean geometry within the real number system. Although the real number system had been thoroughly investigated by mathematicians, no proof of its consistency has been found. Thus, in answer to our question, we state that there is at present no way that we can know for sure that Euclidean geometry is consistent.

The establishment of the relative consistency of hyperbolic geometry placed it on par with Euclidean geometry as a mathematical system. This fact had serious philosophical implications, some of which will be examined in the next chapter.

CHAPTER 6

THE PHILOSOPHICAL IMPLICATIONS OF NON-EUCLIDEAN GEOMETRY

So far as the theories of mathematics are about reality, they are not certain; so far as they are certain, they are not about reality.

Albert Einstein

6.1 INTRODUCTION

In the previous chapter, we have seen that if Euclidean geometry is consistent, then so is hyperbolic geometry. Conversely, it can be shown that if hyperbolic geometry is consistent, then so is Euclidean geometry. Thus, from the standpoint of pure logic, hyperbolic geometry was elevated to the same position as Euclidean geometry. This, together with the fact that other equally consistent geometries were being developed, caused mathematicians to ponder the following, somewhat disturbing questions: Could it be that Euclidean geometry is not necessarily the most accurate description of physical space? What is the basis of our belief that space is Euclidean? Is it the design of our mental constitution or a habit of thought that we have acquired? What is the true geometry of space and how can we ascertain this? Is there a true geometry of space?

In this chapter, an attempt will be made to answer these questions in the light of the new knowledge which became at the disposal of mathematicians, scientists and philosophers. In the process, it will become evident that the discovery of non-Euclidean geometry has had a profound effect on the way we think about our world and our place in it.

6.2 THE TRUE GEOMETRY OF PHYSICAL SPACE

Superficially, the idea that other geometries can rival Euclidean geometry may seem absurd. Architects and engineers have relied on Euclidean geometry since its inception for the construction of bridges, skyscrapers, dams etc. and there is no reason for them to believe that they cannot continue doing so in the future. However, we have to bear in mind that these applications of Euclidean geometry are confined to a relatively small portion of the earth and when larger distances, such as those between celestial bodies, are involved, we may have to look towards an alternative geometry. Experiment has shown that the theory of relativity which is based on a non-Euclidean geometry, is a more accurate description of natural phenomena than the Newtonian theory which is based on Euclidean geometry, when astronomical distances are involved. Trudeau (1987:243) provides food for thought when he contemplates the possibility that space may after all be hyperbolic, but because of the great discrepancy between human and cosmic dimensions, we are unable to detect this.

We recall that Gauss attempted to determine the geometry of physical space by measuring the angles of the triangle formed by three mountain peaks. We recall also that this experiment was inconclusive. In fact, because of experimental error, any empirical method, even one involving stellar triangles, will never be able to provide conclusive evidence that space is Euclidean, only that it is non-Euclidean. Roxburgh, in his article *Is Space Curved?* (1978), makes interesting comments with regard to this experiment. He claims that this experiment is an indication that Gauss and others firmly believed that a particular geometry is an intrinsic property of space that can be determined empirically. The result of the experiment is a statement about the behaviour of light rays relative to other objects in space, and not about space itself. He concludes that a geometry is a mathematical representation of space, and since representations may be changed to suit the circumstances, there is no true or correct geometry of space. This notion about space is clear from Poincaré's response to the question of which geometry is true (Greenberg 1974:250):

> If geometry were an experimental science, it would not be an exact science. It would be subjected to continual revision..... The geometrical axioms are therefore neither synthetic *a priori* intuitions nor experimental facts. They are conventions. Our choice among all possible conventions is guided by experimental facts; but it remains free, and is only limited by the necessity of avoiding every contradiction, and thus it is that postulates may remain rigorously true even when the experimental laws which have determined their adoption are only approximate. In other words, the axioms of geometry are only definitions in disguise. What then are we to think of the question: Is Euclidean geometry true ? It has no meaning. We might as well ask if the metric system is true and if the old system of weights and measures is false; if Cartesian co-ordinates are true and polar co-ordinates false. One geometry cannot be more true than another: it can only be more convenient.

6.3 THE TRUE NATURE OF MATHEMATICS

Another important consequence of the discovery of non-Euclidean geometry was the realisation that mathematics does not offer truths. We recall that, to the ancient Greeks, the postulates of Euclidean geometry were self-evident truths that did not need to be verified. However, the fact that contradictory geometries could be used to describe physical space indicated that mathematics as absolute truth is an illusion. Mathematicians were thus able to pursue their work with much greater freedom. In the words of George Cantor: "The essence of mathematics lies in its freedom" (Eves 1981:81). According to Greenberg (1974:252), the wide range of research in modern mathematics is due to this freedom. However, this freedom does not imply that mathematicians can choose their system of axioms without having some underlying motivation or goal. Greenberg (1974:253) quotes Hermann Weyl:

> The constructions of the mathematical mind are at the same time free and necessary. The individual mathematician feels free to define his notions and to set up his axioms as he pleases. But the question is, will he get his fellow mathematicians interested in the constructs of his imagination? We can not help feeling that certain mathematical structures which have evolved through the combined efforts of the mathematical community bear the stamp of a necessity not affected by the accidents of their historical birth.

Because mathematics is intimately connected with science and philosophy, any radical change in our understanding of mathematics is sure to be experienced in these two domains. When the postulates of Euclidean geometry ceased to be regarded as basic truths, the scientific theories which were developed from them also ceased to be regarded as truths. Moreover, scientists began to question whether they will ever be able to discover the truth about natural phenomena. Whereas the Greeks held man to be merely in the position of one who uncovers the laws of nature, scientists now realise that their theories merely describe the laws of nature from a human perspective. Similarly, philosophers no longer seek the perfect system of government, the perfect economic system etc., but rather those that are the most appropriate under a given set of circumstances.

6.4 CONCLUSION

Non-Euclidean geometry, as a valid alternative to Euclidean geometry, resulted in a serious re-evaluation of fundamental questions pertaining to the nature of physical space and the nature of mathematics amongst others. According to Morris Kline (1963:577), non-Euclidean geometry demonstrates "how powerless the mind is to recognise the assumptions that it makes". Yet at the same time, it demonstrates the heights which the mind can ascend to despite the restraining effects of 'common-sense', 'intuition', and the most popular philosophical doctrines of the time.

In chapter 4, details have been provided on some of the elementary theorems in hyperbolic geometry with the object of showing that they can be studied meaningfully at school level. In chapter 5, the proof of the relative consistency of non-Euclidean geometry implied that it was equal in status to Euclidean geometry from a mathematical perspective. In this chapter, the impact of non-Euclidean geometry on human thinking has been highlighted. The stage is thus set for giving serious consideration to the motivation for including non-Euclidean geometry in the advanced senior secondary mathematics syllabus. This motivation will be outlined in great detail in the next chapter.

CHAPTER 7

THE MOTIVATION FOR STUDYING NON-EUCLIDEAN GEOMETRY AT SCHOOL LEVEL

> Teaching mathematics means caring about your students, caring about the mathematical ideas they are forming, caring about how they are coming to think of themselves in relation to mathematics ......
>
> Robert Davis

7.1 INTRODUCTION

In the preceding chapters, the elementary mathematical - historical aspects of non-Euclidean geometry have been presented with the intention of showing that non-Euclidean geometry is not the exclusive domain of professional mathematicians and students of advanced mathematics at tertiary institutions, but that the content is well within the domain of understanding of the bright student at secondary school. This is not to say that a reduction of the content level is advocated, by which is meant that teachers merely 'tell' students who are intellectually not ready for the study of non-Euclidean geometry the mathematical facts pertaining to it. What is advocated is that intellectually capable students should be provided with opportunities to 'experience' non-Euclidean geometry in the classroom as they had (hopefully) 'experienced' Euclidean geometry so that meaningful learning can occur.

The previous statements may cause an outcry from those who are familiar with the controversy surrounding the inclusion of Euclidean geometry in the syllabus. On the one hand there are those who oppose the inclusion of Euclidean geometry on the grounds that the nature of the content is outmoded, boring and irrelevant, and likely to give rise to conceptual difficulties amongst students. In 1959 the French mathematician

Jean Dieudonné declared that "Euclid must go!", resulting in a variety of alternative geometry courses being proposed. However, to date no consensus has been reached on the merits of these courses. On the other hand there are those who are opposed to Euclidean geometry because of the uninspiring teaching methods which were traditionally associated with it. Niven in Lindquist (1987:38) quotes the reply of the famous geometer H.S.M. Coxeter on being asked why the teaching of geometry has not been very effective in the United States and Canada:

> I think because there was a tradition of dull teaching; perhaps too much emphasis on axiomatics went on for a long time. People thought that the only thing to do in geometry was to build a system of axioms and see how you would go from there. So children got bogged down in this formal stuff and didn't get a lively feel for this subject. That did a lot of harm. And you see if you have a subject badly taught, then the next generation will have the same thing, and so on in perpetuity.

It is thus clear that the place of Euclidean geometry in the syllabus is in the balance. How can the inclusion of non-Euclidean geometry, which appears to be less relevant, more likely to cause conceptual difficulties and more prone to dull teaching methods, then be justified?

In this chapter, the recommendation that non-Euclidean geometry should be included in the advanced mathematics syllabus for standards 9-10 in South African schools will be motivated. At present, the advanced mathematics syllabus provides a framework for the study of mathematics as a seventh subject by those students who show exceptional ability and who will most likely study mathematics at tertiary level.

7.2 NON-EUCLIDEAN GEOMETRY AND THE VAN HIELE LEVELS OF DEVELOPMENT OF GEOMETRIC THOUGHT

According to the Dutch educators, Dina and Pierre Van Hiele, students progress through a series of levels in the development of geometric thought :

Level 1: Visualisation

At this level, students recognise geometric figures by their physical appearance and not by their specific properties. Students functioning at level 1 are thus able to distinguish between examples and non-examples of a geometric figure and to reproduce this figure when asked to do so (Williams in Moodley, Njisane and Presmeg 1992:337).

Level 2: Analysis

At this level, the properties of geometric figures become evident through observation and experimentation. However, relationships between properties are not yet understood. Students at level 2 are able to identify right angles, parallel sides, equal angles etc.(Crowley in Lindquist 1987:2).

Level 3: Informal Deduction

At this level, relationships between properties both within figures and among figures are perceived. Definitions become meaningful at this stage and informal arguments can be followed. However, deduction is not yet fully comprehended so that pupils are unable to construct their own proofs (Crowley in Lindquist 1987:3).

Level 4: Formal Deductions

At this level, the significance of deductive reasoning as a means of verifying geometric knowledge which has been acquired by empirical means is grasped. Concepts such as *undefined terms*, *definitions*, *postulates*, *theorems* and *proofs* are formed and their role in a mathematical system is appreciated (Crowley in Lindquist 1987:3).

Level 5: Rigour

At this level, students perceive axioms as consistent assumptions which need not be 'true' in the ordinary sense, but which have been chosen by a mathematician for a specific purpose. Students can compare and contrast the mathematical systems which are generated by different sets of axioms. A geometry is thus regarded as an abstract system.

Unfortunately, most researchers have neglected level 5 since most of the geometric activities at school are pitched at level 4. However, since the content and the teaching methods applied are the deciding factors in the progress from one level to the next, by not exposing those students who are already operating efficiently at level 4, to content and activities appropriate to level 5, teachers may be depriving them of the opportunity to develop their full geometric potential. By studying non-Euclidean geometry, a student's progress from level 4 to level 5 can be facilitated.

Some capabilities of students at level 5 which can be developed through the study of non-Euclidean geometry will now be discussed.

7.2.1 Understanding the concept of a geometry

In a proposal known as the Erlangen Program, Felix Klein (1849 - 1925) defined a geometry to be the study of those properties of figures which remain invariant under a given group of transformations. Now it is unreasonable to expect high school students to have the same kind of conceptual understanding of a geometry as professional mathematicians have, but it is equally unacceptable that students and often teachers, continue having the notions about geometry that they have at present. This may hamper the development of geometric thinking in the more able student because he may first have to 'unlearn' these misleading notions before learning more advanced geometric concepts.

(a) Realising that Euclidean geometry is 'Euclid's geometry'

Although Euclidean geometry is taught at school, the adjective 'Euclidean' is seldom used by teachers to describe the geometry that they are engaging in with their students. The teachers can hardly be blamed for this. Most of them have had their last encounter with Euclidean geometry at high school where they were exposed to the same narrow viewpoint about geometry which they are now perpetuating. Very few of them have had the opportunity of studying Euclidean geometry from an advanced viewpoint, and even fewer of them are acquainted with some of the non-Euclidean geometries. It is therefore not surprising that when pupils are asked about the kind of geometry they are studying, the response is: "Why, ordinary geometry!" (It is not quite correct to read into this response that students regard the non-Euclidean geometries as extraordinary.) Once a solid intuitive foundation for the study of Euclidean geometry has been established, teachers should facilitate their students' understanding of definitions, postulates and theorems and the role which each of these play in the logical structuring of Euclidean geometry. It should be pointed out to students that the definitions and postulates which form the basis of the theorems which they are studying, are those of Euclid, and if David or Helen in the class were one day to devise a set of postulates fundamentally different from Euclid's, and from which new and exciting theorems can be deduced, they may just find themselves studying 'Davidian' or 'Helenian' geometry.

