COMPACTNESS IN CATEGORIES AND ITS APPLICATION IN DIFFERENT CATEGORIES

by

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Summary

In the paper [HSS] Herrlich, Salicrup and Strecker were able to show that Kuratowski / Mrówka's Theorem concerning compactness for topological spaces could be applied to a wider setting. In this dissertation, which is based on the paper [F_1], we interpret Kuratowski / Mrówka's result in the category **R-Mod**. Chapter One deals mainly with the preliminary definitions and results and we also show that there is a 1-1 correspondence between torsion theories and standard factorisation systems. In Chapter Two we, obtain for every torsion theory T, a theory of T-compactness which is an extension of the definition of compactness found in [HSS]. We then obtain a characterisation of T-compactness under certain conditions on the ring R and torsion theory T. In Chapter Three we examine the class of T-compact R-modules more closely when the ring R is T-hereditary and T-noetherian. We also obtain further characterisations of T-compactness under these additional conditions. In Chapter Four we show that many topological results have analogues in **R-Mod**.

Key Terms.

torsion theory; radical; factorisation structure; hereditary; T-dense; T-closed; T-compact; T-herditary; T-injective; T-noetherian; p-divisible

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INTRODUCTION.

Kuratowski/Mrówka's Theorem states that a topological space X is compact if and only if for every topological space Y, the second projection $\pi_2 : X \times Y \to Y$ is a closed map. In 1987 Herrlich, Salicrup and Strecker published a paper ([HSS]) in which it was shown that this theorem can be interpreted and thus applied in a wider setting than the category of topological spaces. Fay in his paper "Compact Modules" ([F_1]) looked at this categorical interpretation of compactness in the category **R-Mod,** where **R-Mod** denotes the category of R-modules and module homomorphisms. This dissertation is based on the paper [F_1], with a few differences. We note that in [HSS] the definition of compactness was provided relative to a factorisation structure on **X** and many results were proved assuming that **X** was also an hereditary construct (that is, a concrete category over **Set** with the property that that each inclusion of a subset into the underlying set of any **X**-object has an initial lift). However we find that **R-Mod** is not a hereditary construct: Let H′ be the underlying set of H, where H is an R-module. If K is a subset of H′ not containing the identity of H, then the inclusion of K into H′ does not have an initial lift.

In this dissertation we will work mainly in the category **R-Mod**, where R is a ring with unity. After providing some preliminary results and definitions, we show a one-to-one correspondence between torsion theories and standard factorisation structures. It is then possible to obtain, for each torsion theory, a theory of compactness which is an extension of the definition of compactness found in [HSS]. In the case that the torsion theory T is hereditary and the ring R is T-hereditary we obtain a characterisation of T-compact R-modules: an R-module G is T-compact provided that G/T(G) is T-injective. In Chapter Three we look more carefully at the class of T-compact R-modules. We are able to show that this class forms a torsion class for a torsion theory. After demonstrating that the lattice of torsion theories exists, we identify the aforementioned torsion theory in the lattice. We are also able to demonstrate that some of the results in section 4 of [HSS], which are proved for categories which are hereditary constructs, are nevertheless valid in **R-Mod.** After interpreting the notion of Hausdorff spaces in the category **R-Mod**, we then proceed to show that many topological results concerning the relationships between compactness, closedness, and Hausdorffness also hold in this setting.

All definitions and proofs, unless otherwise stated, are taken from the paper [F_1].

CHAPTER ONE

PRELIMINARIES

In this chapter we will first define what is meant by a torsion theory T. It will then be shown that every torsion theory gives rise to a factorisation structure on **R-Mod**. When the factorisation structure satisfies the further condition of being standard, we will see that the converse of the preceding statement also holds.

Although this dissertation is based on the paper [F_1], we find that the paper , "Factorisations, denseness, separation, and relatively compact objects" by Herrlich, Salicrup and Strecker ([HSS]) forms the backbone of this dissertation. We will therefore now look at some of the results from [HSS] that will be referred to in this paper. These are Definitions 1.1, 1.5, 1.6 ; Propositions 1.4 and 1.7 ; and Examples 1.2 and 1.3. This paper, [HSS], has been looked at in detail by M. Siweya in his M.Sc dissertation "On Factorisation Structures, Denseness, Separation and Relatively Compact Objects", University of South Africa, 1994.

All definitions and proofs, unless otherwise stated,are taken from the paper [F_1].

Definition 1.1.

(E, M) is called a factorisation structure on the category $\mathbf{X}$ (where E and M are classes of morphisms) provided that

i) If $e \in E$ and h is an isomorphism, and $h \circ e$ exists, then $h \circ e \in E$. If $m \in M$ and k is an isomorphism, and $m \circ k$ exists, then $m \circ k \in M$.

ii) X has (E, M)- factorisations of morphisms; that is, each morphism f in $\mathbf{X}$ has a factorisation $f = m \circ e$ with $e \in E$ and $m \in M$.

iii) $\mathbf{X}$ has the unique (E, M) diagonalisation property; that is, for each commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "r"'] & B \arrow[d, "s"] \arrow[dl, "d"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

with $e \in E$ and $m \in M$ there exists a unique $d : B \to C$ with $d \circ e = r$ and $m \circ d = s$. ■

Example 1.2.

For any category $\mathbf{X}$ let Iso denote the class of all isomorphisms, and Mor the class of all morphisms. Then (Iso,Mor) and (Mor,Iso) are the trivial factorisation structures. ■

Example 1.3.

Let **Set** and **Grp** denote the categories of sets and functions, and groups and group homomorphisms respectively. In each of the categories **Set**, **Grp** and **R-Mod** (Epi , Mono) is a factorisation structure, where Epi represents the class of epimorphisms and Mono, the class of monomorphisms. (Note that the result for **R-Mod** is not mentioned in [HSS]. However by combining Example 4.5 on page 47 and Theorem 4.8 on page 49 of Blyth, [B_2], we see that this result holds on **R-Mod** as well.) ∎

Later on in this chapter we will define a specific factorisation structure relative to a given torsion theory on the category **R-Mod**.

Proposition 1.4.

If (E,M) is a factorisation structure on **X** , then

i) each of E and M is closed under composition.

ii) each (E, M) - factorisation is unique up to a unique commuting isomorphism. This means that if

$$A \xrightarrow{e_1} A_1 \xrightarrow{m_1} B$$

and

$$A \xrightarrow{e_2} A_2 \xrightarrow{m_2} B$$

are two (E,M)-factorisations of a morphism $f: A \to B$ then A_1 is isomorphic to A_2, that is, there exists an isomorphism $d : A_1 \to A_2$ such that the following diagram commutes.

$$\begin{tikzcd}
 & A_1 \arrow[dd, "d"] \arrow[dr, "m_1"] & \\
A \arrow[ur, "e_1"] \arrow[dr, "e_2"'] & & B \\
 & A_2 \arrow[ur, "m_2"'] &
\end{tikzcd}$$

iii) If $m = n \circ f$ with $m, n \in M$, then $f \in M$.

iv) M is closed under the formation of products and pullbacks. This means that, respectively, if m_1 and m_2 belong to M, then $m_1 \times m_2$ belongs to M and if

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "n"'] & B \arrow[d, "m"] \\
C \arrow[r] & D
\end{tikzcd}$$

is a pullback with $m \in M$, then $n \in M$.

v) A morphism $f \in M$ if and only if for each commutative square

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "r"'] & D \arrow[d, "s"] \arrow[dl, "d"] \\
C \arrow[r, "f"'] & B
\end{tikzcd}$$

(where r and s are arbitrary morphisms while $e \in E$), there exists a (not necessarily unique) $d : D \to C$ such that $r = d \circ e$ and $f \circ d = s$.

vi). A morphism belongs to both E and M if and only if it is an isomorphism. ∎

Definition 1.5.

For any category **X**, let the relation $\sigma \subseteq$ Mor **X** $\times$ Ob **X** given by $e\ \sigma\ Y$ if and only if for each pair of **X**-morphisms f, g with codomain Y, $f \circ e = g \circ e$ implies that $f = g$. Given a class E of **X**-morphisms, E-Sep $= \{Y \mid e\ \sigma\ Y$ for all $e \in E\}$ is called the class of E-separated objects in X. ■

In Chapter 4 we will look more closely at the class E-Sep in the category **R-Mod**, for a particular class E of homomorphisms in **R-Mod**.

Now we need to look at the definition of what is referred to in [HSS] as the "graph" of a morphism, which we obtain from Herrlich and Strecker, [HS], - Definition 18.2, page 115. Note that it is not called a graph in [HS]. This definition will be used, in Proposition 1.7, to produce an equivalent characterisation of E-separated objects.

Definition 1.6.

Let **X** be a category with $f: X \to Y$ any **X**-morphism and $id: X \to X$ the identity morphism on X. Then the graph of f, $\langle id, f \rangle$ is given by the following product diagram

$$\begin{tikzcd} & & X \\ X \arrow[urr, "id"] \arrow[rr, "<id,f>"] \arrow[drr, "f"'] & & X \times Y \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\ & & Y \end{tikzcd}$$

where π_X and π_Y denote the first and second projection morphisms respectively of the product $X \times Y$. ∎

We are now in a position to provide an alternate characterisation of E-separated objects which is obtained from Theorem 2.4, on page 164 of [HSS].

Proposition 1.7.

If (E, M) is a factorisation structure in a finitely complete category $\mathbf{X}$, then for any object Y in $\mathbf{X}$, the following are equivalent :

(i) $Y \in$ E-Sep.

(ii) For each $f: X \to Y$, the graph of f is in M; that is, $<id, f>$ is in M.

(iii) $\Delta_Y : Y \to Y \times Y \in M$, where the morphism Δ_Y is defined via the following product diagram:

$$\begin{tikzcd}
& Y \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Y & & Y \\
& Y \arrow[ul, "id_Y"] \arrow[ur, "id_Y"'] \arrow[uu, "\Delta_Y"'] &
\end{tikzcd}$$

■

As mentioned in the introduction to this dissertation, the main category under study in this chapter is **R-Mod**. However we will need to look at the notions of the kernel and cokernel of a given homomorphism and in order to do that we need to first see what a pointed category (which **R-Mod** is) is.

Definition 1.8. ([AM] , page 78 .)

A category **X** is pointed if there exists an assignment $0_{AB} \in \mathbf{X}(A,B)$ (the set of all **X**-morphisms from A to B) to each pair of objects A, B of **X** subject to the following law :

> Given $f: A \to B$ and $g: C \to D$ (with f,g both **X**-morphisms), then
>
> (i) $0_{BC} \circ f = 0_{AC}$ and,
>
> (ii) $g \circ 0_{BC} = 0_{BD}$. ■

Example 1.9. ([HS], Example 8.11, page 51)

Some examples of pointed categories are :

R-Mod, **Grp**, **Mon** (the category of monoids and monoid homomorphisms), and **pSet** (the category of pointed sets and pointed morphisms, that is, maps which preserve the base point of a pointed set). ∎

We are now ready to define the terms kernel and cokernel. These definitions are central to the notion of exact sequences which are used very often in many of the proofs found in these two chapters.

<u>Definition 1.10.</u> ([HS], page 105, Definition 16.17; and [B_2], page 44)

Let **X** be a pointed category.

If $f: A \to B$ is an **X**-morphism and if 0_{AB} is the unique zero morphism from A to B, then (if it exists) the equaliser of the pair $(f, 0_{AB})$ is called the <u>kernel of</u> f. Thus we have that, given $f: A \to B$ in **X**, a morphism $k: K \to A$ is a kernel of f if and only if

(i) $f \circ k = 0_{KB}$;

(ii) if $g: X \to A$ is such that $f \circ g = 0_{XB}$ then there is a unique morphism $v: X \to K$ such that the diagram

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, "f"] & B \\ & X \arrow[ul, "v"] \arrow[u, "g"'] & \end{tikzcd}$$

commutes.

We will refer to the kernel of f as Ker (f) and write Ker(f) $\cong$ (K,k). If there is no confusion we may write

$\mathrm{Ker}(f) = K$ or maybe just $\mathrm{Ker}\ (f) = k$.

Dually the <u>cokernel of</u> $f\colon A \to B$ is the coequaliser of the pair $(0_{AB}, f)$. Hence we have that a morphism $c\colon B \to C$ is a cokernel of f if and only if

(i) $c \circ f = 0_{AC}$;

(ii) if $b\colon B \to D$ is such that $b \circ f = 0_{AD}$ then there is a unique morphism $u\colon C \to D$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "c"] \arrow[dr, "b"'] & C \arrow[d, "u"] \\ & & D \end{tikzcd}$$

commutes. We will refer to the cokernel of f as $\mathrm{Coker}(f) \cong (C, c)$ and we may sometimes say $\mathrm{Coker}(f) = C$ or just $\mathrm{Coker}(f) = c$. ∎

<u>Example 1.11.</u> ([HS], Example 16.18, page 105)
If $f\colon A \to B$ is a morphism in **Grp** or **R-Mod** and e is the identity element of B, then $(f^{-1}[\{e\}], i) \cong \mathrm{Ker}\ (f)$ and $(B/f[A], \#) \cong \mathrm{Coker}\ (f)$, where i is the inclusion $i\colon f^{-1}[\{e\}] \to A$ and $\#$ is the natural morphism from B to the quotient. ∎

From Blyth, [B_1], page 24, we find the definition of an "exact sequence" in the category **R-Mod**. To enable this definition to be of the form that we will require in this dissertation, we also combine Blyth's definition with a result from Lang, [L_2]. However Lang's result holds for groups. Since every R-module is an abelian group and every module homomorphism is a group homomorphism, the following definition holds in the category **R-Mod** as well.