(b) Distinguishing between an approach to and the nature of a geometry

Usiskin in Lindquist (1987:23) mentions another common misconception amongst students and teachers. Because co-ordinate geometry is presently also studied at school, teachers often erroneously contrast it with Euclidean geometry. By an 'approach' to geometry is meant the perspective from which the content is viewed. For example, in co-ordinate geometry, geometric figures in the plane are represented by equations and the method used to prove theorems is to a large extent algebraic. However, the theorems which are proven are those which were compiled by Euclid and hence the geometry which is being studied is Euclidean geometry. Only when a set of postulates yields theorems other than those of Euclid are we entitled to think in terms of 'another' geometry. Some of the other approaches which have been identified are the following: synthetic, affine, transformation, vector and eclectic (Usiskin in Lindquist 1990:21).

The way in which the Standard 10 Higher Grade and Standard Grade syllabi have been laid out may inadvertently have contributed to the confusion experienced by teachers and students. The following is an extract from the Standard 10 Higher Grade syllabus:

4. EUCLIDEAN GEOMETRY

*4.1 A line parallel to one side of a triangle divides the two other sides proportionally and conversely, if a line divides two sides of a triangle proportionally, it is parallel to the third side. (Theorem)

4.2 Definition of similarity.

*4.3 If two triangles are equiangular, the corresponding sides are proportional and conversely, if the corresponding sides of a triangle are proportional, the triangle is equiangular. (Theorem)

4.4 Equiangular triangles are similar, and if the corresponding sides of two triangles are proportional, the triangles are similar. (Corollaries)

*4.5 The perpendicular drawn from the vertex of the right angle of a right-angled triangle to the hypotenuse, divides the triangle into two triangles which are similar to each other and to the original triangle. (Theorem)

4.6 The theorem of Pythagoras and its converse. (Theorem)

5. ANALYTICAL GEOMETRY IN A PLANE

5.1 The distance between two points.

5.2 The mid-point of a line segment.

5.3 Gradient of a line.

5.4 Equation of a line and its sketch.

5.5 Perpendicular and parallel lines (no proofs).

5.6 Collinear points and intersecting lines.

5.7 Intercepts made by a line on the axes.

5.8 Equations of circles with any given centre and given radius.

5.9 Points of intersection of lines and circles.

5.10 Equation of the tangent to a circle at a given point on the circle.

5.11 Other loci with respect to straight lines and circles.

From the above, the impression is created that the content matter of Euclidean geometry and co-ordinate geometry are independent, and that the study of the theorems of Euclid as it is undertaken at school, is Euclidean geometry and not one of many approaches to Euclidean geometry. It would therefore be technically more correct if the heading 'EUCLIDEAN GEOMETRY' were to have as subheadings 'SYNTHETIC APPROACH' and 'CO-ORDINATE APPROACH'. This will help those teachers who have these misconceptions about Euclidean geometry to become aware of them, which is the first step towards correcting them.

(c) Teaching non-Euclidean geometry as a means of facilitating a conceptual understanding of a geometry

When students study non-Euclidean geometry, they have a better chance to come to understand that a geometry is a way of describing physical reality and, just as different styles of prose can describe the same scene, so different geometries can be used to describe the same reality. No geometry is more correct than any other - a geometry is chosen for its appropriateness under certain circumstances. For example, a bricklayer will use Euclidean geometry to build a house, but a scientist will use Riemannian geometry to investigate the gravitational attraction between bodies in space.

It is highly unlikely than an understanding of a geometry as described above will be acquired at school level without the study of non-Euclidean geometry. More often than not, such an understanding is never acquired.

7.2.2 Comprehending the nature of and need for proof

Extending the frontiers of mathematical knowledge is the main concern of mathematicians. A mathematician has to support his propositions with rigorous arguments in order to convince the rest of the mathematical community of their correctness. Proving theorems is thus an important activity in mathematics. Most of a student's experience with formal proofs is gained through the study of Euclidean geometry. Bell (1978:289) even goes as far as suggesting that the primary reason for studying Euclidean geometry at school is to become well-versed with the methods used in mathematical proofs.

(a) Defining the term 'proof'

A proof is a chain of reasoning wherein explicitly stated assumptions and the laws of logic are used to arrive at a certain conclusion. Proofs may either be direct or indirect. Most of the proofs studied in Euclidean geometry at school are direct. Students generally find indirect proofs much harder to comprehend. This is quite ironic because, according to Usiskin in Lindquist (1987:27), children at the pre-school level can grasp aspects of indirect proof better than those of direct proof. Bell (1978:293) identifies seven types of direct proof and two types of indirect proof which are commonly studied at school:

	Direct Proof		Indirect Proof
1.	Modus ponents	1.	Reductio ad absurdum
2.	Transitivity	2.	Counter example
3.	Modus tollens		
4.	Deduction theorem		
5.	Contraposition		
6.	Proof by cases		
7.	Mathematical induction		

According to Bell (1978:289), the purpose of a proof may either be to verify that a proposition is true, to elucidate the reason for a proposition being true, or to illustrate the logical connection between propositions. De Villiers in Moodley et al. (1992:54) remarks that a proof can also serve the purpose of communicating the results of research in a clear, concise and unambiguous manner. He states further that new mathematical knowledge can also be stumbled upon by means of a proof, the discovery of non-Euclidean geometry being a classic example.

(b) Students' difficulties with proofs

Despite the importance of proofs in mathematics, research by Senk (1985) referred to by Dreyfus and Hadas in Lindquist (1987:48) has shown that very few students leave school with insight into the nature of and need for proofs. For many students, a proof is what results when you write in two columns, the one being a 'Statement' column and the other a 'Reason' column. They cannot distinguish between the premises and the conclusions in a proof and as a result they become guilty of circular reasoning, the occurrence of which causes endless frustration to teachers. They do not realise that there may be alternative, equally valid ways of proving a theorem and may therefore never be involved in comparing and contrasting different proofs. They are unable to identify the fundamental ideas in a proof and can therefore not recognise and appreciate the ingenuity of some proofs. They do not understand that the premises in a proof may be modified to obtain new and exciting results. They seldom see the need for a proof because the truth of the propositions which they are required to prove is quite obvious to them. Thus, for many students at school, proofs are like poems - they do not have to make sense but they have to be learnt by heart.

The content of the geometry course at school coupled with poor teaching strategies are to a large extent the cause of students' misconceptions of proofs. Besides devising better strategies for teaching Euclidean geometry as Dreyfus and Hadas in Lindquist (1987:47-58) have done, the potential which studying non-Euclidean geometry has to enhance students' conceptual understanding of proofs should be exploited.

(c) Teaching non-Euclidean geometry as a means of facilitating students' comprehension of the nature of and need for proofs

During the 2000 years preceding the discovery of non-Euclidean geometry, a wealth of attempts at proving the parallel postulate had been made, ranging from the elementary arguments of the early Greeks to the more sophisticated approaches of Saccheri and Lambert. By being exposed to these, students see how different approaches can be taken to solve a problem. The significance of originality and creativity in proving theorems becomes evident to them, and they therefore become better equipped to make intelligent judgements about the merits of certain proofs. They get to comprehend the fact that circular reasoning renders a proof invalid as had been the case with Proclus' attempted proof of the parallel postulate (see chapter 3). They consequently realise the importance of distinguishing between the premises and the conclusion, and the necessity of identifying those premises which are logically equivalent to the conclusion. In non-Euclidean geometry, the parallel postulate is substituted with an alternative postulate whilst all the other postulates of Euclidean geometry are retained. Students thus see how a slight modification of the premises can lead to new, somewhat startling conclusions.

Whereas very few theorems of the Euclidean geometry studied at school are proven using the *reductio ad absurdum* method, many of the results in non-Euclidean geometry are obtained by this method. Saccheri, Lambert and Legendre had also made extensive use of this method in their efforts to prove the parallel postulate. The *reductio ad absurdum* method entails the acceptance of a proposition on the basis of the contradiction that arises on the assumption of the negation of that proposition. It is essential that students become competent at using this method of proof if they are to become adults who are able and willing to find solutions to some of the most pressing issues in society.

Possibly one of the most important benefits of studying non-Euclidean geometry is that students learn that a proof is a necessary means of verifying a conjecture. Because some of the results in non-Euclidean geometry contradict results in Euclidean geometry with which students have been thoroughly familiarised, they will have to convince themselves that the arguments by which these strange results have been obtained are indeed valid. They are therefore more likely to want to discover the key elements in these proofs than to be satisfied with mere memorisation.

Constructing and presenting valid arguments are not exclusive to mathematicians, but are vital activities in proper social interaction. Because the teaching of Euclidean geometry has not been very successful in making these activities natural ways of communicating effectively, not only better strategies for teaching Euclidean geometry, but also introducing non-Euclidean geometry should be considered as a feasible alternative.

7.2.3 Realising that diagrams play a limited role in geometry

Many experienced teachers advise their students to start tackling a problem by drawing a diagram. Diagrams help students to concretise relationships between objects and thus put them in a favourable position for finding a solution to the problem. Many students thus come to rely heavily on drawing diagrams when solving problems. Niven in Lindquist (1987:4) recommends the use of diagrams in all explanations, especially in proofs, as a means of "enhancing the attractiveness of a first course in geometry".

(a) The problem with diagrams

There are a number of pitfalls associated with a reliance on diagrams which students seldom become aware of. Diagrams may be inaccurate, especially as far as the relative positions of points and lines are concerned as is illustrated in the following well-known argument that 'proves' that all triangles are isosceles (Greenberg 1974:48) :

Suppose ABC is a triangle. Construct the bisector of $\angle A$ and the perpendicular bisector of side BC opposite to $\angle A$. Consider the various cases that may arise.
Case 1 (see figure 7.1):

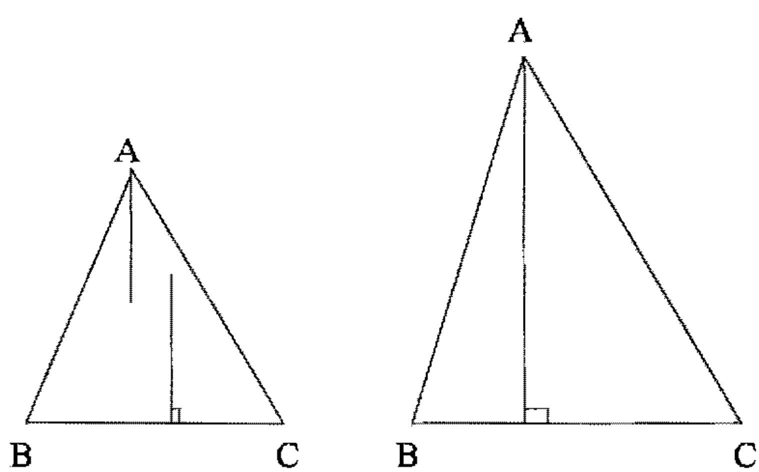


Figure 7.1

The bisector of ∠A and the perpendicular bisector of BC are either parallel or identical. In either case, the bisector of ∠A is perpendicular to BC and hence an altitude of triangle ABC. Therefore triangle ABC is isosceles.

Now suppose that the bisector of ∠A and the perpendicular bisector of BC are neither parallel nor do they coincide. Then they intersect at exactly one point, D, and there are three cases to consider:

Case 2 (see figure 7.2): The point D is inside the triangle.

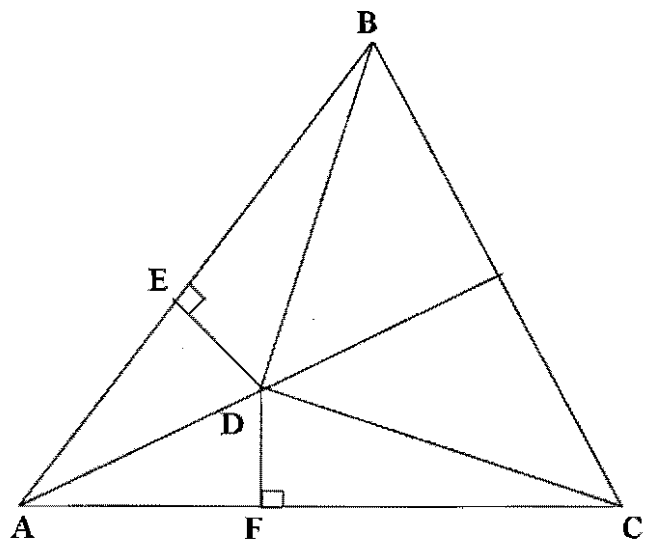


Figure 7.2

Case 3 (see figure 7.3): The point D is on the triangle

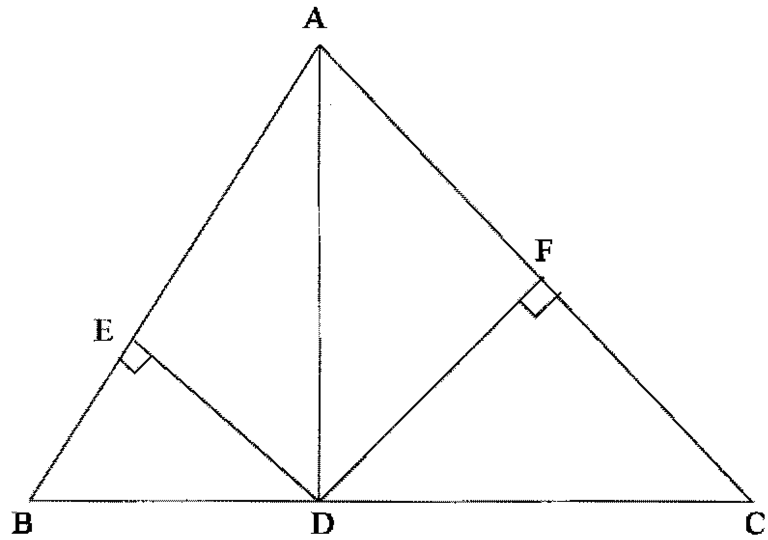


Figure 7.3

Case 4 (see figure 7.4): The point D is outside the triangle

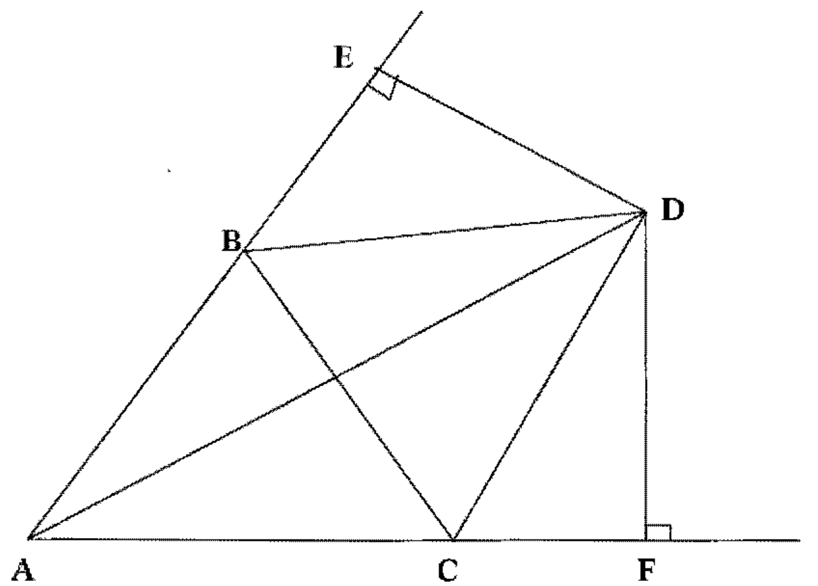


Figure 7.4

For each case, construct DE perpendicular to AB and DF perpendicular to AC. Draw DB and DC. In each case, the following proof now holds:

∠DAE equals ∠DAF since AD bisects ∠A; DA is a common side of triangles AED and AFD; and ∠DEA and ∠DFA are both right angles. Hence triangles AED and AFD are congruent by the angle, angle, side criterion. Therefore the corresponding sides AE and AF are equal. Now DB equals DC because D is on the perpendicular bisector of BC. Also, DE equals DF since D is on the bisector of ∠A, and ∠DEB and ∠DFC are both right angles. Hence triangles DEB and DFC are congruent by the hypotenuse, right angle, side criterion. Hence the sides EB and FC are equal, and since AE and AF are also equal, it follows that sides AB and AC are equal i.e. triangle ABC is isosceles.

An accurately drawn diagram will reveal that the relative positions of D, E, and F are incorrect (see figure 7.5) :

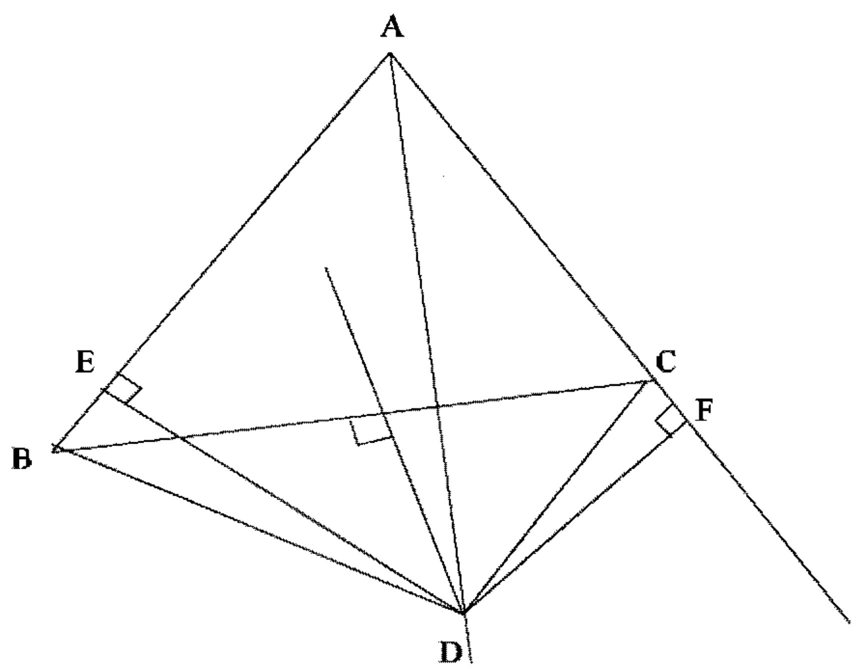


Figure 7.5

Diagrams may also imply more than what can correctly be inferred from the stated assumptions as is the case in each of the following situations:

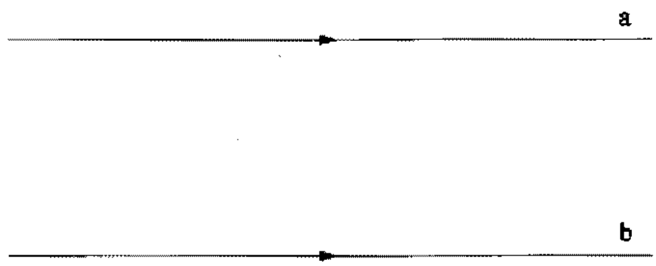


Figure 7.6

From the depiction of a pair of parallel lines as in figure 7.6, students deduce that parallel lines have the equidistance property. Although this is true in Euclidean geometry, it is not the case in non-Euclidean geometry. In hyperbolic geometry, parallel lines either diverge or approach each other asymptotically, and in Riemannian geometry parallel lines do not even exist.

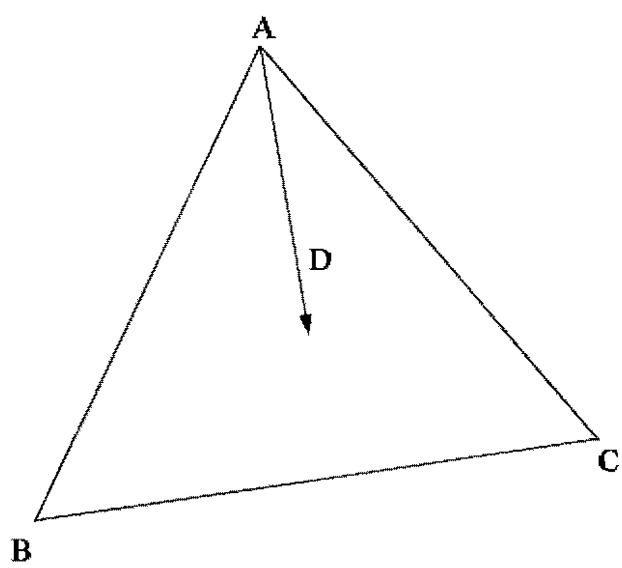


Figure 7.7

From figure 7.7, students conclude that ray AD intersects side BC at a point between B and C. From the *Crossbar Theorem* we know that this will be true, but we cannot just assume that this will be the case by merely studying the diagram. Reference has to be made to the *Crossbar Theorem* or the *Separation postulates* from which it (*the Crossbar Theorem*) can be deduced.

Diagrams may also not cover all the possible situations that may arise. For example, in two of the most widely used textbooks in Western Cape schools the following diagram accompanies the proof of the theorem in Euclidean geometry which states that the opposite angles of a cyclic quadrilateral are supplementary (see figure 7.8):

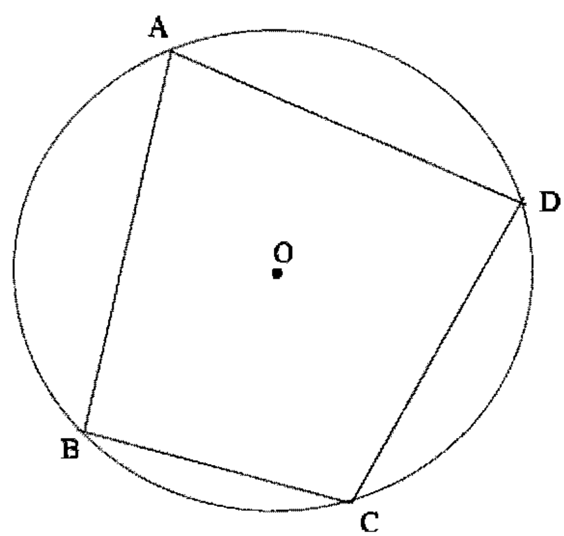


Figure 7.8

Since theorems are general i.e. they become true propositions in any situation in which the premises are true, the diagram in figure 7.9 also has to form part of the proof of the theorem. Often students are not even aware of this kind of omission.

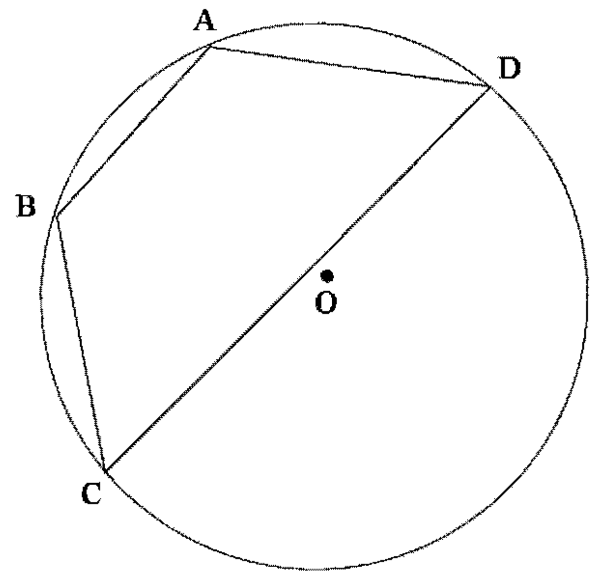


Figure 7.9

(b) Teaching non-Euclidean geometry can expose the limitations of diagrams

When studying non-Euclidean geometry, students learn that diagrams are a mere aid to cognition and cannot always be relied upon to provide an accurate and complete depiction of a situation. The following examples illustrate this:

Figure 7.10

Drawing parallel lines ℓ and m at an angle to each other as is normally the case when representing asymptotic parallels (see figure 7.10), will force students to dissociate the image of oblique lines from the idea of meeting at a finitely distant point.

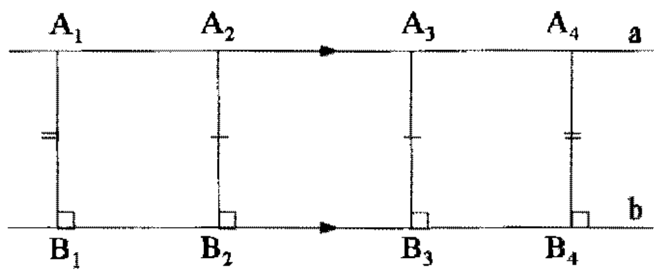


Figure 7.11

The representation of divergent parallel lines as in figure 7.11 requires students to visualise line segments A_1B_1 and A_4B_4 as being longer than line segments A_2B_2 and A_3B_3.

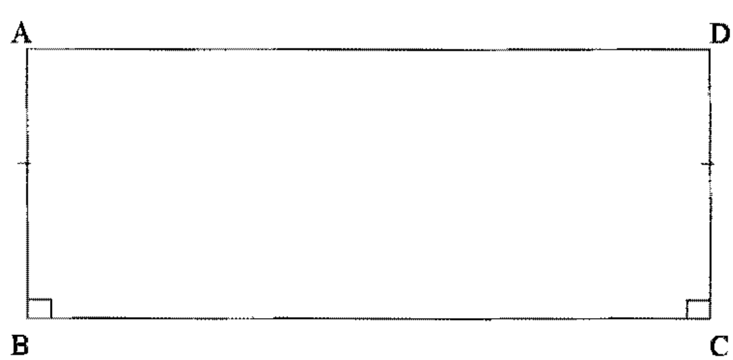


Figure 7.12

In Euclidean geometry, $\angle A$ and $\angle D$ of Saccheri quadrilateral ABCD (see figure 7.12) are right angles with side AD equal in length to side BC. In hyperbolic geometry, the same diagram is used to depict acute angles at A and D with AD longer than BC.

Non-Euclidean geometry has the potential to make students think beyond the diagrams, thereby causing them to make great conceptual leaps and to function at a high level of abstraction. Because Euclidean geometry on the other hand lends itself to representations which are supported by the intuition, this capacity will remain underdeveloped in students who are never exposed to non-Euclidean geometry.

Besides those factors which are explicitly associated with the development of geometric thought of students, there are other factors as well which point to the necessity of studying non-Euclidean geometry at school. An exposition of these will now be given.

7.3 NON-EUCLIDEAN GEOMETRY AND STUDENTS' PERCEPTIONS AND EXPECTATIONS OF MATHEMATICS EDUCATION

To many students, mathematics is a cold, rational, rigid and mysterious discipline which is pursued only by those individuals whose personalities are equally cold, rational, rigid and mysterious. The following quotation in Buerk (1981) (Borasi in Cooney and Hirsch 1990:174) bears this out:

> Math does make me think of a stainless steel wall - hard, cold, smooth, offering no handhold, all it does is glint back at me. Edge up to it, put your nose against it, it doesn't give anything back, you can't put a dent in it, it doesn't take your shape, it doesn't have any smell, all it does is make your nose cold. I like the shine of it - it does look smart, intelligent in an icy way. But I resent its cold impenetrability, its supercilious glare.