<u>Definition 1.12.</u>

Let

$$G' \xrightarrow{f} G \xrightarrow{g} G''$$

be a sequence of homomorphisms in **Grp**. We say that this sequence is <u>exact</u> if $\mathrm{Im}(f) = \mathrm{Ker}(g)$. A sequence of homomorphisms having more than one term, like

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \ldots G_{n-1} \xrightarrow{f_{n-1}} G_n$$

is called <u>exact</u> if it is exact at each joint, that is, if $\mathrm{Im}\,(f_i) = \mathrm{Ker}\,(f_{i+1})$ for each $i = 1, 2, \ldots, n-2$.
We find that exact sequences of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

are called <u>short exact sequences.</u> We can see that the above sequence is exact if and only if f is injective, $\mathrm{Im}(f) = \mathrm{Ker}(g)$, and g is surjective. Furthermore if $H = \mathrm{Ker}(g)$, then the aforementioned sequence is essentially the same as the sequence

$$0 \longrightarrow H \xrightarrow{i} M \xrightarrow{n} M/H \longrightarrow 0$$

where i is the inclusion monomorphism and n is the canonical epimorphism. More precisely, from [L_2], page 14, we find that there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r, "i"] & M \arrow[r, "n"] & M/H \arrow[r] & 0
\end{tikzcd}$$

in which the homomorphisms in the columns are isomorphisms, and the rows are exact. ∎

We now look at the definition of a torsion theory on **R-Mod**. Since we are primarily concerned in this dissertation with the notion of compactness in **R-Mod** and **Grp**, this following definition is vital to the rest of this dissertation. We find in Proposition 1.1.31 that every torsion theory gives rise to a factorisation structure (E , M), on **R-Mod**, which in turn enables us to define compactness on **R-Mod** in a similar manner to that of [HSS]. By combining the definition of a torsion theory found on page 4 of Stenstrom, [S_2], with Propositions 2.1 and 2.2 also of [S_2], we obtain the following definition.

Definition 1.13.

A torsion theory on a category of R-modules consists of a pair of classes ($\underline{T}$, $\underline{F}$) satisfying the following properties :

i) $\underline{T}$ is closed under the formation of direct sums, epimorphic images, and group extensions (See Definition 1.14).

ii) $\underline{F}$ is closed under the formation of products, submodules, and group extensions.

iii) An R-module belongs to $\underline{T}$ provided Hom(A,B) = 0 for every R-module belonging to $\underline{F}$, and dually, A belongs to $\underline{F}$ provided Hom(B,A) = 0 for every R-module B belonging to $\underline{T}$. ■

From Lambek, [L_1], page 2, we have the following definition of "closed under group extensions".

<u>Definition 1.14.</u>

We say that $\underline{T}$ is <u>closed under group extensions</u> if, whenever B is a submodule of an R-module M and both B and M/B are in $\underline{T}$, then so is M. ■

Closely related to the notion of a torsion theory is a radical. We will see later on that in **R-Mod** every radical gives rise to a factorisation structure and vice versa. So let us define exactly what is meant by a radical on **R-Mod**.

<u>Definition 1.15.</u> ([L_1], page 2)

An object function T : **R Mod** $\rightarrow$ **R Mod** is called a <u>radical</u> if it satisfies the following three conditions :

(i) $T(M) \subseteq M$

(ii) If f is a homomorphism from M to N, that is, $f: M \rightarrow N$, then $f[T(M)] \subseteq T(N)$.

(iii) $T[M/T(M)] = 0$.

for all R-modules M and N and all homomorphisms f.

If conditions (i) and (ii) are satisfied, then T is called a preradical. See for example [FOW_1], page 39. Furthermore from Stenstrom, [S2], page 1, we find that a preradical T is idempotent if $T(T(M)) = T(M)$ for every R-module M. ∎

Before going on to provide some examples of radicals on **R-Mod** we need to first of all look at the following definitions which will be required in the discussion of these examples.

Definitions 1.16. ([R_1], pages 102 and 214)

Let R be any ring with identity not equal to zero.

(i) An element s of R is regular if $rs \neq 0$ and $sr \neq 0$ for all $r \neq 0$.

(ii) An element x of an R-module M is divisible by r in R if $rg = x$ for some g in R. The R-module M is divisible by r if each element of M is divisible by r. Finally M is a divisible R-module if M is divisible by every regular element of R. ∎

Example 1.17. ([S_2], page 3)

Let A be an integral domain. For every A-module M let t(M) denote the submodule of M consisting of all elements of M that have finite order, and d(M) the maximal divisible submodule of M. Then both t and d are idempotent radicals.∎

Example 1.18. ([S_2] , page 3)

Let A be an arbitrary ring. For each right A-module M we let s(M) denote the sum of all submodules of M which have no nonzero submodules; and let r(M) denote the intersection of all maximal proper submodules of M. Then s is a preradical while r is a radical. ∎

We are now ready to get back to the original paper, [F_1]. The following result describes the important link that exists between torsion theories and radicals and will be referred to in many of the results in this chapter.

Proposition 1.19. ([L_1], Proposition 0.1, page 2)

Every torsion theory ($\underline{T}$, $\underline{F}$) determines an idempotent radical T in the following way : For any R-module M, let T(M) be the sum of all submodules of M which belong to $\underline{T}$. This is equivalent to allowing T(M) to be the intersection of all submodules K of M for which M/K is in $\underline{F}$. ∎

We will now define exactly what we mean by "torsion" R-modules or classes and "torsion-free" R-modules or classes in this dissertation.

<u>Definition 1.20.</u>

Let ($\underline{T}$, $\underline{F}$) be a given torsion theory. An R-module A is called T-<u>torsion</u> provided TA = A, that is, A belongs to $\underline{T}$. A is called T-<u>torsion-free</u> provided TA = 0, that is, A belongs to $\underline{F}$. We shall call an R-module simply torsion or torsion-free, dropping the label " T- ". $\underline{T}$ will be called the <u>torsion</u> class and $\underline{F}$ the <u>torsion - free</u> class of the torsion theory ($\underline{T}$, $\underline{F}$). We shall refer to the torsion theory ($\underline{T}$, $\underline{F}$) by simply T. ∎

The following type of torsion theory is very important because we will be able to obtain many results in Chapters 2, 3 and 4 of this dissertation when the torsion theory satisfies this extra condition of being hereditary. In fact most of the characterisations of compactness are obtained when the torsion theory concerned is hereditary.

<u>Definition 1.21.</u>

A torsion theory T is called <u>hereditary</u> when its torsion class is closed under the formation of submodules. ∎

We can look at some examples of torsion theories.

<u>Example 1.22.</u>

Let R be an integral domain. Considering the radical t of

Example 1.17, we let

$\underline{T}$ = { M | M is an R-module and t(M) = M }   and

$\underline{F}$ = { M | M is an R-module and t(M) = 0 }.

Then from the remarks preceding Proposition 2.3 on page 6 of Stenstrom, [S_2], we know that ($\underline{T}$, $\underline{F}$) is a torsion theory which is hereditary because the torsion class is closed under formation of submodules. ■

Given any class $\underline{C}$ of R-modules, the following definition shows us how $\underline{C}$ can "generate" or "cogenerate" a torsion theory.

Definition 1.23. ([S_2] , page 5.)

Any given class $\underline{C}$ of R-modules generates a torsion theory in the following way. Let $\underline{T}$ and $\underline{F}$ be the two classes of R-modules described below.

$\underline{F}$ = { F | F is an R-module and Hom (C , F) = 0 for all C $\in$ $\underline{C}$.}

$\underline{T}$ = { T | T is an R-module and Hom (T , F) = 0 for all F $\in$ $\underline{F}$.}

Then ($\underline{T}$, $\underline{F}$) is a torsion theory and $\underline{T}$ is the smallest class of torsion R-modules containing $\underline{C}$.

Dually $\underline{C}$ cogenerates a torsion theory ($\underline{T}$, $\underline{F}$) such that $\underline{F}$ is the smallest class of torsion-free R-modules containing $\underline{C}$, that is

$\underline{T}$ = { T | T is an R-module and Hom (T,C) = 0 for all C $\in$ $\underline{C}$.}

$\underline{F}$ = { F | F is an R-module and Hom (T,F) = 0 for all T $\in$ $\underline{T}$.}

■

To provide a description of a well known hereditary torsion theory we need to first look at the notion of an "essential" submodule of an R-module.

Definition 1.24. ([GH], page 127)
Let M be a submodule of an R-module K such that M has a non-zero intersection with every non-zero submodule of K. Then M is called an essential submodule of K. Note that in [GH] M is referred to as a large submodule of K. In the rest of this chapter we will refer to the corresponding inclusion homomorphism $i: M \to K$ as an essential embedding. ∎

Let us now look at the Goldie torsion theory.

Example 1.25. ([S_2], page 9)
Let R be an arbitrary ring and let $\underline{C}$ be the class of R-modules of the form M/L where L is an essential submodule of M. The torsion theory generated by $\underline{C}$ is called the Goldie torsion theory and it is an hereditary torsion theory. ∎

The following is an example of a torsion theory that is not hereditary.

Example 1.26. ([S_2], page 9)
Let A be an integral domain with classes $\underline{T}$ and $\underline{F}$, where $\underline{T}$ is the class of divisible A-modules, and $\underline{F}$ the class of reduced

A-modules, that is, A-modules which have no nonzero divisible submodules. Then $(\underline{T}, \underline{F})$ is an example of a non-hereditary torsion theory. ∎

To provide a characterisation of hereditary torsion theories we will need to look at the definition of a "left exact" radical first.

<u>Definition 1.27.</u> ($[G_1]$, page 213)
A subfunctor T of the identity functor on **R-Mod** is said to be <u>left exact</u> if and only if $T(N) = N \cap T(M)$ for every submodule N of an R-module M. ∎

<u>Example 1.28.</u> ($[S_2]$, page 3)
The radical t mentioned in Example 1.17 is left exact while d is not. The preradical s of Example 1.18 is also left exact. ∎

The following proposition provides us with another way to describe hereditary torsion theories.

<u>Proposition 1.29.</u>
A torsion theory T is hereditary if and only if the radical T is left exact.

Proof.

See [S_1] , Proposition 2.6 on page 8. ■

It was mentioned earlier on, in the introduction, that we would show that a torsion theory T gives rise to a factorisation structure. To enable us to do that, we need to first look at the definition of a "T-dense" homomorphism and a "T-closed" embedding, for a given torsion theory, T.

Definition 1.30.

A homomorphism having a torsion cokernel is called a T-dense homomorphism. A monomorphism having a torsion-free cokernel is called a T-closed embedding. ■

We now prove that every torsion theory gives rise to a factorisation structure. This result is very important because it enables us to obtain a definition of compactness in Chapter Two. Another reason for its importance is that the result in this section concerning the relationship between idempotent radicals and a certain type of factorisation structure also depends on the following proposition.

Proposition 1.31.

Every torsion theory T determines a (T-dense, T-closed embedding) factorisation structure on **R -Mod**.

Proof.

Let E be the class of T-dense homomorphisms, and M the class of T-closed embeddings. Using Definition 1.1 we first show that condition (i) holds. So suppose that $e: A \to B \in E$ and $h: B \to C$ is an isomorphism. Then we must show that $h \circ e \in E$. That is $C/h \circ e(A)$ must be shown to be a torsion R-module. Now $C \cong B$ and $h(e(A)) \cong e(A)$ because h is an isomorphism. Therefore $C/(h \circ e(A)) \cong B/e(A)$ which is torsion because $e \in E$.

On the other hand if $m: A \to B \in M$ and $h: G \to A$ is an isomorphism we need to show that $m \circ h \in M$. That is, $B/m(h(G))$ must be shown to be torsion-free. Now $h(G) \cong A$ and $A \cong m(A)$ since m is a monomorphism. Therefore $B/m(h(G)) \cong B/m(A)$ which is torsion-free because $m \in M$.

We now show that part (ii) of Definition 1.1 holds. So let $f: A \to B$ be an arbitrary homomorphism and let us consider the following commutative diagram

$$\begin{tikzcd}
& & B/X \arrow[r, "m"] & Q & \\
& A \arrow[r, "f"] \arrow[d, "r"'] & B \arrow[r, "q"] \arrow[u, "n"] & B/f(A) \arrow[r] \arrow[u, "p"] & 0 \\
0 \arrow[r] & f(A) \arrow[r, "k"'] & X \arrow[r, "v"'] \arrow[u, "t"] \arrow[ur, phantom, "PB"] & T[B/f(A)] \arrow[r] \arrow[u, "s"] & 0
\end{tikzcd}$$

Fig. 1.1.

where : $Q = [B/f(A)] / [T(B/f(A))]$,

q is the canonical epimorphism,

r is the epimorphism induced by f,

s is the inclusion homomorphism,

v is the pullback of q along s while t is the pullback of s along q, and

n and p are the canonical epimorphisms.

Let us now look at the derivation of $k: f(A) \to X$, and later, of $m: B/X \to Q$. Consider the following pullback diagram obtained from Fig.1.1. The homomorphism i is the inclusion of $f(A)$ into B while 0 represents the zero homomorphism.

$$\begin{tikzcd}
f(A) \arrow[dr, "k"] \arrow[drr, "0"] \arrow[ddr, "i"'] & & \\
& X \arrow[r, "v"] \arrow[d, "t"] & T(B/f(A)) \arrow[d, "s"] \\
& B \arrow[r, "q"'] & B/f(A)
\end{tikzcd}$$

Now $s \circ 0 = 0$ and $q \circ i = 0$. Therefore by the definition of a pullback, there exists a unique $k: f(A) \to X$ such that $t \circ k = i$ and $v \circ k = 0$. Now i and t are both monomorphisms. Since the first factor of a monomorphism is a monomorphism, we find that k is also a monomorphism. Since v is the pullback of q along s and q is an epimorphism, we have that v will also be an epimorphism. Now Ker $(v) \cong f(A) \cong$ Im (k), hence the last row of Fig. 1.1 is a short exact sequence. Therefore we find from

Definition 1.12 that $T(B/f(A)) \cong X/f(A)$. Hence we see that $Q = [B/f(A)]/[T(B/f(A))] \cong [B/f(A)]/[X/f(A)] \cong B/X$. Thus m, in Fig. 1.1, represents this isomorphism.

Now, since $X/f(A) \cong T(B/f(A))$, we can see that the homomorphism k and thus $k \circ r$ has a torsion cokernel. Furthermore $Q = [B/f(A)]/[T(B/f(A))]$, which is torsion-free because T is a radical. This means that $T(Q) = 0$. Therefore $T(B/X) = 0$. So the homomorphism $k \circ r$ has a torsion cokernel while t has a torsion-free cokernel. This means that $f: A \rightarrow B$ can be factorised into

$$A \xrightarrow{k \circ r} X \xrightarrow{t} B ,$$

which is a (T-dense , T-closed embedding) - factorisation.

We will now show that part (iii) of Definition 1.1 holds. That is, the diagonalisation condition is satisfied. Consider the following commutative diagram :

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "t"'] & B \arrow[r, "n"] \arrow[d, "s"] \arrow[dl, "d"] & B/e(A) \arrow[d, "u"] \\
C \arrow[r, "m"'] & D \arrow[r, "q"'] & D/C
\end{tikzcd}$$

Fig 1.2.

where : e is a T -dense homomorphism ,

m is a T -closed homomorphism ,

t and s are arbitrary homomorphisms making the left hand square commute, and

n and q are the canonical epimorphisms.

We now explain how the homomorphism u is obtained. Because m is a T-closed embedding, the R-module C is isomorphic to a submodule of D. Hence m can be seen as an inclusion homomorphism. Now $q \circ s \circ e = q \circ m \circ t$ because $s \circ e = m \circ t$. But $q \circ m = 0$ since q is the cokernel of m; therefore $q \circ s \circ e$ is equal to zero as well. Hence e(A) is a submodule of the kernel of $q \circ s$.

$$\begin{tikzcd}
A \arrow[r, "e"] & B \arrow[r, "n"] \arrow[dr, "q \circ s"'] & B/e(A) \arrow[d, "u"] \\
 & & D/C
\end{tikzcd}$$

We consider the commutative diagram above where the given homomorphisms are obtained from Fig 1.2. Clearly, n is the cokernel of e. We know that $q \circ s \circ e = 0$. By the definition of cokernel, there exists a unique homomorphism $u: B/e(A) \to D/C$ which makes the above diagram commute. We know that $T(B/e(A)) = B/e(A)$ and $T(D/C) = 0$. By property (ii) of Definition 1.15 we find that $u[T(B/e(A))] \subseteq T(D/C)$. That is $u(B/e(A)) = 0$. This means of course that u is actually the zero homomorphism. Therefore $q \circ s = u \circ n = 0$. In order to obtain the unique diagonal morphism, let us consider another diagram obtained from Fig 1.2.

$$\begin{tikzcd}
C \arrow[r, "m"] & D \arrow[r, "q"] & D/C \\
B \arrow[u, "d"] \arrow[ur, "s"'] & &
\end{tikzcd}$$

We can see that m is the kernel of q. Since $q \circ s = 0$, by the definition of kernel, there exists a unique $d: B \to C$ such that

$m \circ d = s$. Now $m \circ d \circ e = s \circ e = m \circ t$, and since m is a monomorphism $d \circ e = t$. So we have found a unique $d: B \to C$ such that $d \circ e = t$ and $m \circ d = s$.

We have thus shown that (T-dense , T-closed embedding) is a factorisation structure on **R-Mod**. ■

The following result can be seen as a "weak" converse of the above proposition. In fact this following result is building up to a stronger result (namely Corollary 1.35) which will provide a converse to Proposition 1.31 provided that the factorisation structure satisfies some additional conditions.

Proposition 1.32.

Each (E , M) -factorisation system on **R-Mod** (where M consists of monomorphisms) gives rise to an idempotent preradical.

Proof.

Let (E , M) be such a factorisation system on **R-Mod**. We need to find an idempotent preradical arising from this system. Let A be an arbitrary R-module. We define T(A) to be the intermediate object in the (E , M) -factorisation of $0 \to A$, namely,

$$0 \xrightarrow{e} T(A) \xrightarrow{m} A$$

where T(A) is taken to be a submodule of A , and m the corresponding inclusion homomorphism. This can be done since M is assumed to consist of monomorphisms, that is, of injective homomorphisms.

Let $f : K \to N$ be any module homomorphism. We need to show that $f[T(K)] \subseteq T(N)$. Consider the composition

$$0 \xrightarrow{i_1} T(K) \xrightarrow{i_2} K \xrightarrow{f} N$$

where i_1 and i_2 are the inclusion homomorphisms. This can be factorised as

$$0 \xrightarrow{e_1} T(N) \xrightarrow{m_1} N$$

where e_1 belongs to E, and m_1 to M.

Let us look at the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r, "i_1"] \arrow[dd, "e_1"'] & T(K) \arrow[d, "i_2"] \arrow[ddl, "d"'] \\ & K \arrow[d, "f"] \\ T(N) \arrow[r, "m_1"'] & N \end{tikzcd}$$

where the homomorphisms have been described in the previous paragraph.

By Proposition 1.4(v), there exists a unique homomorphism $d: T(K) \to T(N)$ such that $m_1 \circ d = f \circ i_2$ and $d \circ i_1 = e_1$. Then this means that d is just the restriction of f to $T(K)$. Hence f must map $T(K)$ into $T(N)$, that is, $f[T(K)] \subseteq T(N)$. Since the two conditions in Definition 1.15 are satisfied, T is a preradical.

We now show that T is idempotent. Given an R -module A, consider the (E, M) -factorisation

$$0 \xrightarrow{e} T(A) \xrightarrow{m} A$$

of $0 \to A$, and then the (E,M)-factorisation of $e: 0 \to T(A)$, namely,

$$0 \xrightarrow{e_1} T(T(A)) \xrightarrow{m_1} T(A)$$

From Proposition 1.4 (iii) (dual), it follows that $m_1 \in E$. Since also $m_1 \in M$ it follows (from Proposition 1.4 (vi)) that m_1 is an isomorphism, and hence that $T(A) = T(T(A))$. ■

Proposition 1.1 of $[F_1]$ states that torsion theories are in one-to-one correspondence with factorisation structures. However we find that this result cannot be true, unless the factorisation system satisfies a further condition. In $[DG_2]$ it is shown that the result holds when the closure operator resulting from the factorisation structure is "standard". Our factorisation structure would need to satisfy a similar condition. The following definition of a "standard factorisation system" seems to be the missing ingredient of Proposition 1.1 of $[F_1]$ and plays the same role in our proof that the "standard" closure operator does in $[DG_2]$. With this definition we will be able to prove a one-to-one correspondence between the standard factorisation structures and torsion theories.

Definition 1.33.

Let (E, M) be a factorisation system on R-Mod with each member of M being a monomorphism. Given a module A and a submodule B of A, let $C_A(B)$ denote the intermediate object in the (E, M) - factorisation of the inclusion $i: B \to A$ as represented by this diagram

$$\begin{tikzcd} B \arrow[rr, "i"] \arrow[dr, "e"'] & & A \\ & C_A(B) \arrow[ur, "m"'] & \end{tikzcd}$$

where $e \in E$ and $m \in M$, where the latter is the inclusion homomorphism.

Now a factorisation system (E, M) is called standard provided for each R-module K and each submodule N of K it holds that $q(C_K(N)) = C_{K/N}(0)$, where q is the canonical homomorphism $q : K \to K/N$. ■

Note that wherever q is mentioned in the following proposition it refers to the above homomorphism. The following result demonstrates the relationship between standard factorisation systems and idempotent radicals, and hence, torsion theories.

Theorem 1.34.

There is a one-to-one correspondence between idempotent radicals and standard factorisation systems.

Proof

We will first show that every idempotent radical gives rise to a standard factorisation system (E^T, M^T). Thereafter we will prove that every standard factorisation system (E,M) gives rise to an idempotent radical T. It will then be demonstrated that if T gives rise to (E^T, M^T) and (E^T, M^T) gives rise to T′ then T = T′. Finally we show that if (E , M) gives rise to T and T gives rise to (E^T, M^T) then $(E, M) = (E^T, M^T)$.

So suppose that T is an idempotent radical. Let E^T consist of all homomorphisms which have T-torsion cokernel and M^T of all monomorphisms with T-torsion-free cokernels. We have already seen (in Proposition 1.31) that (E^T, M^T) is a factorisation system on **R-Mod**. It must be shown that (E^T, M^T) is standard. Let K be an arbitrary R-module and N a submodule of K. We consider the (E^T, M^T) -factorisations of the following two inclusion homomorphisms $f: N \to K$ and $0 \to K/N$. That is,

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr, "e_1"'] & & K \\ & C_K(N) \arrow[ur, "m_1"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "e_2"'] & & K/N \\ & C_{K/N}(0) \arrow[ur, "m_2"'] & \end{tikzcd}$$

where e_1, $e_2 \in E^T$ and m_1, $m_2 \in M^T$. We need to show that $q(C_K(N)) = C_{K/N}(0)$. We first show that $C_{K/N}(0) = T(K/N)$. We consider the following decomposition of $0 \to K/N$. That is,

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "e"'] & & K/N \\ & T(K/N) \arrow[ur, "m"'] & \end{tikzcd}$$

where m is the inclusion monomomorphism.
The radical T is idempotent, therefore we have that $T(T(K/N)/0) \cong T(T(K/N)) = T(K/N)$. Hence we can see that the homomorphism e has a T-torsion cokernel and is therefore an element of E^T. Also $T((K/N)/T(K/N)) = 0$ because T is a radical. This means that m has a T-torsion-free cokernel and is an element of M^T. So (e,m) is an (E^T,M^T) - factorisation of the homomorphism $0 \to K/N$. But (e_2,m_2) is also an (E^T,M^T) - factorisation of $0 \to K/N$, therefore it is true that $C_{K/N}(0) \cong T(K/N)$.
We now consider the (E^T,M^T) - factorisation of the inclusion homomorphism $f\colon N \to K$. Let us consider the following diagram which is similar to Fig 1.1. Here X represents the intermediate object of the (E^T,M^T) -factorisation of f and is therefore equal to $C_K(N)$.

$$\begin{tikzcd}
& & K/X \arrow[r, "m"] & Q = (K/N)/T(K/N) & \\
& N \arrow[r, "f"] \arrow[d, "r"'] & K \arrow[u, "n"] \arrow[r, "q"] & K/N \arrow[u, "p"'] \arrow[r] & 0 \\
0 \arrow[r] & f(N) \arrow[r, "k"] & X \arrow[u, "t"] \arrow[r, "v"] & T(K/N) \arrow[u, "s"'] \arrow[r] & 0
\end{tikzcd}$$

It has already been mentioned in Proposition 1.31 that the homomorphisms n, p, q and v are epimorphisms while k, t, and s are monomorphisms and m is an isomorphism. We also have f and r as monomorphisms. So it turns out that $f(N)$ is actually N. Now $s \circ v(X) = q \circ t(X)$. Since t and s are just the inclusion homomorphisms, we have $q(X) = v(X) = T(K/N)$. This means that $q(C_K(N)) = T(K/N)$ which in turn shows that $q(CK(N)) = C_{K/N}(0)$. In other words every indempotent radical T gives rise to a standard factorisation structure.

Conversely if $(E\ ,M)$ is a given standard factorisation structure on **R-Mod** we define a preradical T as in the proof of Proposition 1.32. It was shown in the latter that T is an idempotent preradical. To prove that T is a radical we will show that condition (iii) of Definition 1.15 is satisfied.

Let us look at the (E,M) -factorisations of the two inclusion homomorphisms $i\colon T(K) \to K$, and $0 \to K/T(K)$, where K is any R-module. The first diagram shows the factorisation of $i\colon T(K) \to K$ while the second diagram demonstrates the factorisation of $0 \to K/T(K)$.

$$\begin{tikzcd} T(K) \arrow[rr, "i"] \arrow[dr, "e_1"'] & & K \\ & T(K) = C_K(T(K)) \arrow[ur, "m_1"'] & \end{tikzcd}$$

(Note that $e_1 \in E$: If $e\colon 0 \to T(K)$, then $e_1 \circ e = e$ and it follows from the dual of Proposition 1.4(iii) that $e_1 \in E$.)