7.3.1 Identifying the nature and origin of students' beliefs

According to Borasi in Cooney and Hirsch (1990:176), numerous researchers have tried to clarify the nature of students' beliefs about mathematics by employing various methodologies such as open-ended questionnaires, in-depth interviews, student journals and videotapes of problem-solving sessions. They have even gone to the extent of analysing students' metaphors pertaining to mathematics. The beliefs which became evident from these studies could be categorised as follows:

(a) The scope of mathematical activity: The efforts of both research mathematicians and students are directed towards finding the correct answer to a specific problem, the only difference being that the problems which mathematicians are absorbed in are far more complex than the problems which students are required to solve at school. Problems in mathematics are normally clear and unambiguous and as such have exact and predictable solutions.

(b) The nature of mathematical activity: 'Doing' mathematics entails recalling and applying appropriate mathematical skills which have been acquired by drill and practice methods to solve problems in various contexts.

(c) The nature of mathematical knowledge: Personal judgements and preferences have played no role in the development of mathematics, neither does mathematics offer individuals the 'space' to show their preferences and exercise their judgements. In mathematics there is a distinct line separating that which is true from that which is false.

(d) The origin of mathematical knowledge: The task of mathematicians is merely to uncover that which has since time immemorial existed in its final form. Students passively receive this knowledge, and they in turn have to preserve it for future transmission.

We now turn to the controversial issue of how these beliefs had developed in students. Social conditioning in the form of the media etc. and the limited intellectual capacities of students are undeniably contributing factors. However, Bishop in Bishop, Mellin-Olsen and Van Dormolen (1992:208), and Borasi in Cooney and Hirsch (1990:177), are both of the opinion that these beliefs are unknowingly and unintentionally fostered in the mathematics classroom. (Bishop in Bishop et al. (1992:208) refers to a "hidden curriculum" in mathematics.) How often do teachers not emphasise the importance of practising skills, because from experience they have learnt that this is what counts in examinations. The number of 'Solve for x', 'Factorise', 'Find the product', 'Simplify' and 'Sketch the graph' questions in most examination papers, even external examination papers, certainly justifies this viewpoint. How often do teachers not refrain from exposing their students to alternative solutions to a problem out of fear of 'confusing' them and consequently being labelled 'bad' teachers. How often do teachers not exclude interesting aspects pertaining to the history and philosophy of mathematics from their lessons because they believe they can ill afford this 'diversion' if they are to fulfil all the requirements of the syllabus.

The beliefs which have been referred to represent a distorted view of the nature of mathematics and according to Borasi in Cooney and Hirsch (1990:177), "could prove dysfunctional to students' learning of mathematics". Because the beliefs are most frequently adhered to firmly yet unconsciously, researchers have identified the need for students to be provided with opportunities in the classroom whereby they can become aware of these beliefs, reflect on them, and be presented with plausible alternatives (Borasi in Cooney and Hirsch 1990:179). Merely providing students with information about important events in the history of mathematics, the true nature of mathematics or what their attitude towards mathematics should be, will not be very effective in bringing about a fundamental change in these beliefs. Studying non-Euclidean geometry can provide students with a 'hands-on' opportunity to alter these beliefs.

7.3.2 Teaching non-Euclidean geometry as a means of helping students to reconceive their beliefs

From the history of non-Euclidean geometry, students learn that finding the right answer is not the essence of mathematical activities. For 2000 years the best mathematical minds of the times had attempted to find a proof of the parallel postulate but had failed to do so. Moreover, the models for non-Euclidean geometry which had been constructed by Poincaré and others provided conclusive evidence that no such proof could ever be found. However, this does not render the efforts of Wallis, Saccheri, Lambert and others futile, because they had succeeded in reformulating the postulate and providing insight into the nature of the geometries that would result if it were negated. The frontiers of mathematical knowledge were thereby extended, and the actual discovery of non-Euclidean geometry became inevitable.

In non-Euclidean geometry, the fact that context is very important in mathematics is highlighted. For example, the well-known proposition which states that the sum of the angles of a triangle equals two right angles is not true in general - it is true in Euclidean geometry but false in non-Euclidean geometry. The context of a proposition therefore plays a crucial role in determining its truth or falsity. Non-Euclidean geometry also illustrates that, contrary to popular belief, mathematics has a lot to do with personal judgement. The definitions, the choice of postulates, the wording of the postulates, particularly the parallel postulate, the arrangement of the theorems, and even some proofs in Euclidean geometry are evidence of Euclid's preferences and judgements. Yet, students seldom get this impression from their textbooks or their teachers. However, when non-Euclidean geometry is contrasted with Euclidean geometry, it becomes clear that the postulates of Lobachevsky, Gauss and Bolyai, or of Riemann, are as much a matter of personal judgement as the postulates of Euclid actually are.

Because Euclidean geometry agrees so well with our own interpretation of physical reality, we are inclined to believe that mathematics is merely the revelation of that which has always existed. However, after studying non-Euclidean geometry, few people are likely to dispute the fact that mathematics is a creation of the human mind and an expression of the freedom of the human spirit. In fact, non-Euclidean geometry was labelled 'a logical curiosity' by its early critics (Kline 1980:88).

Although the study of non-Euclidean geometry can help students to reconceive their views relating to the nature and origin of mathematics, which are potentially counterproductive to their learning of mathematics, a few words of caution have to be stated: If students are merely going to be 'filled to the brim' with non-Euclidean geometry, it is unlikely that they will emerge with views which are vastly different to those which they already hold. This viewpoint is evident from the following remarks by Bishop in Bishop et al. (1992:206) :

> The pupil is not to be thought of as a receptive vessel for mathematical knowledge. On the contrary the pupil is the person who must decontextualise and reconstruct the mathematical knowledge from the contextualised situation offered in the classroom. Although we can glibly talk of 'knowledge transmission', and although it is clearly possible to recognise similar knowledge existing in consecutive generations, it is the pupils in any one generation who are busy recreating and reconstructing the mathematical knowledge of their parents' generation and who in their turn structure and recontextualise the mathematical knowledge into situations within which their children's generations can do their own de-contextualising and re-creation of the knowledge.

7.4 NON-EUCLIDEAN GEOMETRY AND THE REAL WORLD

Although Euclid's definition of a line as a curve which lies evenly between its points was found to be unsatisfactory (for reasons which have already been outlined in chapter 2), very few people have difficulty in visualising a straight line. To most people, the reasonable and convenient physical interpretation of the straight line is the ruler's edge or the stretched string - under this interpretation the postulates of Euclidean geometry naturally become true. However, there are other equally plausible interpretations of the term which are linked to the procedures which have been adopted to determine whether a line is straight. Moreover, under some of these interpretations the postulates of Euclidean geometry cease to be true. Three such interpretations of the term *straight line* as described by Barker (1964:49) and Kline (1963:568) will now be discussed briefly.

A straight line can be considered to be the shortest distance between two points. According to this interpretation, to determine whether a line is straight, we would have to investigate if it is the shortest path between its two endpoints. We thus require a device for measuring distances, such as the metre - stick, which we lay down end-to-end each time. If prevailing conditions somehow cause the length of the metre - stick to vary accordingly, as is the case when a metre - stick made out of metal is subjected to great changes in temperature, then the path along which it can be laid down the fewest number of times will not be a straight line in the conventional sense. This was the interpretation which was used by Poincaré in his construction of a model for hyperbolic geometry.

Another common perception of a straight line is that of the path travelled by a light ray. However, light rays do not travel in straight lines. Light rays are bent when entering media of different refractive index and when passing through strong gravitational fields. If the paths travelled by light rays are regarded as straight lines, then the resulting geometry will be non-Euclidean. This was the interpretation which Einstein adopted when developing the theory of relativity.

A very natural and useful interpretation of a straight line is that of a great circle on the surface of the sphere. Under this interpretation, lines are not infinitely long, but are of a fixed length even though there are no identifiable endpoints. Any two points do not determine a unique line, because two points which are diametrically opposite will not lie on a unique great circle. There are no parallel lines in this geometry because any two great circles will necessarily meet. These properties resulting from the interpretation of a line as a great circle on the sphere were the postulates which Riemann used for his non-Euclidean geometry. (Kline 1963:570) makes a very interesting point when he remarks that, even though Riemannian geometry applies so naturally to the surface of the earth, it was not contemplated by the Greek geometers because the Greeks, influenced by the Egyptians and the Babylonians, had adopted the stretched string or the ruler's edge as their interpretation of the straight line.

It is important for students to know that Euclid relied heavily on our intuitive understanding when he defined the term *straight line*, and that the term is open to various interpretations in daily life. And under some of these interpretations, non-Euclidean geometry becomes a valid description of the world we live in.

7.5 NON-EUCLIDEAN GEOMETRY AND THE MATHEMATICAL COMPETENCY OF TEACHERS

It is common knowledge that in many South African schools, especially those schools which had formerly been disadvantaged, the mathematics and science teachers do not have very good qualifications in their subjects. A number of commendable efforts are being made to assist teachers with their presentation of the syllabus material, for example the work being done by the Maths Education Project of the University of Cape Town. However, not enough attention is being given to increasing the mathematical competency of teachers.

Perhaps the viewpoint that good mathematicians seldom make good teachers holds society back from demanding a high level of mathematical competency from those entrusted with the education of its children. But how far can such teachers take their students? How can they expect their students to conduct open-ended investigations or to devise alternative, equally valid proofs etc., if they themselves are either unable or unwilling to undertake such mathematical activities? Usiskin in Lindquist (1987:26) states quite emphatically: "We will not be able to work from problems to solutions in school geometry without knowledgeable teachers".

7.5.1 The generally inadequate level of mathematical competency amongst teachers

To substantiate the belief that many teachers are not mathematically competent enough, two actual examples will be related. (There are numerous other examples which could also be related, but these two will particularly emphasise the point that is being made.) The first example involves a question from a standard 7 examination paper:

Use the sketch below to prove that ST is a straight line.

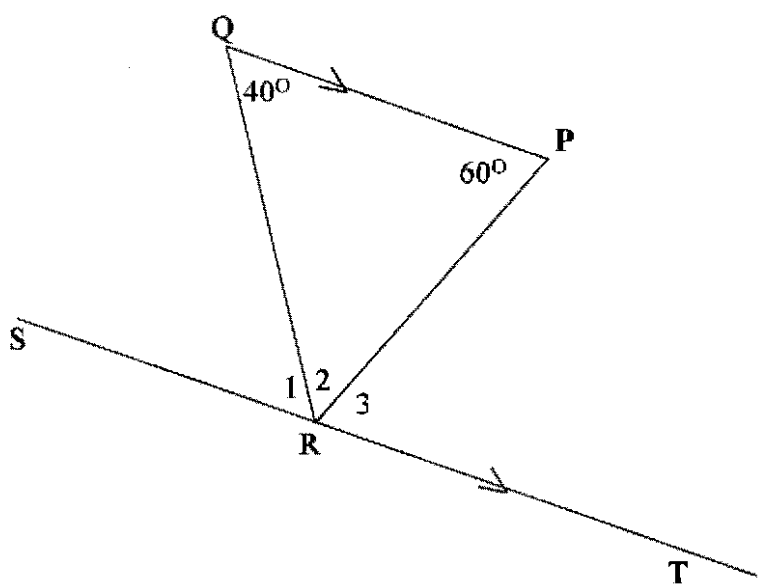


Figure 7.13

The argument given in the memorandum went as follows:

$\angle R_1 = 40°$ (alternate to $\angle Q$)

$\angle R_3 = 60°$ (alternate to $\angle P$)

$\angle R_2 = 80°$ (sum of angles of ΔPQR is $180°$)

$\Rightarrow \angle R_1 + \angle R_2 + \angle R_3 = 180°$

$\Rightarrow$ ST is a straight line

The previous argument clearly illustrates a case of circular reasoning. It is a cause for concern the such questions are being set in examinations. An even greater cause for concern however, is what is possibly being taught in the classrooms of these teachers.

The second example relates to a question from a recent standard 10 external examination:

f(x) is a quadratic expression. f(x) = 0 has equal roots, namely :

$$x = \frac{12 \pm \sqrt{192 - 48k}}{2k - 2}, \qquad k \in R$$

If further it is known that f(- 1) = 27, determine f(x) in the form $ax^2 + bx + c$.

At a workshop at which teachers were given the opportunity to comment on the paper, a number of teachers claimed that, from the quadratic formula, it is evident that 'a' is 3 and 'b' is -12. Although 'a' and 'b' do turn out to be 3 and -12, respectively this reasoning can, upon simplification, of the right hand side also erroneously lead to 'a' and 'b' being taken as -6 and 1½ respectively. It is unacceptable that teachers remain oblivious to such subtleties.

7.5.2 Teaching non-Euclidean geometry as a means of encouraging teachers to become mathematically more competent

Granted that many teachers lack the necessary competence in mathematics, how can the inclusion of non-Euclidean geometry in the school syllabus possibly contribute towards changing this state of affairs?

If teachers have to teach non-Euclidean geometry, then they will be obliged to study or to have studied the subject themselves. From the previous discussions in this chapter, it is clear that if teachers study non-Euclidean geometry, then they will be mathematically more competent than they would have been if they had not had any exposure to it. Even if non-Euclidean geometry does not become part of the school syllabus, it is vital that teachers become acquainted with it if they wish to teach Euclidean geometry meaningfully. Yet, not all colleges and universities offer non-Euclidean geometry as part of the mathematics course for prospective teachers. In-service courses therefore have to be designed to give teachers the opportunity to study topics in mathematics such as non-Euclidean geometry, which are on the periphery of the current school syllabus but fundamentally linked to it. This will enable teachers to teach with greater confidence which is essential if they wish to rely on their knowledge and experience rather than authoritarian tactics as the means of ensuring successful learning.