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "e_2"'] & & K/T(K) \\ & T(K/T(K)) = C_{K/T(K)}(0) \arrow[ur, "m_2"'] & \end{tikzcd}$$

Since (E,M) is a standard factorisation system , we know that $q(C_K(T(K))) = C_{K/T(K)}(0)$. Hence $q(T(K)) = T(K/T(K))$. But $q(T(K))=0$, therefore $T(K/T(K)) = 0$, that is, T is a radical.

We now show the one-to-one correspondence between idempotent radicals and standard factorisation systems .

Firstly let T be an idempotent radical. This gives rise to a standard factorisation system (E^T,M^T), as described already, which in turn gives rise to a certain idempotent radical T', as described previously. It has to be shown that $T = T'$. Choose any R-module K. Now $T'(K)$ is found as the intermediate object in the (E^T,M^T)-factorisation of $0 \to K$, that is,

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "e'"'] & & K \\ & T'(K) \arrow[ur, "m'"'] & \end{tikzcd}$$

where $e' \in E^T$ and $m' \in M^T$.

Now let us look at the following diagram

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "e"'] & & K \\ & T(K) \arrow[ur, "m"'] & \end{tikzcd}$$

where m denotes the inclusion homomorphism.

We see that $T((T(K))/0) = T(T(K)) = T(K)$ because T is idempotent. Hence $e \in E^T$. Considering m, we find that $T(K/m(T(K))) \cong T(K/T(K)) = 0$. Thus $m \in M^T$. Thus the above diagram also gives an (E^T,M^T) - factorisation of $0 \to K$. Hence $T(K) \cong T'(K)$ for all R-modules K.

On the other hand if (E,M) is a standard factorisation structure we obtain (as proven earlier on) an idempotent radical, T, associated with (E,M). We then form the standard factorisation system (E^T, M^T) where E^T is the class of homomorphisms with T-torsion cokernels while M^T is that class of monomorphisms which have T-torsion-free cokernels. We need to show that $(E,M) = (E^T,M^T)$. Let K and N be two R-modules with $f: K \to N \in M^T$. This means that f is a monomorphism and K can be seen as a submodule of N. We can thus form the R-module N/K. Then N/K will be T-torsion-free, that is, $T(N/K) = 0$. We now consider the (E, M)-factorisation of the homomorphism $f: K \to N$ which is

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "e"'] & & N \\ & L \arrow[ur, "m"'] & \end{tikzcd}$$

Fig. 1.3

(where L is of course just $C_N(K)$, assuming that f is inclusion) and $e: K \to L \in E$ and $m: L \to N \in M$; and, the factorisation of the homomorphism $0 \to N/K$ which is

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "e'"'] & & N/K \\ & T(N/K) \arrow[ur, "m'"'] & \end{tikzcd}$$

with $e' \in E$ and $m' \in M$.

By the definition of the idempotent radical T, the intermediate object in the (E,M)-factorisation of $0 \to N/K$ is $T(N/K)$. Since (E , M) is a standard factorisation system, we have that $q(L) = T(N/K) = 0$. Hence $L = q^{-1}(T(N/K)) = K$. Now if we consider Fig 1.3 again,

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "id"'] & & N \\ & L = K \arrow[ur, "m"'] & \end{tikzcd}$$

we find that this is another (E , M) -factorisation of the homomorphism $f\colon K \to N$. Hence the above homomorphism is the same as $m\colon L \to N$. Thus $f\colon K \to N \in M$. Therefore $M^T \subseteq M$.

On the other hand if the homomorphism $f\colon K \to N \in M$, then

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "id"'] & & N \\ & K \arrow[ur, "f"'] & \end{tikzcd}$$

is an (E,M) - factorisation system , where id is the identity on K. Since the factorisation system is standard, $q(K) = T(N/K)$. But $q(K) = 0$, therefore $T(N/K) = 0$. This means that the homomorphism $f\colon K \to N$ has a T -torsion-free cokernel and will therefore belong to M^T. Hence $M \subseteq M^T$.

We can thus see that $M = M^T$. The dual of Proposition 1.4(v) shows that $(E,M) = (E^T,M^T)$. ∎

As a corollary to Theorem 1.34 we can prove the following result which is very close to Proposition 1.1 of [F_1].

Corollary 1.35.

Torsion theories are in one-to-one correspondence with standard factorisation structures.

Proof.

From Proposition 2.3 on page 6 of [S_2], we know that torsion theories are in one-to-one correspondence with idempotent radicals. On the other hand from Theorem 1.34 we know that idempotent radicals are in one-to-one correspondence with standard factorisation structures. It therefore follows that torsion theories are in one-to-one correspondence with standard factorisation structures. ∎

CHAPTER TWO

COMPACT R-MODULES

In this chapter we will define the notion of compactness in the category **R-Mod**. We will see that this definition depends on the factorisation system discussed in Proposition 1.31. It has been mentioned in the introduction that in [HSS] compactness was defined for an hereditary construct but **R-Mod** is not such a construct. However our definition of compactness will still enable us to get results similar to those found in section 4 of [HSS]. One of our results in this chapter is Theorem 2.18 in which a characterisation of T-compactness is obtained under certain conditions on the ring R and torsion theory T. We will also consider in Corollary 2.26 the application of this theorem in the category **Ab**, consisting of abelian groups and the homomorphisms between them.

We start with the definition of an "injective" R-module. We will then look at a special kind of R-module, called a "T-injective" R-module which will enable us to obtain characterisations of compactness in **R-Mod.** We will then obtain various characterisations of "T-injective" R-modules in this chapter.

Definition 2.1. ([L_2] , page 102)
An R-module Q is called injective if it satisfies the

following condition :

Given any R-module M and a submodule M′ of M, and a homomorphism $f: M' \to Q$ there exists an extension of this homomorphism to M. That is, there exists $h: M \to Q$ making the following diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, "f"'] & M \arrow[dl, "h"] \\ & Q & \end{tikzcd}$$

commute , where i is the inclusion homomorphism. ∎

Let us look at an example of an injective R-module which is familiar to us.

<u>Example 2.2.</u> ([GH] , page 115)

The Z-module Q is an injective left Z-module, where Z is the ring of integers and Q is the group of rationals. ∎

Let us now look at a specific type of R-module associated with a given torsion theory T. As mentioned in the beginning of the section this definition will enable us to provide characterisations of T-compact R-modules.

<u>Definition 2.3.</u> ([L_1] , Proposition 0.5. , page 8)

An R-module C is called <u>T-injective</u> (where T is a torsion theory) provided C is injective relative to the class of T-dense embeddings (that is, embeddings which have a torsion cokernel). This means that C is a T-injective R-module if, given an R-module B with a submodule A such that B/A is torsion, any homomorphism $f: A \to C$ can be extended to a homomorphism $g: B \to C$ so that the following diagram

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

(where i represents the inclusion) is commutative. (Note that in [L_1], Lambek refers to T-injective R-modules as divisible R-modules.) ∎

We can see clearly that every injective R-module is T-injective. The following proposition will enable us to determine whether a certain R-module is T-injective by just considering the T-dense left ideals of the ring R. The proof of this proposition has been adapted from Golan, [G_1], page 77.

Proposition 2.4.

Let T be any hereditary torsion theory. If M is an R-module such that any homomorphism from a T-dense left ideal I of R to M can be extended to a homomorphism from R to M, then M is T-injective.

Proof.

Let N be a T-dense submodule of an R-module N′ and let

$\alpha: N \to M$ be a homomorphism from N to M. We need to show that α can be extended to N'. Consider the set, S, of all pairs (W, β), where W is a submodule of N' containing N and β is a homomorphism from W to M extending α as shown in the following diagram

$$\begin{tikzcd} N \arrow[r, "i_1"] \arrow[d, "\alpha"'] & W \arrow[r, "i_2"] \arrow[dl, "\beta"] & N' \\ M & & \end{tikzcd}$$

where i_1 and i_2 are the usual inclusion homomorphisms.

Since N is a submodule of N' containing N and α is a homomorphism from N to M, we see that $(N, \alpha) \in S$. So S is non-empty. We can partially order S by setting $(W, \beta) \le (W', \beta')$ if and only if $W \subseteq W'$ and β is the restriction of β' to W. We will now show that S has the property that every totally ordered subset of S has a maximal element. Let K be a non-zero subset of S which is totally ordered. Suppose that $K = \{ (W_i, \beta_i) \mid i \in I \}$. Then set $W = \cup_{i \in I} W_i$ and define $\beta: W \to M$ by $\beta(x) = \beta_i(x)$ if $x \in W_i$. Then β is a well-defined homomorphism : Suppose that x is an element of W such that x belongs to both W_i and W_j. Then we may assume that $W_i \subseteq W_j$ without loss of generality. So $\beta(x) = \beta_i(x) = \beta_j(x)$ since β_i is the restriction of β_j to W_i. Let $x \in W$ and $r \in R$. Then $x \in W_i$ for some i and $rx \in W_i$ as well. Thus $\beta(rx) = \beta_i(rx) = r\beta_i(x) = r\beta(x)$. If $x_1 \in W_k$ and $x_2 \in W_j$, say, then either $W_k \subseteq W_j$ or $W_j \subseteq W_k$ because K is totally ordered. Suppose that $W_j \subseteq W_k$. Then $\beta(x_1+x_2) = \beta_k(x_1 + x_2) = \beta_k(x_1) + \beta_k(x_2) = \beta_k(x_1) + \beta_j(x_2) =$

$\beta(x_1) + \beta(x_2)$. Thus β is a well defined homomorphism and (W,β) will be a maximal element of K, showing that every totally ordered subset of S has a maximal element. By Zorn's Lemma, S has a maximal element, say, (W_0, β_0). If $W_0 = N'$, then we are done. So suppose that $W_0 \neq N'$, that is, suppose that W_0 is properly contained in N' and let x be an element of N' which does not belong to W_0. Then x belongs to N' but not to N. Let $A = \{r \in R \mid rx \in N\}$. Since T is hereditary and N is a dense submodule of N' we can apply Proposition 4.1.(4), page 29 of [G_1] to find that A is a T-dense left ideal of R (Note that in Golan's notation A would be written as $(N:x)$). Let $I = \{r \in R \mid rx \in W_0\}$. Then $A \subseteq I$ and since $R/I \cong (R/A)/(I/A)$, we see that R/I is an epimorphic image of the torsion R-module R/A and R/I will therefore be torsion. Thus I is a T-dense left ideal of R. Consider the homomorphism $\sigma\colon I \to M$ defined by $\sigma(a) = \beta_0(ax)$. Then by the hypothesis of the Proposition, σ can be extended to a homomorphism $\phi\colon R \to M$ as demonstrated in the following diagram.

$$\begin{tikzcd} I \arrow[r] \arrow[d, "\sigma"'] & R \arrow[dl, "\phi"] \\ M & \end{tikzcd}$$

Fig 2.1

Now define the homomorphism $\beta_1\colon (W_0 + Rx) \to M$ by $\beta_1(w_0 + rx) = \beta_0(w_0) + \phi(r)$. To see that β_1 is well defined, suppose that $w_1 + r_1x = w_2 + r_2x$, where w_1 and w_2 are elements of W_0 while r_1

and r_2 belong to R. Now $w_1 - w_2 = (r_2 - r_1)x$ implies that $r_2 - r_1 \in I$ by the definition of I. Therefore $\phi(r_2 - r_1) = \sigma(r_2 - r_1) = \beta_0((r_2 - r_1)x) = \beta_0(w_1 - w_2)$. This means that $\beta_0(w_2) + \phi(r_2) = \beta_0(w_1) + \phi(r_1)$ which implies that $\beta_1(w_1 + r_1x) = \beta_1(w_2 + r_2x)$. Thus β_1 is a well defined homomorphism. The following diagram illustrates the situation

$$\begin{tikzcd}
N \arrow[r, "m_1"] \arrow[d, "\alpha"'] & W_0 \arrow[r, "m_2"] \arrow[dl, "\beta_0"] & W_0 + Rx \arrow[dll, "\beta_1"] \\
M & &
\end{tikzcd}$$

where m_1 and m_2 are the inclusion homomorphisms. We will now show that β_1 properly extends β_0. Suppose that $w \in W_0$. Then $\beta_1(w) = \beta_1(w + 0x) = \beta_0(w) + \phi(0) = \beta_0(w) + 0 = \beta_0(w)$. So we see that β_0 and β_1 coincide on W_0. Now x is an element of $W_0 + Rx$ but does not belong to W_0. So β_1 properly extends β_0 and this contradicts the maximality of (W_0, β_0). Hence $W_0 = N'$. ∎

We now look at the definition of the "injective hull" of an R-module. We will then look at the notion of a "T-injective hull". These definitions will enable us to prove some compactness results later on in this chapter.

<u>Definition 2.5.</u> ([GH], pages 127 and 129)

Let $\hat{C}$ be an injective R-module containing the R-module C.

Then $\hat{C}$ is called the <u>injective hull</u> of C if C is an essential submodule of $\hat{C}$. ∎

We note first of all that every R-module has an injective hull (See Section 7.4 of [GH]). We find that, for a submodule C' of an R-module C containing an R-module A to be an injective hull of A, it is necessary that C' be maximal among those submodules, K, of C for which $A \rightarrow K$ is an essential embedding (Observation 5, page 129 of [GH]). Furthermore we note that that if C' and C'' are two injective hulls of an R-module A, then there is an isomorphism between C' and C'' (Observation 8, page 131 of [GH]). So we can see that an injective hull of an R-module is essentially unique.

Here is an example of an injective hull.