7.3 CONCLUSION

In this chapter, it has been attempted to show that non-Euclidean geometry has the potential to play a very significant role in the mathematics syllabus. The study of non-Euclidean geometry can facilitate the progress from level 4 to level 5 as described by the Van Hieles in their model of the development of geometric thought, because through it the concept of a geometry can be clarified, the nature of and the necessity for proofs can be demonstrated, and the restricted role which diagrams play in the process of cognition can become evident. False perceptions pertaining to the origin and nature of mathematics can be destroyed and plausible alternatives can be presented through the study of non-Euclidean geometry. Non-Euclidean geometry is a true description of the world we live in, and is arguably even more so than Euclidean geometry proposes to be.

It has to be stressed that the study of non-Euclidean geometry is not advocated for all students in the senior phase. This is in line with the Van Hieles' assertion that the development of geometric thought is sequential i.e. only those students who are at a required level can genuinely progress to the next level. However, efforts should be made so that all students who have been identified as being at the required level, have the opportunity of studying non-Euclidean geometry as part of the advanced mathematics syllabus. This implies that additional resources will have to be made available to schools which have been historically disadvantaged.

The key to realising all the potential advantages of the study of non-Euclidean geometry which have been outlined in this chapter is the use of appropriate teaching strategies. This will be the focus of the next chapter.

CHAPTER 8

STRATEGIES FOR TEACHING NON-EUCLIDEAN GEOMETRY

> Mathematics is a way of thinking that involves mental representations of problem situations and of relevant knowledge, that involves dealing with those mental representations, and that involves using heuristics. It may make use of written symbols, but the real essence is something that takes place within the student's mind.
>
> Robert Davis

8.1 INTRODUCTION

In the previous chapter, an attempt was made to provide and substantiate reasons why non-Euclidean geometry should be included in the advanced mathematics syllabus. These reasons provide the guidelines according to which the content will be unfolded to the student and influence the choice of teaching strategies to be used.

According to Bell (1978:222), a teaching-learning strategy is "a particular procedure for teaching a specific topic or lesson". Numerous strategies have been developed, amongst them being the expository, discovery, laboratory and inquiry strategies. In teaching a topic such as non-Euclidean geometry, it is advisable to use a variety of teaching strategies so that the potential risks associated with a particular strategy could be avoided. For example, since teaching non-Euclidean geometry mainly concerns teaching new concepts and principles, the content matter is most suited to discovery strategies. However, too frequent use of discovery strategies coupled with unrealistic expectations on the part of teachers, can lead to feelings of frustration and inadequacy even amongst good students.

A valid criticism of the teaching strategies which are most commonly used in schools is that they aim at realising only the lower levels of cognition such as knowledge, comprehension and application as identified by Benjamin Bloom (1956) and David Krathwohl (1964). Also, affective objectives, which describe behaviours that indicate changes in students' attitudes, are seldom taken into consideration when planning teaching strategies. (A possible reason for this is that society places greater emphasis on high achievement in mathematics than positive attitudes towards it.) An attempt will therefore be made to illustrate that the higher levels of cognition such as analysis, synthesis and evaluation, as well as favourable attitudes towards mathematics, can be facilitated through appropriate teaching strategies.

Since ensuring that students have the pre-requisite knowledge and skills is in essence the first step towards successfully teaching non-Euclidean geometry, strategies to this effect will be detailed in this chapter. Strategies for introducing and developing the content matter, and for continuously evaluating the teacher's presentation and the students' learning thereof, will naturally constitute the key issues in this chapter.

8.2 PRE-REQUISITE KNOWLEDGE AND SKILLS FOR STUDYING NON-EUCLIDEAN GEOMETRY AND MATCHING STRATEGIES TO ENSURE THAT THESE ARE FUNCTIONAL IN STUDENTS

To be motivated sufficiently for studying non-Euclidean geometry, and to participate actively in the lessons, students have to exhibit the competencies associated with the level of formal deduction. The following are some of the competencies which Crowley in Lindquist (1987) and Williams in Moodley, Njisane and Presmeg (1992) discuss:

8.2.1 Pre-requisites pertaining to Euclidean geometry

Students should :

(a) comprehend the concepts *undefined term*, *definition*, *postulate*, *theorem* and *proof*, and the role which each plays in an axiomatic system such as Euclidean geometry

(b) distinguish clearly between the premises and the conclusions in a theorem

(c) comprehend the implications of necessary and sufficient conditions

(d) identify information implied by a given figure, yet at the same time realise that the apparent features of a figure cannot form the basis of a proof

(e) construct valid proofs for both familiar and unfamiliar relationships

(f) devise alternative proofs of a theorem, and form opinions with respect to the clarity, simplicity and ingenuity of each

8.2.2 Pre-requisites pertaining to logical arguments in general

Students should:

(a) distinguish between deductions which are logically valid and those which are logically flawed

(b) comprehend that a deduction may be valid even though the initial assumptions may be 'false' in the ordinary sense

(c) comprehend the concept *counter-example*, and that one counter-example is sufficient to refute a conjecture

(d) comprehend the concept *contradiction*, and that the existence of a contradiction implies the rejection of the hypothesis from which it was derived

(e) comprehend the concept *consistency*, and that consistency is an absolute requirement of any well-chosen system of axioms

(f) comprehend the concept *negation*, and be able to formulate the negation of any given statement, particularly when quantifiers, disjunctions or conjunctions are involved

(g) comprehend the concept *logical equivalence*, and that equivalent statements perform the same logical function i.e. the same theorems can be deduced from them

8.2.3 Strategies to confirm that the pre-requisites pertaining to Euclidean geometry are met

Students can be set the following items:

(a) Which of the following statements is a definition, a postulate, a theorem? From which of these (if any) can those which you have classified as theorems (if any) be deduced?

(i) Through a point not on a given line, a unique line can be drawn parallel to the given line
(ii) Any two non-parallel lines intersect at a unique point
(iii) A line is a curve which lies evenly between its endpoints
(iv) A unique line can be drawn between any two points
(v) Parallel lines are lines which do not intersect
(vi) If two parallel lines are cut by a transversal, then the alternate interior angles are equal

(b) Rewrite the following statements in IF .... THEN .... form :

(i) Vertically opposite angles are equal

(ii) The sum of the interior angles of a triangle equals two right angles

(iii) The diagonals of a rhombus bisect each other at right angles

(iv) Equiangular triangles have corresponding sides which are in proportion

(c) Write a definition of a square that begins as follows :

(i) A square is a quadrilateral ....

(ii) A square is a parallelogram ....

(iii) A square is a rhombus ....

(iv) A square is a rectangle ....

(d) State whether the following pairs of triangles in figures 8.1 - 8.3 are necessarily congruent or not. Provide reasons for your answers.

(i)

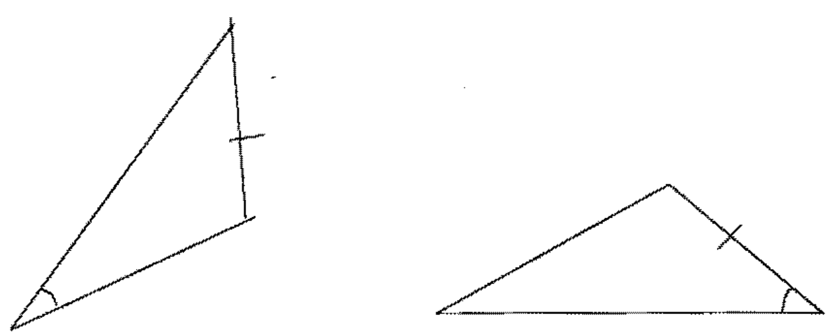


Figure 8.1

(ii)

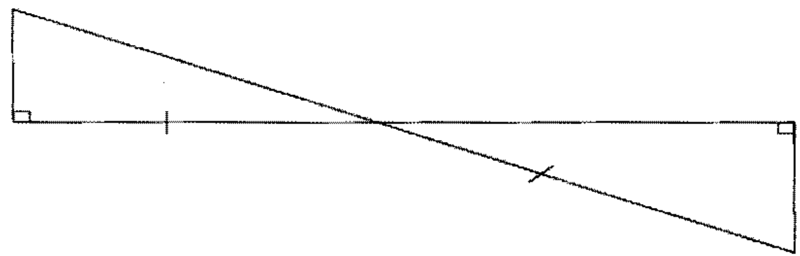


Figure 8.2

(iii)

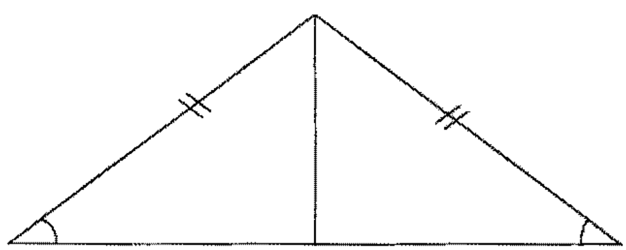


Figure 8.3

(e) Prove that the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides. (See figure 8.4.)

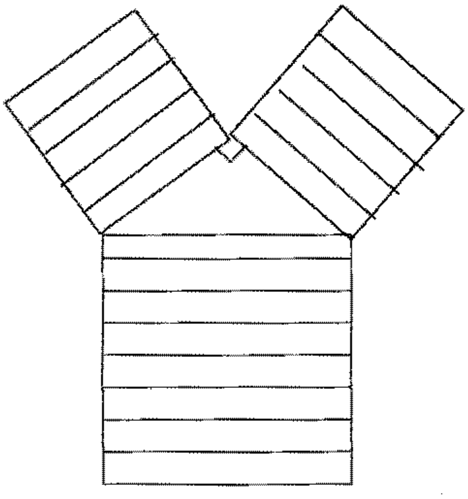


Figure 8.4

Now investigate analagous results for :

(i) semi -circles

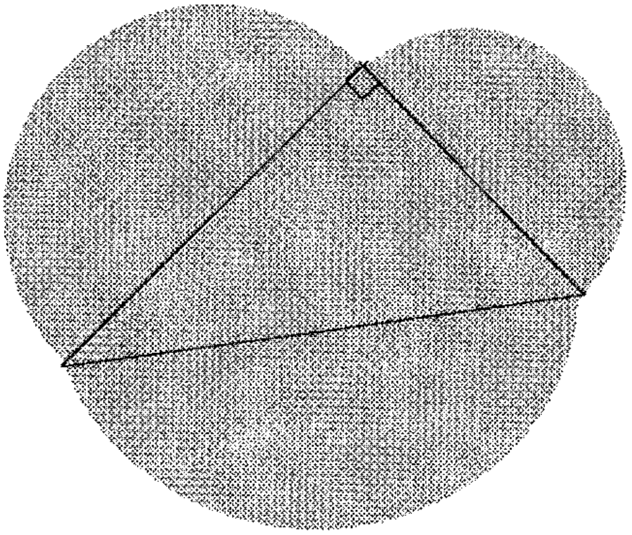


Figure 8.5

(ii) equilateral triangles

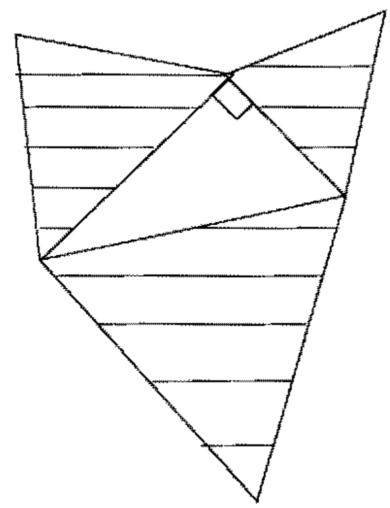


Figure 8.6

(f) Study the proofs of the Theorem of Pythagoras given below and answer the questions that follow.