<u>Example 2.6.</u> ([GH], pages 136 and 98)
Let R be a principal ideal domain, that is, it is an integral domain such that each of its ideals is cyclic. Let Q be the field of quotients of R. Then Q is an injective hull of each of its nonzero submodules. In particular, Q is an injective hull of R. As a particular case of this we note that the ring Q, of rational numbers is an injective hull of the Z-module Z, where Z is the ring of integers under addition. ∎

When T is hereditary we can deduce a further property of the torsion-free class concerning the injective hull of a torsion-free R-module. The following result will be required in the proof of a characterisation of T-compact R-modules (Theorem 2.18).

Proposition 2.7. ($[L_1]$, Proposition 03, page 4)
If T is an hereditary torsion theory then the torsion - free class is closed under injective hulls as well.. ■

Let T be an hereditary torsion theory. We would like to find the T-injective hull, D(C), of any R-module C in the sense that D(C) would be the maximal T-injective R-module containing C such that C is an essential submodule of D(C). In order to do that we need to consider the following.

Proposition 2.8.
Let T be an hereditary torsion theory. Let C be any R-module and let D(C) be the intersection of all T-closed submodules of $\hat{C}$ which contain C. Then D(C) is a T-closed submodule of $\hat{C}$. Furthermore D(C) can be uniquely obtained from the equation $D(C)/C = T(\hat{C}/C)$.

Proof.
We first show that D(C) is T-closed in $\hat{C}$. It suffices to show that any intersection of T-closed submodues of an R-module K is T-closed in K. So let $\{K_i \mid i \in I$, where I is some index set$\}$ be any set of T-closed submodules of K. Set $K' = \cap_{i \in I} K_i$. Fix $i_0 \in K$. Then there is a monomorphism from K/K' to $\Pi_{i\in I} (K/K_i)$ which

sends an element $k + K'$ to the element of $\prod_{i\in I} (K/K_i)$ whose only non-zero component is the i_0th component which is $(k + K_{i_0})$. Now $\prod_{i\in I} (K/K_i)$ is torsion-free being the product of torsion-free R-modules. It therefore follows that K/K' is torsion-free, as it can be seen as a submodule of a torsion-free R-module. We have thus shown that $D(C)$ is the minimal T-closed submodule of $\hat{C}$ which contains C.

We will now show that $D(C)/C = T(\hat{C}/C)$. Suppose that $T(\hat{C}/C) = K/C$ for some submodule K of $\hat{C}$ containing C. Now $\hat{C}/K \cong (\hat{C}/C)/(K/C)$ which is torsion-free. Thus K is a T-closed submodule of $\hat{C}$ which contains C. Hence $D(C) \subseteq K$. If this inclusion is strict then $K/D(C)$ is a nonzero submodule of the torsion-free R-module $\hat{C}/D(C)$ and is therefore torsion-free. On the other hand $K/D(C)$ is an epimorphic image of K/C since $K/D(C) \cong (K/C)/(D(C)/C)$ and is therefore torsion. This means that $K = D(C)$. ∎

<u>Definition 2.9.</u>

Let C be any R-module and T an hereditary torsion theory. Then the R-module $D(C)$ described in Proposition 2.8 is called the <u>T-injective hull of</u> C. ∎

Note that our definition here differs from the one found in $[F_1]$ in that Fay, in $[F_1]$ defines $D(C)$ for all R-modules C and torsion theories T and we are restricting our definition to hereditary torsion theories only. In fact Proposition 2.8 and Definition 2.9 are valid if T is not an hereditary torsion theory. However when proving that D(C) is T-injective in Proposition 2.10 we use the result of Proposition 2.4 whose hypothesis requires that T be an hereditary torsion theory. So in order for the term "T-injective" hull to make sense we will specify that the torsion theory concerned must be hereditary. The following proposition proves that D(C) is T-injective when T is hereditary and provides three uniquely determining features of the T-injective hull of a particular R-module. Some parts of the proof of the following proposition have been adapted from Chapter 8 of [G_1].

Proposition 2.10.

Let T be an hereditary torsion theory and let D(C) be the T-injective hull of an R-module C. Then D(C) is uniquely determined by the following three facts :

(i). D(C) is T-injective.

(ii). D(C)/C is torsion.

(iii). The inclusion $C \to D(C)$ is an essential embedding.

Proof.

We will first show that D(C) satisfies these three properties.

(i). To show that D(C) is T-injective we will use Proposition 2.4. So let I be a T-dense left ideal of R with α a homomorphism from I to D(C). We need to show that the homomorphism α can be extended to a homomorphism from R to D(C). By the injectivity of $\hat{C}$, there exists a homomorphism

$\phi: R \to \hat{C}$ such that the following diagram is commutative

$$\begin{tikzcd}
I \arrow[r, "i_1"] \arrow[d, "\alpha"'] & R \arrow[ddl, "\phi"] \\
D(C) \arrow[d, "i_2"'] & \\
\hat{C} &
\end{tikzcd}$$

where i_1 and i_2 are the inclusion homomorphisms from I to R and $D(C)$ to $\hat{C}$ respectively. We will show that the restriction of ϕ to I is just α by showing that $\phi(R)$ is contained in $D(C)$. Thus ϕ will be the extension of α, from R to $D(C)$. Let $x = \phi(1)$, where 1 is the multiplicative identity of R, and $(D(C) : x) = \{ r \in R \mid rx \in D(C)\}$. Then $(Rx + D(C))/D(C)$ $\cong Rx/(Rx \cap D(C)) \cong Rx/((D(C) :x)x) \cong R/(D(C):x)$. Now if $i \in I$, then $ix = i\phi(1) = \phi\ (i.1) = \phi(i) = \alpha(i)$. Thus $ix \in D(C)$ and it follows that $I \subseteq (D(C):x)$. Then $R/(D(C):x)$ is an epimorphic image of the torsion R-module R/I because $R/(D(C) :x) \cong (R/I)/((D(C) :x)/I)$ and will therefore be torsion. Thus $(Rx + D(C))/D(C)$ is torsion as well. Now $Rx \subseteq \hat{C}$ since $\phi(R) \subseteq \hat{C}$. Hence $(Rx +D(C))/D(C)$ is a submodule of $\hat{C}/(D(C))$ which is torsion-free by the proof of Proposition 2.8. Therefore $(Rx + D(C))/D(C)$ is torsion-free and thus $(Rx + D(C)) = D(C)$. This means that $Rx \subseteq D(C)$. Thus $\phi(R) \subseteq D(C)$.

(ii). From Proposition 2.8 we know that $D(C)/C = T(\hat{C}/C)$. It then follows that $T(D(C)/C) = T(T(\hat{C}/C)) = T(\hat{C}/C) = D(C)/C$.

(iii). Every non-zero submodule of $D(C)$ would be a non-zero submodule of $\hat{C}$. Since C is an essential submodule of $\hat{C}$, it follows that C has a non-zero intersection with every non-zero submodule of $\hat{C}$. Thus C has a non-zero intersection with every non-zero submodule of $D(C)$.

Now we need to prove that $D(C)$ is uniquely determined by these three facts. So suppose that A is an R-module having C as a submodule, and satisfying the three properties mentioned, that is, A is T-injective, A/C is torsion and the inclusion of C into A is an essential embedding. We will show that A and $D(C)$ coincide. We can firstly deduce that the injective hull, $\hat{A}$, of A and the injective hull, $\hat{C}$, of C coincide: From Observation 2 on page 128 of [GH], we know that if C is an essential extension of B and B is an essential extension of A, then C is an essential extension of A. Now applying this to the fact that $C \to A$ is an essential embedding, we see that the inclusion of C into $\hat{A}$ must be an essential embedding. This means that $\hat{A}$ will be an injective hull of C as well. But by the remarks following Definition 2.5, we see that $\hat{A}$ and $\hat{C}$ will be isomorphic. So we can say that $\hat{C}$ is the injective hull of the T-injective R-module A. From Proposition 8.2 on page 77 of $[G_1]$, we see that A is in fact a T-closed submodule of $\hat{C}$. (Note that in $[G_1]$, the notion of a τ-pure submodule is equivalent to our notion of a T-closed submodule of a given R-module.) This enables us to deduce that $D(C) \subseteq A$ because we know from Proposition 2.8 that $D(C)$ is the smallest T-closed submodule of $\hat{C}$ containing C. On the other hand, $A \subseteq \hat{A}$ implies that $T(A/C) \subseteq T(\hat{A}/C) = T(\hat{C}/C) = D(C)/C$. It therefore follows that $A/C = T(A/C) \subseteq D(C)/C$ and we have that $A \subseteq D(C)$. Hence $A = D(C)$. ■

Here is a characterisation of T-injectivity of a given R-module in terms of its injective hull.

<u>Proposition 2.11.</u>

Let T be an hereditary torsion theory. Then an R-module C is T-injective if and only if $\hat{C}/C$ is torsion-free.

<u>Proof</u>

See Proposition 8.2, page 77, [G_1]. ■

We now provide a characterisation of T-injectivity of a given R-module in terms of its T-injective hull.

<u>Proposition 2.12.</u>

Let T be an hereditary torsion theory. Then an R-module C is T-injective if and only if $C = D(C)$.

<u>Proof.</u>

If C is T-injective, then C is T-closed in $\hat{C}$ by Proposition 2.11, that is, $T(\hat{C}/C) = 0$. Since $D(C)/C = T(\hat{C}/C)$, we find that $D(C) = C$. The converse assertion is clear. ■

Let us look at a characterisation of T-injectivity in terms of the functor $Ext^1(\_, C)$ for an R-module C. This result will be used in the proof of an important example in Chapter 3.

Proposition 2.13.

Let T be an hereditary torsion theory. An R-module C is T-injective provided $Ext^1(A,C) = 0$ for every torsion R-module A.

Proof.

The proof of this may be found on page 29 of $[S_2]$, Proposition 6.2. ∎

We now look at some other alternate ways of showing that an R-module is T-injective. The following lemma is useful because it provides another characterisation in terms of the T-dense left ideals of the ring R and is an extension of Proposition 2.4. Some parts of the proof of this proposition have been adapted from page 8 of $[L_1]$.

Proposition 2.14

The T-injective Test Lemma.

Let T be an hereditary torsion theory. The following are equivalent for an R-module C.

(i) C is T-injective.

(ii) C is injective relative to the T-dense left ideals of R (that is, those left ideals, K, of R for which $T(R/K) = R/K$). This means that any homomorphism $f: K \to C$ can be extended to a homomorphism $g: R \to C$ as shown in this following diagram where i is the inclusion homomorphism.

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "f"'] & R \arrow[dl, "g"] \\ C & \end{tikzcd}$$

(iii). For every T-dense left ideal I of R and homomorphism $f: I \to C$, there exists an $x \in C$ such that, for each $i \in I$, $f(i) = ix$.

Proof.

We will first show that (i) $\Longrightarrow$ (iii). So suppose that C is a T-injective R-module. To show that condition (iii) holds, let $f: I \to C$ be a homomorphism from a T-dense left ideal, I, of R to C. We need to find an $x \in C$ such that $f(i) = ix$ for each $i \in I$. Now consider the following diagram

$$\begin{tikzcd} I \arrow[r, "i_1"] \arrow[d, "f"] & R \arrow[ddl, "\phi"] \\ C \arrow[d, "i_2"] & \\ \hat{C} & \end{tikzcd}$$

where $\hat{C}$ is the injective hull of C, and i_1 and i_2 are the inclusion homomorphisms from I to R and C to $\hat{C}$ respectively. Because $\hat{C}$ is injective, there exists a homomorphism $\phi: R \to \hat{C}$ which extends $i_2 \circ f$. Let $x = \phi(1)$, where 1 is the multiplicative identity element of R. If $i \in I$, then $i = i.1$. Thus $f(i) = f(i.1) = i.f(1) = ix$ for each $i \in I$. We know that $x \in \hat{C}$. Now $Ix \subseteq C$ because $\phi(I) \subseteq C$. Therefore $I \subseteq (C:x)$, where $(C:x) = \{ r \in R \mid rx \in C \}$. Then we have that $(Rx + C)/C \cong Rx/(Rx \cap C) \cong Rx/((C:x)x) \cong R/(C:x)$. Now $R/(C:x) \cong (R/I)/(C:x)/I)$ which is an epimorphic image of the torsion R-module R/I. Therefore $(Rx + C)/C$ is torsion. But $x \in \hat{C}$, therefore $(Rx + C)/C \subseteq \hat{C}/C$ which is torsion-free by Proposition 2.11. Hence $(Rx + C)/C$ is also torsion-free. This implies that $(Rx + C)/C = \{0\}$. It then follows that $Rx \subseteq C$, that is, $x \in C$.