PROOF A

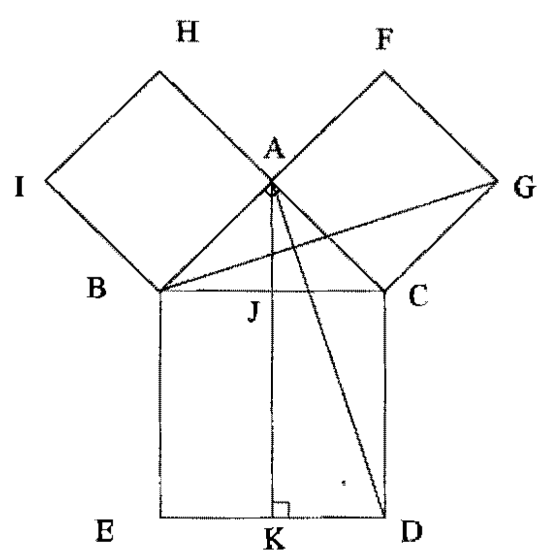


Figure 8.7

Δ BCG $\equiv$ Δ DCA (S, A, S)

Rectangle JCDK = 2 Δ DCA (Same base, equal heights)

Square AFGC = 2 Δ BCG (Same base, equal heights

$\Rightarrow$ Rectangle JCDK = Square AFGC

Similarly, Rectangle BJEK = Square HABI

$\Rightarrow$ Square AFGC + Square HABI = Square BCDE

PROOF B

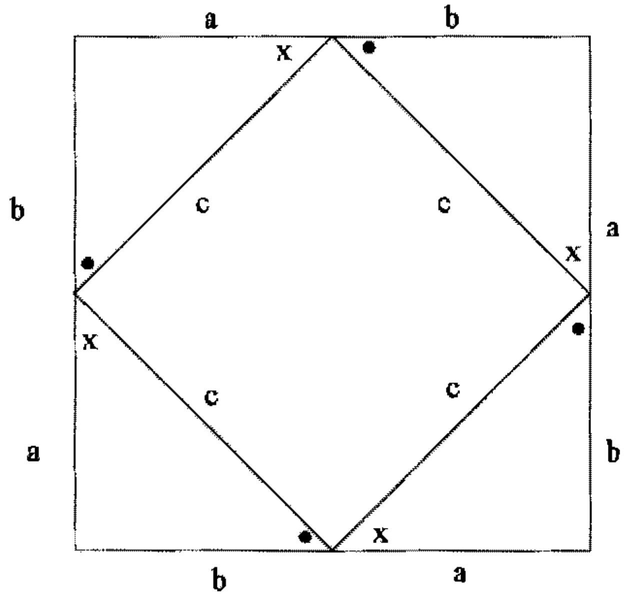


Figure 8.8

The area of the larger square equals $(a + b)^2$ and also $c^2 + 4$ (½ab)

$\Rightarrow (a + b)^2 = c^2 + 4\,(\tfrac{1}{2}ab)$

$\Rightarrow a^2 + b^2 = c^2$

PROOF C

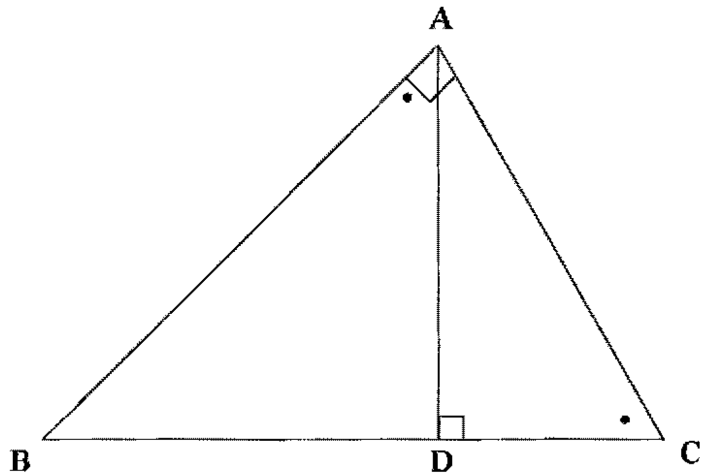


Figure 8.9

Δ ABC is similar to Δ DBA

$$\Rightarrow \frac{AB}{BC} = \frac{DB}{BA}$$

i.e. $AB^2 = BC.\ DB$

Similarly, $AC^2 = CD.\ BC$

$$\begin{aligned}\Rightarrow AB^2 + AC^2 &= BC.\ DB + CD.\ BC\\ &= BC\ (DB + CD)\\ &= BC.\ BC\\ &= BC^2\end{aligned}$$

(i) Which one of these proofs is, in your opinion, easiest to understand?

(ii) Which one is, in your opinion, the most ingenious?

(iii) What is your opinion about using an algebraic approach in a proof in geometry?

8.2.4 Strategies to confirm that the pre-requisites pertaining to logical arguments in general are met

Students can be set the following items:

(a) Comment on the validity of each of the following deductions:

(i) Tulips are pretty
Flowers are pretty
∴ Tulips are flowers

(ii) Roses are flowers
Flowers have thorns
∴ Roses have thorns

(iii) Daffodils are yellow
Sunflowers are yellow
∴ Daffodils are sunflowers

(NB. Although (i) and (ii) above test the same logical principle, the idea is to get students to realise that an argument may be invalid even though the conclusion may be true)

(b) Choose the correct conclusion in each case :

(i) Carnations are red
Here is a pink carnation

A. ∴ Carnations are not red

B. ∴ Carnations are pink

C. ∴ Carnations are not at all red

(ii) This flower is a daisy
This flower is a violet

A. ∴ It is neither a daisy nor a violet

B. ∴ It is both a daisy and a violet

C. ∴ It is either a daisy or a violet

(c) Negate each of the following statements :

(i) All orchids are exquisite
(ii) There is a red rose
(iii) Marigolds and dahlias grow in the garden
(iv) Petunias are purple or white flowers

8.3 STRATEGIES FOR CONTEXUALISING THE NEW CONTENT MATTER

Because non-Euclidean geometry was discovered in the attempt to strengthen the logical foundation of Euclidean geometry, it is appropriate to introduce it by means of a critical examination of the foundation of Euclidean geometry. This can be achieved by discussions on GEOMETRY, EUCLID, and EUCLIDEAN GEOMETRY, which can either be teacher-led or group-based.

8.3.1 Points which should emerge from a discussion on geometry

(a) The Egyptians and the Babylonians surveyed the land for practical purposes
(b) The Greeks termed this science 'geometry' which means 'earth measurement'
(c) Unlike the Egyptians and the Babylonians, the Greeks used deduction to arrive at their conclusions
(d) The Greeks studied geometry for its aesthetic value rather than its potential for application
(e) The results which the Greeks obtained by deduction were in agreement with their experiences which were confined to a small part of the earth

8.3.2 Questions for further research by students

(a) Why did the Greeks specifically choose to study geometry rather than algebra or arithmetic?

(b) Were there any prominent Greek women geometers?

(c) Why was deduction rather than induction so highly rated by the Greeks?

(d) What effect has the Greek preference for deduction had on the way you study geometry at school?

Students can then be divided into groups, each of which has to present a short written piece on the life and work of Euclid.

8.3.3 Themes which should be highlighted in a presentation on Euclid

(a) Euclid as geometer and teacher

(b) Euclid as compiler of the work of his ancestors

(c) The logical arrangement of the Elements

(d) The Elements as the most durable and influential textbook in the history of mathematics

Those groups which are particularly eager can be encouraged to compare and contrast the presentation of one of the theorems in their school textbooks with that in Sir Thomas Heath's translation of the Elements. These findings can be presented to the class.

Students are now ready to comprehend that Euclidean geometry is the study of figures in space based on the definitions and postulates of Euclid. A statement which could provoke lively debate in the class is the following: Euclidean geometry is far too abstract to be studied at school, and is of no benefit to students once they leave school.

8.3.4 Questions for a worksheet on the postulates of Euclid

(a) Which one of the following is the most accurate description of the term *postulates*?

- (i) self-evident truths
- (ii) arbitrarily chosen assumptions

(iii) assumptions chosen with a particular aim in mind

(b) List the postulates of Euclid.

(c) Do you think it is always possible to do the constructions as stipulated in postulates 1 - 3? Provide reasons for your answer.

(d) Why does Euclid state explicitly that all right angles are equal? Consider the fact that Euclid defined a right angle as an angle which is equal to its supplement.

(e) (i) Rephrase postulate 5 beginning with:
If two lines are parallel, ...

(ii) Is this postulate intuitively obvious?

(iii) Can it be verified by human experience?

(iv) Write down any theorem that is deduced from it.

(v) Write down any theorem that is independent of it.

By means of a lecture-demonstration, the history of the parallel postulate can now be related to the students.

8.3.5 Pertinent points in the history of the parallel postulate

(a) The parallel postulate was not self-evident

(b) Research on the postulate began during the time of Euclid

(c) The aim of the research was to secure the logical foundation of Euclidean geometry

(d) Three approaches were taken, namely to attempt a proof of the postulate, to find a substitute postulate which was more acceptable, and to investigate the geometry which resulted if the postulate were negated

(e) The most prominent researchers were Proclus, Wallis, Saccheri, Lambert and Legendre

(f) It is ironic that Saccheri, believing that Euclid had to be "vindicated from all defects", did not realise the importance of his discoveries

(g) After efforts spanning 2000 years the problem remained unsolved

(h) Klügel, Kästner, and others began expressing their doubts about the provability of the postulate

A few simple attempted proofs of the parallel postulate such as those by Ptolemy and Proclus can be demonstrated to the students, and they can be challenged to find the flaw in each. The paradoxical argument of the ancient Greeks which has been presented in chapter 3, is sure to stimulate interest in the students. As a more challenging exercise, students can be asked to prove that Wallis' postulate, or the Euclidean theorem about the sum of the angles of a triangle, is equivalent to the parallel postulate.

8.3.6 Questions for a worksheet on the fundamental figures used in the investigations of Saccheri and Lambert

(a) A Saccheri quadrilateral is a quadrilateral ABCD with AD and BC perpendicular to the base DC, and AD equal to BC. AB is called the summit, and $\angle A$ and $\angle B$ the summit angles. (See figure 8.10.)

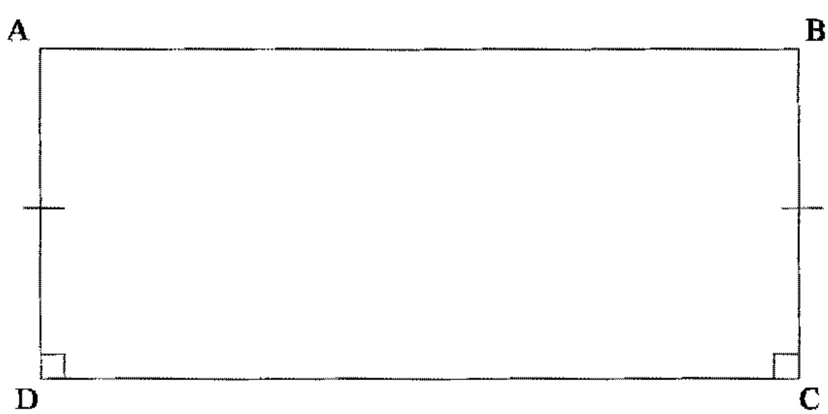


Figure 8.10

(i) Prove that the summit angles are equal.

(Hint : Draw AC and BD, and then use congruency)

(ii) Prove that the line segment joining the midpoints of the summit and the base is perpendicular to both of these sides.

(Hint : Draw ED, EC, AF and BF, and then use congruency)

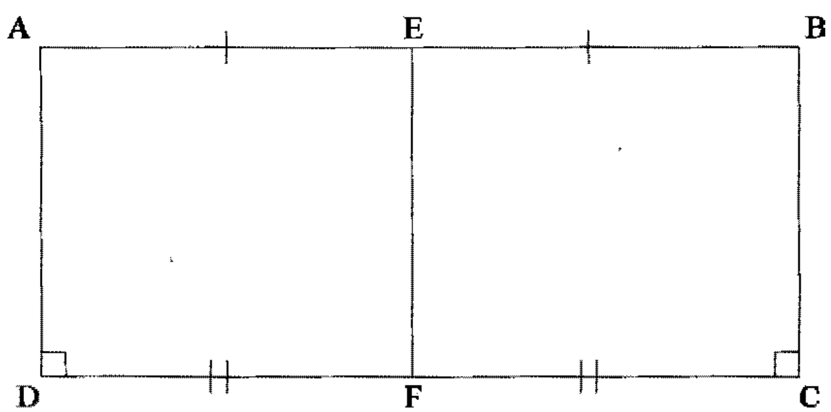


Figure 8.11

(b) Now suppose that the Saccheri quadrilateral in Figure 8.11 is cut along line segment EF. The resulting quadrilaterals are called Lambert quadrilaterals. Consider one of these:

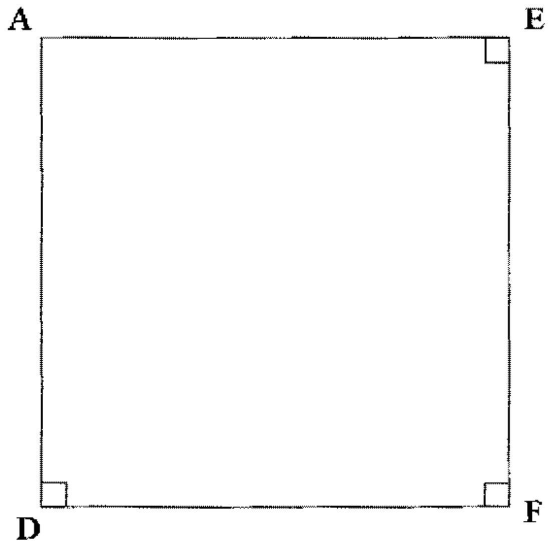


Figure 8.12

(i) Prove that if $\angle A$ is acute, then AE > DF

(Hint : Assume AE < DF and derive a contradiction.

Do the same for AE = DF)

(ii) Prove the analogous result if $\angle A$ is obtuse.

(iii) Suppose that the two Lambert quadrilaterals above are rejoined at EF. What can you deduce about the relationship between the summit and base of a Saccheri quadrilateral from (b) (i), and (b) (ii)?

(iv) Describe a quadrilateral which is both Saccheri and Lambert.

8.4 STRATEGIES FOR INTRODUCING THE CONCEPT OF AN ALTERNATIVE GEOMETRY

Students can be introduced to the concepts of an alternative geometry by means of a discovery strategy. Some of these ideas are based on the workshop presentations of Professor Henderson from Cornell University.

8.4.1 Student activities for introducing the concept of an alternative geometry

Students can be divided into small groups for the following activities.

(a) For the first activity, each of these groups should be given a tennis ball, some elastic bands, and the following set of instructions and questions: (Consider the interpretation of a straight line segment as the shortest distance between two points.)

(i) Mark any two points on your tennis ball.
Manipulate one of the elastics around the tennis ball so that it passes through the two points. Hence describe the lines on the surface of the tennis ball.

(ii) Investigate the properties of these lines by manipulating a few more elastic bands.

(iii) How would you define an angle on the surface of the tennis ball?

(iv) Form a triangle and investigate its properties.

(v) Can you form a rectangle? Explain clearly why you can /cannot form a rectangle.
(Hint : Refer to the properties you have established in (ii))

(vi) Can you form two similar, non-congruent triangles?

(b) For the next activity, each group should be provided with a felt-tipped pen and with two funnels which have been glued at their rims. The resulting surface is called a pseudosphere. (See figure 8.13.)

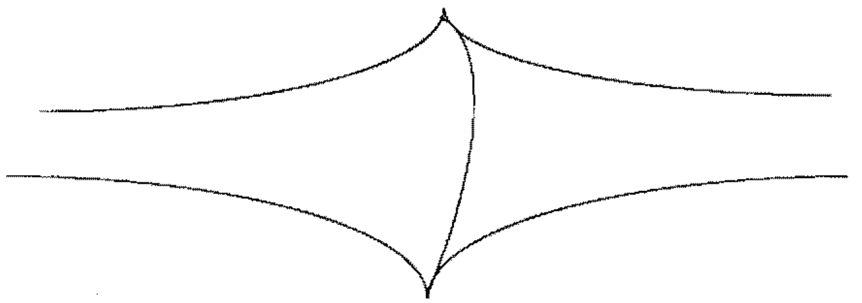


Figure 8.13

The following set of questions and instructions can be given:

(i) Mark any two points, A and B, on the joined rims of the two funnels. Trace the path of shortest distance from A to B.

(ii) Locate a third point, C, anywhere on the surface of the funnels except the joined rims, and trace the path of shortest distance from B to C.

(iii) On the basis of your findings in (i) and (ii), describe the lines on the pseudosphere.

(iv) Investigate some properties of these lines.

(v) How would you define an angle on the pseudosphere?

(vi) Investigate the properties of triangles on the pseudosphere

(vii) Draw a Saccheri quadrilateral. What is the nature of the summit angles? Can you link this observation up with any of your other observations?

(c) Now choose the correct answer(s) from those in brackets for each of the following:

(i) Any two points lie on (exactly one, more than one) line

(ii) Lines are of (fixed, infinite) length

(iii) From any point not on a line, (no, exactly one, more than one) perpendicular(s) can be drawn to the given line

(iv) Through a point not on a line, (no, exactly one, more than one) line(s) can be drawn parallel to the given line

(v) The distance between two parallel lines (remains constant, increases, decreases)

(vi) The sum of the angles in a triangle is (less than, equal to, more than) two right angles

(vii) The exterior angle of a triangle is (less than, equal to, more than) the sum of the interior opposite angles

(viii) The sum of the angles of a convex quadrilateral is (less than, equal to, more than) four right angles

(ix) The summit angles of a Saccheri quadrilateral are (acute, right, obtuse) angles

(x) Triangles which are equiangular are (similar, congruent)

(d) Comment intelligently on the implications of your answers in (c).

8.5 DEVELOPING HYPERBOLIC GEOMETRY RIGOROUSLY

Now that the intuitive foundations for non-Euclidean geometry have been established, an expository strategy can be used to further the development of the content matter. The geometries investigated by the previous activities should be named, their postulates should be compared and contrasted with those of Euclidean geometry, and the mathematicians responsible for their discovery should be discussed very briefly. Because of the intellectual maturity of the students and the strangeness of the results which have been observed, they would want to have these results confirmed by rigorous demonstrations. However, only hyperbolic geometry should be developed formally as the mathematical content of Riemannian geometry is too complex for students at this level.

The extent of the development should be as has been indicated in chapter 4, although a different logical sequence could be followed, such as that of Faber (1983), Meschkowski (1964) or Trudeau (1987). Students should be encouraged to analyse the proofs and to construct their own valid proofs for the corollaries of the major theorems.

8.5.1 Questions for a worksheet on the theorems and proofs in hyperbolic geometry

(a) State some of the theorems in hyperbolic geometry and their Euclidean counterparts

(b) Have the proofs of these theorems :

 (i) verified results which you doubted?

 (ii) verified results which you accepted without question?

 (iii) served no purpose?

(c) Carefully study the proof given below:

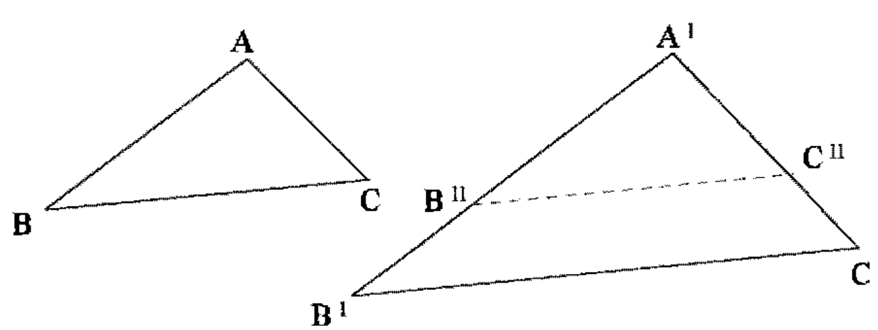


Figure 8.14

Suppose that triangles ABC and $A^1B^1C^1$ are similar but not congruent. (See figure 8.14.) Then no pair of corresponding sides is equal, for then the triangles would be congruent by the angle, side, angle criterion. Thus at least two sides of one triangle must be greater than the two corresponding sides in the other triangle, say $A^1B^1 > AB$ and $A^1C^1 > AC$. Hence there exists points B^{11} on A^1B^1 and C^{11} on A^1C^1 such that $A^1B^{11} = AB$ and $A^1C^{11} = AC$. By the side, angle, side criterion, triangles ABC and $A^1B^{11}C^{11}$ are congruent, and hence the pairs of corresponding angles $\angle ABC$ and $\angle A^1B^{11}C^{11}$, and $\angle ACB$ and $\angle A^1C^{11}B^{11}$ are equal. From similarity however, $\angle ABC$ equals $\angle A^1B^1C^1$, and $\angle ACB$ equals $\angle A^1C^1B^1$, so that $\angle A^1B^{11}C^{11}$ equals $\angle A^1B^1C^1$, and $\angle A^1B^{11}C^{11}$ equals $\angle A^1C^1B^1$. Now $\angle A^1B^{11}C^{11} + \angle B^1B^{11}C^{11} = \angle A^1C^{11}B^{11} + \angle C^1C^{11}B^{11}$ = two right angles. Since $\angle A^1B^{11}C^{11} = \angle A^1B^1C^1$ and $\angle A^1C^{11}B^{11} = \angle A^1C^1B^1$, the sum of the interior angles of quadilateral $B^{11}C^{11}C^1B^1$ equals four right angles, contradicting the result in hyperbolic geometry that the sum of the angles in a convex quadilateral is less than four right angles.

(i) Which theorem is being proved here?

(ii) Identify the *definitions*, *postulates* and *theorems* which are used in the proof of this theorem.

(iii) Identify the method of proof used.

(iv) State and prove a corollary of this theorem.

(v) Devise an alternative proof for this theorem.

After the mathematical content has been discussed, students can be asked to research the important philosophical implications of the discovery of non-Euclidean geometry, and to present a written piece of about 2 pages.

8.5.2 Issues which can be debated in class on the basis of students' research on the philosophical implications of non-Euclidean geometry

(a) Euclidean geometry is the correct description of physical space

(b) Non-Euclidean geometry is merely a logical curiosity

(c) The true geometry of space can be determined empirically

(d) Mathematics does not offer truths, only theories

8.6 CONCLUSION

Although all students should be provided with the opportunity of acquiring a new perspective of the world they live in through the study of non-Euclidean geometry, the mathematical complexity of the subject dictates otherwise. According to the Van Hiele model of the development of geometric thought, students have to be at a required level before genuine progress can be made to the next level. Also, if students are at a particular level but are being taught at a different level, the desired learning may not occur. Crowley in Lindquist (1987:4) states quite emphatically that "if the teacher, instructional materials, content, vocabulary, and so on, are at a higher level than the learner, the student will not be able to follow the thought processes being used". This implies that it is absolutely essential that extensive assessment should be done to identify those students who are at level 4 and who can therefore advance to level 5 by studying non-Euclidean geometry. A Van Hiele based test can be used to help teachers in this regard.

Teachers are also encouraged to facilitate the attainment of the higher levels of cognition by their students. This can be achieved by posing probing questions, by choosing topics for debates which students are likely to become absorbed in, by encouraging independent research on some of the issues at hand and the oral or written presentation thereof, and by providing students with opportunities to interact with manipulative materials and to reflect on their findings.

Although the issue of continuous evaluation is not new, it has currently come under the spotlight due to the new regulations governing evaluation in mathematics for standards 6-9 in South African schools. Whereas continuous evaluation was left to the discretion of the teachers in the past, it is now compulsory to allocate 25% - 50% of the final mark in mathematics on the basis of it. Despite the fact that a number of workshops are currently being conducted to address this issue, many teachers remain confused as to what exactly is expected from them and their students. In this respect, the afore-mentioned strategies have also been presented with the aim of illustrating some potentially useful methods of continually evaluating students' assimilation of the new ideas.

The most significant point which it is hoped emerges from the presentation of these strategies, is that students should be provided with opportunities to construct new knowledge on the basis of the knowledge which they already possess. Traditionally, the student has been perceived as a passive receiver of knowledge transmitted by the teacher. According to the constructivist theory of Piaget, the student engages in a process of constructing knowledge which involves the incorporation of new information and experiences into the mental structure, and the subsequent re-organisation of existing knowledge to accommodate the new information. The role of the teacher thus becomes that of organiser of a learning environment in which such knowledge construction will become possible.

It is evident that thorough planning plays a pivotal role in ensuring that the strategies which have been selected in accordance with the aims of teaching non-Euclidean geometry, do in fact lead to their realisation.

CHAPTER 9

CONCLUSION

> A new scientific truth does not triumph by convincing its opponents and making them see light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.
>
> Max Planck

9.1 INTRODUCTION

The current interim core syllabus in mathematics for the senior secondary phase aims at fostering the following general teaching and learning aims (Department of Education 1995) :

(1) to develop independent, confident and self-critical citizens

(2) to develop critical and reflective reasoning ability

(3) to develop personal creativity and problem solving capabilities

(4) to develop fluency in communicative and linguistic skills e.g. reading, writing, listening and speaking

(5) to encourage a co-operative learning environment

(6) to develop the necessary understanding, values and skills for sustainable individual and social development

(7) to understand knowledge as a contested terrain of ideas

(8) to contextualise the teaching and learning in a manner which fits the experience of the pupils

When the specific conclusions which have been arrived at in this study are examined in the light of the above-mentioned aims, it becomes apparent that the study of non-Euclidean geometry can contribute to their realisation i.e. non-Euclidean geometry has the potential to play a very significant role indeed in the senior secondary mathematics syllabus.

9.2 SUMMARY AND CONCLUSIONS

To substantiate the above claim, it is necessary to provide a summary of that which has been attempted in this study and the conclusions which have been drawn.

9.2.1 The historical aspects of non-Euclidean geometry and the lessons which students can learn from them

The compilation of the Elements around 300 B.C. was undoubtedly a great intellectual achievement in the history of mathematics, because it introduced the concept of deduction as a necessary means of verifying knowledge which has been obtained by empirical methods. Euclid formulated a number of definitions, common notions and postulates to constitute the foundation of his system. To the ancient Greeks, these common notions and postulates were 'self-evident truths' which were supported by their limited experiences on a small portion of the earth. The theorems which followed logically from the definitions, the common notions and the postulates, were therefore also truths in the ordinary sense. Euclidean geometry thus acquired the status of 'the true interpretation of physical space'.

However, the unrivalled position of Euclidean geometry was marred by the parallel postulate because it made an assertion involving infinite distances which could not possibly be verified by human experience. The great anxiety which the parallel postulate had created amongst mathematicians resulted in extensive research being undertaken in an attempt to secure the foundations of Euclidean geometry. Three approaches were made to the problem of the parallel postulate. The first was to attempt to deduce it from the other definitions, common notions and postulates, thereby giving it the status of a theorem in Euclidean geometry. The second was to find a substitute postulate that would be more acceptable to the mathematical community in terms of clarity and self-evidence. The third was to investigate nature of the geometry that would result if the parallel postulate were negated, but all the other postulates of Euclidean geometry were retained.

There is evidence that the research on the parallel postulate had already commenced in the time of Euclid. Through the ages, numerous major and minor mathematicians tackled the problem zealously. Each expanded and modified the ideas of his predecessor - the cumulative development of mathematics is evident from this. Of these efforts, the work of the Jesuit priest, Saccheri, is undoubtedly the most significant. Saccheri's novel idea was to prove the parallel postulate by means of the *reductio ad absurdum* method. In the process however, he came upon strange theorems - he had in fact stumbled upon non-Euclidean geometry. Regrettably, since Saccheri was unable to draw the correct conclusions from his discoveries, he is not credited with the discovery of non-Euclidean geometry.

Two thousand years of attempts to prove the parallel postulate or to formulate a satisfactory substitute postulate by some of the most able mathematicians of the times ended in dismal failure. The inexplicability of this unpleasant reality kindled the thought that the parallel postulate was unprovable from the other definitions, common notions and postulates. Research therefore became geared towards investigating geometries resulting from the negation of the parallel postulate. The discovery of non-Euclidean geometry thus became a logical inevitability.