Now assume that condition (iii) holds. We will show that (i) holds. Suppose that A is a T-dense submodule of an R-module B and $f: A \to C$ is any homomorphism. By the definition of T-injectivity, we need to extend f to B, that is we must find a homomorphism $\psi: B \to C$ so that the restriction of ψ to A is just

f. Let M be the set of all pairs (W,w) where W is a submodule of B containing A and w is a homomorphism from W to C extending f as shown below (with i_1 and i_2 the inclusion homomorphisms).

$$\begin{tikzcd}
N \arrow[r, "i_1"] \arrow[d, "f"'] & W \arrow[r, "i_2"] \arrow[dl, "w"] & B \\
C & &
\end{tikzcd}$$

We now have a situation similar to that found in the first part of the proof of Proposition 2.4. Similarly by Zorn's Lemma we find that M has a maximal element, (K,g), say. That is, we can extend $f: A \to C$ to $g: K \to C$ so that f cannot be extended any further and $A \subseteq K \subseteq B$. The following diagram gives us a picture of the situation,

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "f"'] & K \arrow[r, "i_2"] \arrow[dl, "g"] & B \\
C & &
\end{tikzcd}$$

where i_1 and i_2 are just the inclusion homomorphisms. We claim that K = B. Otherwise there exists $b \in B$, $b \notin K$, and we consider the left ideal $D = \{ r \in R \mid rb \in K \} = (K : b)$. Now $B/K \cong (B/A)/(K/A)$, and thus B/K can be seen as an epimorphic image of the torsion R-module B/A, and B/K will therefore be torsion. So K is a T-dense submodule of B. By Proposition 4.1(4) on page 29 of [G_1], we see that D is a T-dense left ideal of R. Consider the homomorphism $\phi: D \to C$ defined by $\phi(d) = g(db)$. By (iii) there exists $x \in C$ such that $g(db) = \phi(d) = dx$ for all

$d \in D$. Thus for all $r \in R$ such that $rb \in K$, we will have that $g(rb) = rx$. We may therefore extend g to $h: K + Rb \to C$ by $h(k+rb) = g(k) + rx$. We will now show that h properly extends g. Suppose that k is any element of K. Then $k = k + 0b$. So $h(k) = h(k + 0b) = g(k) + 0x = g(k)$. So h extends g. Since the element b of Rb does not belong K we can see that h properly extends g and this is a contradiction. Therefore we must have that $K = B$.

From Proposition 2.4 we know that (ii) $\Longrightarrow$ (i). We now need to only show that (i) $\Longrightarrow$ (ii). But this is clear from the definition of T-injectivity since every T-dense left ideal K, of R can be seen as a T-dense submodule of the R-module R. ∎

We will need the following result in the proofs of many of the results and examples which follow both in this chapter and chapters 3 and 4. Here we have restricted T to be hereditary because Lambek, in [L_1], which we refer to, requires that T be hereditary.

<u>Lemma 2.15.</u>

Let T be an hereditary torsion theory , and let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of R-modules. Then

(i). if B is torsion-free and A is T-injective, then C is torsion-free.

(ii). if B is T-injective and C is torsion-free, then A is T-injective.

(iii). if A and C are T-injective, then so is B.

Proof.

The proof of this can be found in Proposition 06 on page 9 of $[L_1]$. ∎

We are now ready to provide a definition of T-compactness in **R-Mod**. As mentioned before, this is a very important part of this dissertation. We will see that this definition of compactness relies on our notion of T-closed embeddings which in turn arises from the (T-dense, T-closed embedding) - factorisation system of Proposition 1.31. This definition of compactness will allow us to obtain results in **R-Mod** and **Ab** which are analagous to some well known results concerning compactness in general topology.

Definition 2.16.

Following [HSS], we call an R-module C T-compact provided for every R-module Z, the second projection homomorphism $\pi_2: C \times Z \to C$ preserves T-closed submodules. This means that if A is a T-closed submodule of $C \times Z$ (that is, the inclusion $i: A \to C \times Z$ is a T-closed embedding), then the epimorphic image of A under π_2 is a T-closed submodule of Z.

We have mentioned in the introduction that under reasonable assumptions the class of T-compact R-modules forms the torsion class for a torsion theory. Here is one of the "reasonable assumptions", that will be needed for that statement which will be proved in Chapter 3. This definition will also be required for Theorem 2.18.

Definition 2.17.

A ring R is called T-hereditary provided the epimorphic image of every T-injective R-module is T-injective. ■

The following theorem offers two characterisations of T-compact R-modules under some assumptions on the ring R and the torsion theory T. This theorem differs somewhat from Theorem 2.3 of [F_1], in that firstly, T is assumed to be hereditary for the whole theorem. The reason for this assumption is because the proof of this theorem depends on Lemma 2.15 which requires that T be hereditary. Secondly, this theorem provides an extension of Theorem 2.3 of [F_1] when R has the additional property of being T-hereditary. In fact the characterisations contained in this theorem will be needed for the proof of most of the theorems following in the next two chapters.

Theorem 2.18.

Let T be an hereditary torsion theory and C any R-module. Consider the following statements :

(i). C/D is T-injective for each T-closed submodule D of C.

(ii). $C/T(C)$ is T-injective.

(iii). C is T-compact.

1. Then (ii) $\Longleftarrow$ (i) $\Longleftrightarrow$ (iii).

2. If R is T-hereditary, then (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Proof.

1. The proof of (i) $\Longrightarrow$ (ii) is obvious. To prove that (i) $\Longrightarrow$ (iii) we assume that C/D is T-injective for each T-closed submodule D of C and we will show that C is T-compact. Let Z be an arbitrary R-module and A any T-closed submodule of $C \times Z$. Let D be the pullback of $A \to C \times Z$ along $C \to C \times Z$, where the first homomorphism is the inclusion while the second homomorphism is the first injection. Let B be the image of A in Z under π_2. To show that C is T-compact we will show that B is a T-closed submodule of Z. We have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D \arrow[r, "i_1"] \arrow[d, "m_1"'] & A \arrow[r, "p_2"] \arrow[d, "m_2"'] & B \arrow[r] \arrow[d, "m_3"'] & 0 \\
0 \arrow[r] & C \arrow[r, "in_1"] \arrow[d, "n_1"'] & C \times Z \arrow[r, "\pi_2"] \arrow[d, "n_2"'] & Z \arrow[r] \arrow[d, "n_3"'] & 0 \\
0 \arrow[r] & C/D \arrow[r, "\psi"] \arrow[d] & (C \times Z)/A \arrow[r, "\phi"] \arrow[d] & Z/B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Fig 2.2.

where : in_1 and π_2 are the first injection and second projection homomorphisms,

i_1 and p_2 are the restrictions of in_1 to D and π_2 to A respectively,

m_2, and m_3 are the inclusions of the respective R-modules while

n_1, n_2, and n_3 are the canonical epimorphisms.

We find that $m_1 : D \rightarrow C$ is a T-closed embedding being the pullback of a T-closed embedding (Proposition 1.4(iv)). So m_1 is a monomorphism. Also i_1: $D \to A$ is the pullback of a monomorphism and is therefore a monomorphism. Note that in the rest of this proof a square bracket ([]) denotes the equivalence class of an element in the relevant factor group. The homomorphisms ψ and ϕ are defined as follows:

If $[c] \in C/D$, then $\psi\,[c] = [in_1(c)] \in (C \times Z)/A$, while for $[(c,z)] \in (C \times Z)/A$, $\phi\,[(c,z)] = [\pi_2((c,z))] \in Z/B$.

The columns in the diagram are clearly exact. By the definition of exactness, we find that the first and second rows are exact. From the definitions of ψ and ϕ, it is also clear that $\mathrm{Im}\,(\psi) = \mathrm{Ker}\,(\phi)$. Now D is formed as a pullback and can be therefore described as $D = in^{-1}(A)$. We now apply Theorem 4.3 on page 34 of [B_1] to deduce that ψ is a monomorphism. If $[z]$ is any element of Z/B then $\phi\,[(0,z)] = [\pi_2((0,z))] = [z]$. So ϕ is onto and is therefore an epimorphism. Hence the third row is also exact. Now D can be seen as a T-closed submodule of C because m_1 is a T-closed embedding. By the hypothesis of the theorem C/D is T-injective. From Lemma 2.15(i) applied to the third row of Fig.2.2 we find that Z/B is torsion -free. Hence C is T-compact.

To show that (iii) $\Longrightarrow$ (i), we assume that C is T-compact, and we will show that C/D is T-injective whenever D is a T-closed submodule of C. Let Q be the injective hull of C/D and let f denote the composite

$$\begin{tikzcd} C \arrow[r, "n"] \arrow[rr, bend right, "f"'] & C/D \arrow[r, "i"] & Q \end{tikzcd}$$

where n is the canonical epimorphism while i denotes the inclusion homomorphism. We note that in the category **R-Mod** the direct sum and direct product of a finite number of R-modules coincide. We can now consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "\langle 1, f\rangle"] \arrow[d, "n"'] & C \times Q \arrow[r, "\{f, -1\}"] \arrow[d, "\pi_2"'] & Q \arrow[r] \arrow[d, "g"'] & 0 \\ 0 \arrow[r] & C/D \arrow[r, "i"] & Q \arrow[r, "q"] & Q/(C/D) \arrow[r] & 0 \end{tikzcd}$$

Fig. 2.3.

where : $C \times Q$ is the direct sum /direct product of the R-modules C and Q,

q denotes the canonical epimorphism,

1 denotes the identity homomorphism,

-1 denotes multiplication by -1,

g is defined as follows: For $a \in Q$, $g(a) = -q(a)$ and

$\langle 1, f\rangle$ is obtained from the following direct product diagram (See [HS], Definition 18.2, page 115)

$$
\begin{tikzcd}
 & C \\
C \arrow[ur, "1"] \arrow[r, "{\langle 1, f \rangle}"] \arrow[dr, "f"'] & C \times Q \arrow[u, "\pi_C"'] \arrow[d, "\pi_Q"] \\
 & Q
\end{tikzcd}
$$

We can see that for each $c \in C$, $\langle 1, f\rangle(c) = (c, f(c))$ because it satisfies $\pi_Q \circ \langle 1, f\rangle = f$ and $\pi_C \circ \langle 1, f\rangle = 1$.

The homomorphism $\{f, -1\}$ is obtained from the following coproduct diagram (See [HS], Definition 18.3, page 116)

$$
\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "in_1"] \\
Q & C \times Q \arrow[l, "{\{f, -1\}}"'] \\
 & Q \arrow[ul, "-1"] \arrow[u, "in_2"']
\end{tikzcd}
$$

Now we find that $\{f, -1\}$ is defined by $\{f, -1\}(c, q) = (f(c) + (-q))$ because it satisfies $\{f, -1\} \circ (in_1) = f$ and $\{f, -1\} \circ in_2 = -1$. Now going back to Fig.2.3 we look at the left hand square. We have $\pi_2 \circ \langle 1, f\rangle = f$ and $i \circ n = f$. So we see that the left hand square commutes. Let us consider the right hand square. If $(c,a) \in C \times Q$, then $q \circ \pi_2((c,a)) = q(a) = [a]$, the equivalence class of a in $Q/(C/D)$. Also $(g \circ \{f,-1\})((c,a)) = g(f(c) - a) = q(a - f(c)) = [a - f(c)]$, the equivalence class of $a-f(c)$ in $Q/(C/D)$. Now $f(c) = i \circ n(c) = n(c)$ which belongs to C/D. Therefore $[a - f(c)] = [a] - [f(c)] = [a] - 0 = [a]$. Hence the right hand square commutes as well. We will now once again consider the first row of Fig.2.3. We will show that $\{f,-1\} \circ <1,f> = 0$. Let $c \in C$. Then we have that $\{f,-1\}(<1,f>(c)) = (\{f,-1\})(c,f(c)) = f(c) - f(c) = 0$. Hence $Im(<1,f>) \subseteq Ker(\{f,-1\})$. If $(c,q) \in Ker(\{f,-1\})$, then $\{f,-1\}((c,q)) = 0$ implies that $f(c)-q = 0$ which implies that $f(c) = q$. So $(c,q) = (c,f(c)) = <1,f>(c)$. So $(c,q) \in Im(<1,f>)$. Thus $Ker(<1,f>) \subseteq Im(\{f,-1\})$ and hence $Ker(<1,f>) = Im(\{f,-1\})$. Since $\pi_c \circ <1,f> = 1$ and the first factor of a monomorphism is a monomorphism, it follows that $<1,f>$ is a monomorphism. We also have that $\{f,-1\} \circ in_2 = -1$ and by the dual of the previous statement, we find that $\{f,-1\}$ is an epimorphism. This implies that the first row of Fig 2.3 is exact. It is quite clear that the second row is also exact. We know that T is hereditary and C/D is torsion-free. By Proposition 2.7 we find that Q is torsion-free. We know that $Q \cong (C \times Q)/(<1,f>(C))$ because the first row of Fig 2.3 is exact (See Definition 1.12). Thus $<1,f>(C)$ is T-closed in $C \times Q$. Now C is T-compact, therefore the epimorphic image of $<1,f>(C)$ under π_2 will be a T-closed submodule of Q. But this image is just $f(C)$ which will be isomorphic to C/D. Hence $Q/(C/D)$ is torsion-free. Let $D(C/D)$ be the T-injective hull of C/D. We know from Definition 2.9 and Proposition 2.8 that

$(D(C/D))/(C/D)$ is equal to $T(Q/(C/D))$ which is equal to 0. Hence $D(C/D) = C/D$ which implies that C/D is T-injective.

2. Here we need to only show that (ii) $\Longrightarrow$ (i), that is, we show that if R is T-hereditary and $C/T(C)$ is T-injective, then C/D is T-injective for every T-closed submodule D of C. Now C/D is torsion-free. We consider the canonical epimorphism, $n: C \to C/D$ and look at $n(T(C))$. Then $n(T/(C))$ must be a torsion submodule of the R-module C/D, since torsion R-modules are closed under epimorphic images. But $T(C/D) = 0$. Therefore $n(T(C)) = 0$ so we have that $T(C) \subseteq Ker(n) = D$. So we can now form the factor module $D/T(C)$. We see that $C/D \cong (C/T(C))/(D/T(C))$ which is an epimorphic image of the T-injective R-module $C/T(C)$. Since R is T-hereditary, the R-module C/D will be T-injective.∎

We would now like to apply Theorem 2.18 to the category of abelian groups. This is done in Corollary 2.26 where we first characterise injective groups in **Ab** before we obtain a characterisation of T-compact abelian groups. We first need to prove some preliminary results. The following five definitions will be required for these preliminary results as well as for the main characterisation of T-compact abelian groups.