Gauss, Lobachevsky and Bolyai displayed great vision and courage in expounding the elementary ideas in non-Euclidean geometry - they had to free themselves from the shackles of 'commonsense', 'intuition' and the philosophy of Kant which was dominant at the time. The discovery of non-Euclidean geometry was received with ridicule and contempt. Its opponents cherished the hope that it would be shown to be inconsistent, because then its potential to be an alternative, equally valid mathematical system would be destroyed. However, the construction of models for hyperbolic geometry within Euclidean geometry extinguished these hopes forever - hyperbolic geometry is consistent if and only if Euclidean geometry is so, and since the consistency of Euclidean geometry was unquestionable, the consistency of hyperbolic geometry was affirmed. In the light of this new knowledge, the scientific world became pre-occupied with fundamental questions about the nature of space and the nature of mathematics amongst others : Is space necessarily Euclidean? If not, what is the true nature of space and how can we ascertain this? Is there a true geometry of space? Are the axioms in a mathematical system 'truths'? If not, what are the constraints on mathematicians when formulating the axioms for their particular systems?

There are a number of lessons that students can learn from the history of non-Euclidean geometry. Firstly, it is necessary to be familiar with the historical circumstances prevailing at the time of a discovery to be able to understand and appreciate its magnitude - the work of Gauss, Bolyai and Lobachevsky stands out because mathematicians were trapped in the spirit of the times. Secondly, personal judgements and preferences have played a crucial role in the development of mathematics - for example, Euclid selected certain definitions, common notions and postulates to constitute the basis of his system, and displayed ingenuity in his particular formulation of the parallel postulate. This is also evident from the wealth of substitute postulates which had been proposed, and the numerous attempted proofs of the parallel postulate, ranging from the elementary ideas of Proclus to the more intricate arguments of Saccheri and Lambert. Thirdly, mathematical activity entails more than just finding the correct solution to a problem - the reformulation of a problem, placing it in context, and critically reflecting on it are important activities in mathematics as is evident from the research on the parallel postulate. In fact, problems may not always have exact and predictable solutions, as is illustrated by the fact that the consistency of hyperbolic geometry relative to Euclidean geometry implied that the parallel postulate was unprovable from the other definitions, common notions and postulates. Fourthly, independent thinking, courage and perseverance are admirable human qualities which will pay off in the long run. For example, although Lobachevsky published extensively on non-Euclidean geometry - he dictated his last work after he had become blind - he received no acclaim for it during his lifetime. However in the present, hyperbolic geometry is also called 'Lobachevskian' geometry in his honour. Fifthly, ideas cannot simply be rejected on the grounds that they are contrary to popular belief and experience, but have to be put to the test by logical reasoning. Lastly, and most importantly, Euclidean geometry is but one of many valid interpretations of physical space which becomes inappropriate when astronomical distances are considered.

9.2.2 The elementary mathematical aspects of non-Euclidean geometry and the competencies which they engender in students

It is rather surprising that hyperbolic geometry, the discovery of which has had such profound philosophical implications, is based on a very elementary idea, namely that of replacing the parallel postulate with its negation, the hyperbolic postulate, whilst retaining all the other postulates of Euclidean geometry.

Hyperbolic Postulate: There exist a line l and a point P not on l, such that there are at least two distinct lines parallel to l passing through P.

All the theorems of Euclidean geometry which are independent of the parallel postulate are therefore theorems of hyperbolic geometry. Some additional theorems are the following :

(a) The sum of the angles in a triangle is less than two right angles.

(b) Rectangles do not exist.

(c) If two triangles are similar, then they are congruent.

(d) The theorem of Pythagoras is false.

(e) A line parallel to a second line is either asymptotic to it, or has a common perpendicular with it.

There are a number of mathematical competencies which a rigorous development of hyperbolic geometry can engender in students. Firstly, a geometry is understood in the abstract. Secondly, the distinction between the nature of and an approach to a geometry becomes clear. For example, in analytical geometry, the points in the Euclidean plane are interpreted as pairs of co-ordinates. However, the theorems which can be deduced are those of Euclidean geometry, and thus analytical geometry is a specific approach to Euclidean geometry and not an alternative geometry as such. Thirdly, a need for verifying conjectures is created, because, unlike in Euclidean geometry, some of the theorems in hyperbolic geometry are not intuitively obvious. Fourthly, it becomes apparent that the validity of an argument is not determined on the basis of the 'truth' or 'falsity' of the underlying assumptions, but on the correct application of the laws of logic to these assumptions - the hyperbolic postulate, rather than the proofs, is the cause of our discomfort with the results in hyperbolic geometry. Fifthly, the limitations of diagrams are realised - diagrams aid cognition, "but they are not the whole story - not by a long shot" (Niven in Lindquist 1987:41). For example, in hyperbolic geometry, asymptotic parallel lines are normally depicted by oblique lines, thereby forcing the dissociation of the image of oblique lines with the notion of meeting at some finitely distant point. Lastly, it is seen that under certain interpretations of the term straight line, the postulates of Euclidean geometry cease to hold. This can be illustrated by the fact that if a straight line is interpreted as a great circle on the surface of the sphere, then the resulting geometry will be non-Euclidean -

the properties of such lines are in fact the postulates which Riemann used for his geometry.

9.2.3 Teaching strategies which will enhance the learning of non-Euclidean geometry

The implementation of suitable teaching strategies is crucial to the realisation of those factors which originally motivated the study of non-Euclidean geometry at school level.

Since the development of geometric thought occurs sequentially, students have to exhibit the competencies associated with the level of formal deduction before they can study non-Euclidean geometry meaningfully. This implies that teachers have to take the responsibility of devising a means of assessing their students which is in accordance with the Van Hiele model. For example, students should be able to comprehend the concepts *undefined term*, *definition*, *postulate*, *theorem* and *proof*, and their interrelationship in an axiomatic system such as Euclidean geometry. This competency can be confirmed by setting the students a test item which requires them to classify certain statements either as definitions, postulates or theorems, and to determine from which of these can those which they have classified as theorems be deduced.

Since the new content matter primarily involves new concepts and principles, the most appropriate teaching strategy is the discovery strategy. However, the discovery strategy is fraught with dangers such as the frustration and despondency which may result in students if they repeatedly fail to make a discovery. The use of a variety of strategies will alleviate this problem. For example, the foundations of Euclidean geometry can be suitably introduced by means of a discussion which may either be teacher-led or group-based, whereas the concept of an alternative geometry may be more appropriately introduced by having students interact with manipulative materials and reflect on their findings. This will enable students to construct new knowledge on the basis of their prior knowledge and experiences, and to re-organise these in accordance with the new knowledge.

Sadly, since most examinations in mathematics at school level mainly require the regurgitation of facts and procedures, teachers concentrate on using those strategies which aim at realising only the lower levels of cognition. Students cannot be blamed for not being creative, independent and original, because they are not being given the opportunities in class to develop these qualities. Teachers should therefore encourage debates, written and oral presentations, quizzes and research amongst their students. These are also potentially useful methods of continuously evaluating the students' grasp of the new content.

In all of this, thorough planning plays a pivotal role. The potential benefits for students who study non-Euclidean geometry, such as acquiring a new 'way of seeing', far outweigh the demands which successfully teaching non-Euclidean geometry will make on their teachers.

9.3 RECOMMENDATIONS FOR THE FUTURE

(1) A trial course in non-Euclidean geometry, which could possibly incorporate the content matter and the teaching strategies proposed in this study, should be implemented in the advanced senior secondary mathematics syllabus. The findings from such a course should be used to alter or refine the proposed content matter.

(2) Pre-service and in-service courses for teachers should concentrate on both the didactic and content aspects involved in teaching non-Euclidean geometry. The same can be said about all other topics in mathematics which fall beyond the scope of the syllabus, but which will enable teachers to put the syllabus contents in better perspective.

(3) All students who have been identified as having the pre-requisite knowledge and skills for studying non-Euclidean geometry should be provided with the opportunity of doing so, irrespective of whether they come from previously disadvantaged schools or not. This implies that additional resources such as bursaries or loan schemes, which will enable teachers at these schools to improve their qualifications and to upgrade their skills, will have to be made available.

(4) Despite the many problems that education in South Africa has been plagued with, teachers should become imbued with a spirit of optimism and determination. They should encourage the attainment of high levels of cognition by their students. They should incorporate aspects of the history of mathematics into their teaching to enable their students to learn some valuable lessons in life and the noble qualities of dedication and perseverance. They should evaluate their students on a continuous basis. They should make a special effort to foster positive attitudes towards mathematics and mathematical activities in their students so that more students will be drawn towards careers in mathematics.

With reference to chapter 1, on being asked what the sum of the angles in a triangle is, a person who has gained new insights through studying non - Euclidean geometry, will not answer thoughtlessly that it is 180°, but will first enquire about the nature of the geometry that is being referred to, or the premises that are being used.

We, as teachers and mathematics educators, should fully exploit the opportunity which the study of non - Euclidean presents for empowering our students with a new vision of the world and their place in it.

BIBLIOGRAPHY

Barker, Stephen F. 1964. *Philosophy of Mathematics*. Englewood Cliffs, N.J. : Prentice Hall Inc.

Bell, Frederick H. 1978. *Teaching and Learning Mathematics (In Secondary Schools)*. Dubuque, Iowa : Wm. C. Brown Company

Bishop, Alan J. 1992. Mathematical Values in the Teaching Process, in *Mathematical Knowledge : Its Growth through Teaching*, edited by Alan J. Bishop, Stieg Mellin - Olsen and Joop van Dormolen. Dordrecht, the Netherlands: Kluwer Academic Publishers: 195 - 213.

Bonola, Roberto. 1955. *Non - Euclidean geometry : A critical and historical study of its development*. New York: Dover Publications Inc.

Borasi, Raffaella. 1990. The Invisible Hand Operating in Mathematics Instruction: Students' Conceptions and Expectations, in *Teaching and Learning Mathematics in the 1990s*, edited by Thomas J. Cooney and Christian R. Hirsch. NCTM Inc.: 174 - 182.

Bühler, W.K. 1981. *Gauss : A Biographical Study*. New York: Springer - Verlag Inc.

Crowley, Mary L. 1987. The Van Hiele Model of the Development of Geometric Thought, in *Learning and Teaching Geometry, K-12,* edited by Mary Montgomery Lindquist. NCTM Inc.: 1-16.

De Villiers, M. 1992. Inductive and deductive reasoning: logic and proof, in *Mathematics Education for in-service and pre-service teachers*, edited by M. Moodley, R.A. Njisane and N.C. Presmeg. Pietermaritzburg: Shuter and Shooter: 45-59.

Dreyfus, Tommy and Hadas, Nurit. 1987. Euclid May Stay - and Even Be Taught, in *Learning and Teaching Geometry, K-12*, edited by Mary Montgomery Lindquist. NCTM Inc.: 47-58.

Eves, Howard. 1981. *Great moments in Mathematics (after 1650)*. The Mathematical Association of America.

Faber, Richard L. 1983. *Foundations of Euclidean and Non - Euclidean geometry*. New York: Marcel Dekker Inc.

Gray, Jeremy. 1979. *Ideas of Space: Euclidean, Non - Euclidean and Relativistic.* Oxford: Clarendon Press.

Gray, Jeremy. 1987. The Discovery of Non - Euclidean Geometry, in *Studies in the History of Mathematics, Vol. 26*, edited by Esther R. Phillips. The Mathematical Association of America.

Greenberg, Marvin J. 1974. *Euclidean and Non - Euclidean geometries : Development and History.* San Francisco: W.H. Freeman and Company.

Kline, Morris. 1963. *Mathematics - A Cultural Approach.* Reading, Massachusetts: Addison Wesley Publishing Company Inc.

Kline, Morris. 1972. *Mathematical Thought from Ancient to Modern Times.* New York: Oxford University Press.

Kline, Morris. 1980. *Mathematics : The Loss of Certainty.* New York: Oxford University Press.

Meschkowski, Herbert. 1964. *Non - euclidean geometry.* New York: Academic Press Inc.

Meschkowski, Herbert. 1965. *Evolution of Mathematical Thought.* San Francisco: Holden-Day Inc.

Moise, Edwin E. 1989. *Elementary Geometry from an Advanced Standpoint.* Reading, Massachusetts: Addison-Wesley Publishing Company.

Niven, Ivan. 1987. Can Geometry Survive in the Secondary Curriculum?, in *Learning and Teaching Geometry, K-12*, edited by Mary Montgomery Lindquist. NCTM Inc.: 37-46.

Reid, Constance. 1963. *A long way from Euclid.* New York: Thomas Y. Crowell Company.

Roxburgh, I.W. 1978. Is Space Curved?, in *The Encyclopaedia of Ignorance*, edited by Ronald Duncan and Miranda Weston-Smith. Oxford: Pergamon Press.

Saccheri, Girolamo. 1733. *Euclides Vindicatus.* Edited and translated by George Bruce Halsted, 1986. New York: Chelsea Publishing Company.

South Africa. Department of Education. 1995. *Interim Core Syllabus for Mathematics Higher Grade Standards 8-10.*

Trudeau, Richard J. 1987. *The Non - Euclidean Revolution.* Boston: Birkhäuser.

Usiskin, Zalman. 1987. Resolving the continuing Dilemmas in School Geometry, in *Learning and Teaching Geometry, K-12*, edited by Mary Montgomery Lindquist. NCTM Inc.: 17-31.

Williams, G. 1992. Geometry : discovery and proof, *in Mathematics Education for in-service and pre-service teachers*, edited by M. Moodley, R.A. Njisane and N.C. Presmeg. Pietermaritzburg: Shuter and Shooter: 336 - 351.