Definition 3.12 on page 11 and Corollary 4.2 on page 19 of [W_1] act as a basis for the following definition of ours. Note that although Warfield, in [W_1] considers nilpotent groups only, the definition is valid for abelian groups because every abelian group is nilpotent.

<u>Definitions 2.19.</u>

Let A be any abelian group. If p is any prime we will define a functor $t_p : \mathbf{Ab} \to \mathbf{Ab}$ as follows.

(i) $t_p(A)$ is defined to be the subgroup of A consisting of all the elements of A whose order is a power of p. The subgroup $t_p(A)$ will be called the <u>p-torsion subgroup of</u> G.

(ii) If $f: A \to B$ is any homomorphism between abelian groups, then $t_p(f)$ is just the restriction of f to $t_p(A)$ with codomain restriction to $t_p(B)$. We will refer to t_p as the <u>p-torsion subgroup functor.</u> It is quite clear that t_p is a radical on **Ab**. ■

<u>Definition 2.20.</u> ([F_2], page 4 and [FW_4] page 170)

Let A be any abelian group and $n > 0$ any positive integer. The idempotent preradical $[n]$ defined as follows

$$A[n] = \{a \mid a \in A,\ na = 0\}$$

will be referred to as the <u>n-socle</u>, following the terminology found on page 170 of [FW_4]. ■

<u>Definition 2.21.</u> ([F_2], page 25)

An abelian group A is said to to be <u>n-bounded</u> if each of the elements, a of A, satisfy $na = 0$. ■

Definitions 2.22. ($[G_2]$, page 6)

Let G be any abelian group.

(i) We say that G is divisible if for every $x \in G$ and every non-zero integer n, there exists a $y \in G$ with $ny = x$.

(ii) Let p be any prime. We say that an abelian group G is p-divisible if for every $x \in G$ and positive integer n there exists a $y \in G$ such that $p^n y = x$. ∎

The following is a very useful result that we will require for the proof of Theorem 2.25 as well as in Chapter 3.

Proposition 2.23.

Let T be an hereditary torsion theory on the category **R-Mod**. Then T commutes with direct sums.

Proof.

Suppose that G is an R-module which is a direct sum of its subgroups, A_i, that is $G = \Sigma_i A_i$ and $A_i \cap A_j = \{0\}$ when $i \neq j$. Then we know that each element g, of G has a unique representation $g = a_1 + a_2 + \ldots + a_j + \ldots$, with each $a_i \in A_i$. For each A_i, we can define a function $\Pi_{A_i} : G \rightarrow A_i$ by $\Pi_{A_i}(g) = a_i$, where $a_i \in A_i$ is the ith element in the direct sum representation of g. Then each Π_{A_i} is a homomorphism. Since T is a hereditary torsion theory on **R-Mod** we can apply condition (ii) of Definition 1.15 to deduce that $\Pi_{A_i}(T(G)) \subseteq T(A_i) \subseteq T(G)$ for each i. We claim that $T(G) = \Sigma_i (A_i \cap T(G))$. Note that if $i \neq j$, then $((A_i \cap T(G)) \cap (A_j \cap T(G)) = \{0\}$ because A_i and A_j intersect trivially. Let x be an element of $T(G)$. Then x belongs to G, therefore x has a unique direct sum representation $x = a_1 + a_2 + \ldots\ldots$. Fixing i we have $a_i = \Pi_{A_i}(x)$. But $\Pi_{A_i}(T(G)) \subseteq T(G)$ implies that $a_i \in T(G)$. Thus $a_i \in (A_i \cap T(G))$. This holds for each i. Since $x = a_1 + a_2 + \ldots$, we see that $x \in \Sigma_i\ (A_i \cap T(G))$. This shows that $T(G) \subseteq \Sigma_i\ (A_i \cap T(G))$. Since it is clear that $\Sigma_i (A_i \cap T(G)) \subseteq T(G)$, we see that $T(G) = \Sigma_i\ (A_i \cap T(G))$. We know that $A_i \cap T(G) = T(A_i)$ because T is hereditary. Thus $T(\Sigma_i A_i) = T(G) = \Sigma_i (A_i \cap T(G)) = \Sigma_i (TA_i)$. ∎

The following result will be required in the proof of Theorem 2.25. It has been adapted for abelian groups from the paper "Characterisations of quasi-splitting modules" by S.V Joubert ($[J_1]$).

Lemma 2.24.

If $\underline{E}$ is an hereditary class of abelian groups closed under direct sums and homomorphic images, then either $\underline{E} = |\mathbf{Ab}|$ or $\underline{E} \subseteq \underline{T_0}$, the class of all t-torsion abelian groups. (Note that t is the torsion radical of Example 1.17.)

Proof.

We first note that in Example 1.17, t was referred to as a radical on the category **R-Mod.** It is clear that t is also a radical on the category **Ab**. Now suppose that $\underline{E} \not\subset \underline{T}_0$. Then there exists a group M belonging to $\underline{E}$ which has an element x of infinite order. Then <x> , the cyclic subgroup generated by x, is isomorphic to Z, where Z is the group of integers under addition. Since $\underline{E}$ is hereditary it follows that Z must belong to $\underline{E}$. Now take any $G \in |\mathbf{Ab}|$. Every group is the epimorphic image of a free group, and so there exists an epimorphism $\oplus_m Z \to G$. Now $\oplus_m Z \in \underline{E}$ because $\underline{E}$ is closed under direct sums and so $G \in \underline{E}$ because $\underline{E}$ is closed under homomorphic images. Thus $\underline{E} = |\mathbf{Ab}|$. ∎

We are now ready to prove an important result that will enable us to obtain a characterisation of compactness for abelian groups. Note that this proof has been adapted from the paper "Maximal functorial topologies on abelian groups" by Fay and Walls ($[FW_4]$).

Theorem 2.25.

Let T be an hereditary torsion theory on **Ab**. Then there exists a set of primes P′ such that $T = \sum_{p \in P'} t_p$ or $T = 1_{\mathbf{Ab}}$, where $1_{\mathbf{Ab}}$ is the identity functor on **Ab**.

Proof.

Suppose that Q, the group of rationals under addition, belongs to the torsion class of T. Then Z, the group of integers under addition, would belong to the torsion class of T as well because Z is a subgroup of Q and T is hereditary. This means that $TZ = Z$. Now the torsion class of T satisfies the requirements of the class $\underline{\Xi}$ of Lemma 2.24. Since Z is a t-torsion-free group (where t is the torsion radical of Example 1.17), we deduce from Lemma 2.24 that $T = 1_{\mathbf{Ab}}$, the identity functor on **Ab**. Suppose that Q does not belong to the torsion class of T. Note that Q has no non-zero fully invariant proper subgroup (by "fully invariant subgroup" we mean a subgroup which is mapped onto itself by every endomorphism on Q): To see this let A be any proper subgroup of Q. Let $r \neq 1$ be any rational number not contained in A. Then the endomorphism $f : Q \rightarrow Q$ defined by $f(q) = rq$ does not satisfy the condition that $f(A) \subseteq A$. Note further that if $g: Q \rightarrow Q$ is any endomorphism of Q, then $g(T(Q)) \subseteq T(Q)$ by part (ii) of Definition 1.15. Hence T(Q) must be a fully invariant subgroup of Q. Thus we have that $T(Q) = 0$. Let t denote the torsion radical of Example 1.17. Then we deduce that $T \leq t$, in the lattice of all preradicals on **Ab**, where t is the torsion radical of Example 1.17. We know that such a lattice exists from page 136 of [S_1]. From [W_1], Theorem 4.3, page 19 we obtain the result that $t(A) = \sum_{p\in P} t_p(A)$, for every abelian group A, where P is the set of all primes. Furthermore we know that the radical T is determined by $T(A) = \sum\{B \mid B \subseteq A \text{ and } T(B) = B\}$ - see Proposition 1.19. We claim that $T = \sum_{p\in P} T\circ t_p$: We will first show that $T_p = T\circ t_p$ is an idempotent radical, where $T\circ t_p$, the composition of radicals T and t_p, is defined by $T\circ t_p(A) = T(t_p(A))$ for any abelian group A (See [S_2], page 1).

T_p is idempotent:

If A is any abelian group, then $T_p^2(A) = (T\circ t_p)^2(A)$. The latter is then equal to $(T\circ t_p)\circ(T\circ t_p)(A)$ which in turn is equal to $T[t_p(T\circ t_p)(A)]$ where the different brackets have been used just to clarify the term and have no special meaning. Now $T\circ t_p(A)$ is a p-torsion group, therefore $t_p(T\circ t_p(A)) = T\circ t_p(A)$. So $T[t_p(T\circ t_p)(A)] = T(T\circ t_p(A)) = (T\circ T)[t_p(A)] = T(t_p(A))$ because T is idempotent. Thus $T_p^2(A) = T(t_p(A)) = T\circ t_p(A) = T_p(A)$.

T_p is a radical:

We have $T_p(A/T_p(A)) = (T\circ t_p)(A/T\circ t_p(A)) = T[t_p[A/(T\circ t_p(A))]]$. We know from [$S_2$], Lemma 1.2, page 2, that if r is a radical, then $r(G/H) = (r(G))/H$ for any abelian group G and subgroup H contained in $r(G)$. Now applying this result with respect to the radical t_p our first term reduces to $T[t_p(A)/(T\circ t_p(A))]$. Then applying the same result again with respect to the radical T we are now left with the term $(T[t_p(A)])/(T\circ t_p(A))$ which is equal to $(T\circ t_p(A))/(T\circ t_p(A))$ which of course is zero. The other properties of a radical follow easily since T and t_p are both radicals.

To show that $T = \sum_{p\in P} T\circ t_p$, we need only consider those abelian groups, B, for which $T(B) = B$ or $\sum T\circ t_p(B) = B$ because an idempotent radical, that is, a torsion theory, is completely determined by its torsion class - Proposition 1.19. Suppose that $T(B) = B$. Since $T \leq t$, this implies that $t(B) = B$ and, thus, $\sum_{p\in P} t_p(B) = B$. Now for each prime p, $t_p(B)$ is a subgroup of B $= T(B)$. Since T is a hereditary torsion theory, it is true that $T(t_p(B)) = t_p(B)$. Then $\sum_{p\in P} t_p(B) = B$ implies that $\sum_{p\in P} T\circ t_p(B) = B$. On the other hand suppose that B is an abelian group such that $\sum_{p\in P} T\circ t_p(B) = B$. From Proposition 2.23 we know that if T is a hereditary torsion theory then T commutes with direct sums. Thus $T(\sum_{p\in P} t_p(B)) = \sum_{p\in P} T\circ t_p(B) = B$. This then implies that $T(B) = B$. Thus $T = \sum T\circ t_p$. To identify T we therefore need to only determine $T_p = T\circ t_p$ for each prime p.

For the rest of the proof of this theorem, if r is a preradical on **Ab** let S_r denote the set of all abelian groups, A, for which $rA = A$. The class S_r will be referred to as the stabilizer class of the preradical r. When r is an idempotent radical (torsion theory), then S_r is just the torsion class of the torsion theory r. We know that a torsion theory (idempotent radical) on **Ab** is completely determined by its torsion class. So we will only consider the torsion class of T_p in order to determine T_p for each prime, p. Since $T_p \leq t_p$ for each prime, p, we find that $S_{T_p} \subsetneq S_{t_p}$. If $S_{T_p} = \{0\}$, then $T_p = 0$, the zero functor. So we assume that S_{T_p} contains non-zero groups. Now we find that one of the following possibilities must hold for S_{T_p} :

(i) S_{T_p} has only divisible groups.

(ii) S_{T_p} has only reduced groups. Note that reduced groups are groups that have no non-zero divisible groups ([H], page 198).

(iii) S_{T_p} has both reduced and divisible groups.

For our proof we can further subdivide cases (ii) and (iii) on the basis of whether :

(a) $Z/p^kZ \in S_{T_p}$ for all $k \in N$, or

(b) $Z/p^kZ \in S_{T_p}$ for $k = 1,2,3,\ldots\ldots n$ and $Z/p^{n+1}Z \notin S_{T_p}$

Now considering the preceding statement and above possibilities we are left with the following four cases:

<u>Case 1: S_{T_p} contains only divisible p-groups.</u>

From $[F_4]$, Theorem 19.1, page 64 we know that every divisible group, G, is a direct sum of the form G = T + F, where T is t-torsion and F is t-torsion-free and t is the torsion radical of Example 1.17. We will digress a bit just to look at the group $C(p^\infty)$ which is the infinite multiplicative group consisting of the p^kth roots of unity, where k runs through all the natural numbers (See $[F_4]$, page 23). (Other books sometimes refer to $C(p^\infty)$ as Z_{p^∞}.) Note that each element of $C(p^\infty)$ has order a power of p. Now getting back to our proof, we see from the proof of Theorem 19.1 of $[F_4]$ that T is a direct sum of copies of $C(p^\infty)$, and F is a direct sum of copies of Q, the group of rationals under addition. If $G \in \mathcal{S}_{T_p}$, then $G \in \mathcal{S}_{t_p}$ which means that G is a p-torsion group. Thus G must consist only of direct sums of copies of $C(p^\infty)$, that is, $F = 0$. Then G is a divisible p-torsion group and $T_p(G) = d \circ t_p(G)$, where d is the divisible radical of Example 1.17. In this case it follows that $T_p = d \circ t_p$.

<u>Case 11 : $Z/p^nZ \in \mathcal{S}_{T_p}$ for all n.</u>

Suppose that $T_p(Z/p^nZ) = Z/p^nZ$ for all n. If $G[p^n] = G$ for any abelian group, G, then G is p^n-bounded and G will therefore be a direct sum of cyclic groups ($[F_2]$, Theorem 17.2). In this case the cyclic groups will be of the form Z_{p^r}, $r \leq n$ and each $Z_{p^r} \in \mathcal{S}_{T_p}$ since $\mathcal{S}_{T_p}$ is closed under formation of subgroups. Thus $T_pG = G$. Therefore $[p^n] \leq T_p$ for all n, where $[p^n]$ is the p^n-socle of Definition 2.20. So we have $t_p = \Sigma_n[p^n] \leq T_p \leq t_p$. Hence $T_p = t_p$ in this case.

<u>Case 111 : $\mathcal{S}_{T_p}$ contains only reduced groups and $Z/p^kZ \in \mathcal{S}_{T_p}$ for $k = 1,2,\ldots n$, but $Z/p^{n+1}Z \notin \mathcal{S}_{T_p}$.</u>

As above for Case ii, we obtain $[p^k] \leq T_p$ for $k = 1,2,\ldots n$. If an abelian group, A, belongs to $\mathcal{S}_{T_p}$ let us consider $A/p^{n+1}A$. If $p^n(A/p^{n+1}A) \neq 0$ then this means that $A/p^{n+1}A$ has an element of order p^{n+1}. If a is that element of order p^{n+1} then $\langle a \rangle \in \mathcal{S}_{T_p}$ because $\mathcal{S}_{T_p}$ is closed under formation of subgroups. But $\langle a \rangle \cong Z_p{}^{n+1}$. So $\langle a \rangle \in \mathcal{S}_{T_p}$ contradicts the fact that $Z/p^{n+1}Z$ is not an element of $\mathcal{S}_{T_p}$. Thus $p^n(A/p^{n+1}A) = 0$. This implies that $p^nA = p^{n+1}A$ and so p^nA is divisible. Since A is reduced this implies that $p^nA = 0$ and therefore $A \in \mathcal{S}_{[p^n]}$. Thus $\mathcal{S}_{T_p} \subseteq \mathcal{S}_{[p^n]}$. This means that $T_p \le [p^n]$, which in turn implies that $T_p = [p^n]$ in this case.

Case iv : S_{T_p} contains reduced and divisible groups and $Z/p^kZ \in S_{T_p}$ for $k = 1,2,\ldots n$ but $Z/p^{n+1}Z \notin S_{T_p}$

By the argument above of Case iii, every reduced group in S_{T_p} is p^n-bounded. Since every group can be written as a sum of a divisible and reduced group (Theorem 21.3, page 100, [F_2]), we see that all the groups G in S_{T_p} have the form $G = D + B$, where D is divisible and B is p^n-bounded. Then

$(d\circ t_p + [p^n])(D + B) = d\circ t_p(D + B) + [p^n](D + B) = D + B$ because D is a divisible p-group and B is a p^n-bounded group. Therefore $G = D + B \in S_{d\circ t_p+[p^n]}$. Thus $T_p \le d\circ t_p + [p^n]$. Going the other way if we set $D = 0$, that is, G is reduced, we have as for Case ii that $[p^n] \le T_p$. If we set $B = 0$, that is, G is a divisible group, we will have that $d\circ t_p \le T_p$. So $[p^n] + d\circ t_p \le T_p$ and therefore in this case we have $T_p = d\circ t_p + [p^n]$.

By looking at all four possibilities, we have the following situation :

$$T_p = \begin{cases} 1. \quad [p^{n_p}] & n_p = 0,1,\ldots\ldots \infty \quad \text{or} \\ 2.\ d\circ t_p\ ; \quad \text{or} & \\ 3.\ d\circ t_p + [p^{n_p}] & n_p = 0,1,\ldots\ldots\infty \end{cases}$$

where $[p^{n_p}]$ denotes the p^{n_p} socle, $[p^0] = 0$ and $[p^\infty] = t_p$, the p-torsion functor.

However we know that d, the divisible radical is not hereditary (Example 1.28). Since T_p is hereditary we can eliminate cases 2 and 3). So $T_p = [p^{n_p}]$, where n_p can take on the values $0, 1, \ldots \infty$. It is quite clear that $[p^n]$ is a radical only when $n = \infty$. But we have already shown that T_p is a radical. So we see that there is a subset P' of P (namely those primes for which $n_p = \infty$) such that $T = \sum_{p \in P'} t_p$. ∎

We are now finally ready to apply our compactness results to the category of abelian groups.

Corollary 2.26.

Consider the category of abelian groups and let T be an hereditary torsion theory. Let t_p denote the p-torsion subgroup functor. Let P denote the set of primes such that $T = \sum_{p \in P} t_p$. Then, an abelian group, G, is T-compact if and only if $G/T(G)$ is p-divisible for each prime, p, that belongs to P. If P is the set of all primes, then G is T-compact exactly when $G/T(G)$ is divisible.

Proof.

To investigate the T-compactness of abelian groups, we need to first determine what the T-injective abelian groups are. In order to do that we will first characterise the T-dense left ideals of Z, the ring of integers. Let I be a left ideal of Z. Then $I = \{0\}$ or $I = mZ$ for some non-negative $m \in Z$. We want to know what conditions m must satisfy in order for $I = mZ$ to be a T-dense left ideal of Z. Now $Z/mZ \cong Z_m$. If $I \neq \{0\}$, then I will be a T-dense left ideal if and only if

$T(Z_m) = Z_m$, that is, $\Sigma t_p(Z_m) = Z_m$. Now we know that there is a set of distinct primes $\{p_1, p_2, \ldots, p_k\}$ and positive integers $r_1, r_2, \ldots, r_k$ such that $m = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$. From Lemma 2.3, page 77 of [H] we know that $Z_m \cong Z_{p_1^{r_1}} + Z_{p_2^{r_2}} + \ldots + Z_{p_k^{r_k}}$. Then for each i there exists a subgroup G_i of Z_m such that $Z_m = G_1 + \ldots + G_k$, and each $G_i \cong Z_{p_i^{r_i}}$. We will now show that each G_i is equal to $t_{p^i}(Z_m)$. Let us consider the group G_j for some $j \in \{1,2,\ldots,k\}$. If $a \in G_j$, then the order of a divides $p_j^{r_j}$, which means that the order of a is a power p_j. Hence $a \in t_{p^j}(Z_m)$. On the other hand if $a \in t_{p^j}(Z_m)$, then there exists a positive integer t such that $p_j^t a = 0$. Since $a \in Z_m$, there would be a decomposition of a into $a = a_1 + a_2 + \ldots + a_k$ with each $a_i \in G_i$. Then we would have that $0 = p_j^t a = p_j^t a_1 + p_j^t a_2 + \ldots + p_j^t a_k$. By the uniqueness conditon of a direct sum this would imply that for each i, we must have $p_j^t a_i = 0$. But for each i, the order of a_i must divide $p_i^{r_i}$ and the $\gcd(p_j, p_i) = 1$ whenever $i \neq j$. Hence $p_j^t a_i = 0$ implies that $a_i = 0$ if $i \neq j$. Therefore $a = a_j$, that is, a belongs to G_j. So we see that for each i, the R-module G_i is actually $t_{p^i}(Z_m)$. We can therefore write $Z_m = \sum_{i=1}^{k} t_{p^i}(Z_m)$. We thus have two cases to now consider :

<u>(i) Each p_i in the decomposition of m belongs to the set P.</u>
We have that $Z_m = \sum_{i=1}^{k} t_{p^i}(Z_m) \subseteq T(Z_m)$. But $T(Z_m) \subseteq Z_m$. Therefore in this case we find that $T(Z_m) = Z_m$.

<u>(ii) There is a prime ,p_s say, appearing in the decomposition of m which does not belong to the set P.</u>
Let $x = p_1^{r_1} p_2^{r_2} p_{s-1}^{r_{s-1}} p_{s+1}^{r_{s+1}} \ldots p_k^{r_k}$. Then x belongs to Z_m and x has

order $p_s^{r_s}$. Therefore $x \in t_{p_s}(Z_m)$. Suppose that x can be represented as a direct sum, that is,

$x = x_1 + x_2 + \ldots + x_{s-1} + 0 + x_{s+1} + \ldots + x_k$ with each $x_i \in t_{p^i}(Z_m)$, and not all of the $x_i = 0$. Then

$x = 0 + 0 + \ldots + x + \ldots + 0$ is another representation of x as a direct sum which is different to the first representation. This contradicts the uniqueness condition of a direct sum representation. Hence x cannot be represented as a direct sum of $t_{p^i}(Z_m)$, for $i \neq s$ and $i \in \{1,2,\ldots k\}$. That is, $x \notin \sum_{i=1, i\neq s}^{i=k} t_{p^i}(Z_m)$ which implies that $x \notin \sum_{p\in P} t_p(Z_m) = T(Z_m)$. So in this case we see that $T(Z_m) \neq Z_m$.

Hence we see that I is a T -dense left ideal of Z if and only if $I = \{0\}$ or $I = mZ$, when m is a product of some powers of some primes belonging to the set P, that is when $m = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$ with each $p_i \in P$ and the r_i are nonnegative integers.

It will now be shown that an abelian group C is T-injective if and only if C is p-divisible for each $p \in P$. To this end let C be a T-injective abelian group. Then C is a Z -module, where Z is the ring of integers. If $p \in P$ and n is any positive integer, then p^nZ is a T -dense left ideal of Z. Let $c \in C$. Define $f: p^nZ \to C$ by $f(p^nz) = zc$. Then it is clear that f is a homomorphism in Z. By Proposition 2.14 (iii), there exists an element b of C such that $f(x) = xb$ for every $x \in p^nZ$. In particular (taking $z = 1$ in the definition of f) we have that the element $c = f(p^n) = p^nb$. Thus we see that C is p-divisible for any $p \in P$.

On the other hand suppose that C is a p-divisible abelian group for each $p \in P$. Let I be a T-dense left ideal of Z with $f: I \to C$ any homomorphism in Z. Then $I = mZ$ for some $m = p_1^{r_1} p_2^{r_2} \ldots . p_k^{r_k}$, where each p_i belongs to the set P and the r_i are positive integers or $I = \{0\}$. Since I is p-divisible for each $p \in P$, we see that C will be m-divisible. Now $f(m) \in C$ implies that there exists an element c of C such that $f(m) = mc$. If $I = \{0\}$, then $m = 0$ and $f(m) = f(0) = 0 = 0c$. Thus in either case, there exists an element c of C such that $f(m) = mc$. If $x \in I$, we then have that $x = mz$ for some $z \in Z$. Hence $f(x) = f(mz) = zf(m) = zmc = xc$.

$$\begin{tikzcd} I \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[dl, "t_c"] \\ C & \end{tikzcd}$$

Considering the above diagram with i the inclusion of I into Z, we see that the Z-morphism $t_c : Z \to C$ given by $t_c(r) = rc$ extends f to Z. Therefore by Proposition 2.14 (ii), the group C is T-injective. So it is true that the abelian group C is T-injective if and only if it is p-divisible for each $p \in P$.

Note that if C is an abelian p-divisible group, then every epimorphic image of C is p-divisible: Suppose that $f: C \to K$ is an epimorphism. Let $p \in P$ and n be any natural number. If $k \in K$, then there exists an element c of C such that $f(c) = k$. Now C is p-divisible, therefore there exists an element c_1 of C

satisfying $p^n c_1 = c$. Then $f(p^n c_1) = f(c) = k$ implies that $p^n f(c_1) = k$ which in turn shows that K is p-divisible. Now Z is a T-hereditary ring and T is a hereditary torsion theory. So we can apply Theorem 2.18 to deduce that an abelian group C is T-compact if and only if $C/T(C)$ is T-injective. Hence an abelian group C is T-compact if and only if $C/T(C)$ is p-divisible for each $p \in P$. For the case of P being the set of all primes, the group C is T-compact exactly when $C/T(C)$ is divisible. ■

CHAPTER THREE

T-NOETHERIAN AND T-HEREDITARY RINGS.

In this chapter we examine the class of T-compact R-modules a little more closely. We would like to see what sort of results we obtain when the ring R satisfies some additional properties and T is hereditary. We have already seen, in the previous chapter, that the class of T-compact R-modules is responsible for the behaviour of T-compact abelian groups. We will show that under some additional assumptions on the ring R and the torsion theory, T, the class of T-compact R-modules forms the torsion class of a torsion theory. In fact we are able to show how this torsion theory relates, in the lattice of torsion theories, to the torsion theories determined by the class of T-injective R-modules and the original theory T. These results will also enable us to obtain another characterisation of T-compact R-modules under some additional conditions on R and T.

Our first definitions are that of a "noetherian" and a "T-noetherian" ring. As mentioned above, these are additional assumptions that we will sometimes make on the ring R so that we can obtain some important results.

Definitions 3.1.

(i). We call the ring R noetherian if every left ideal is finitely generated.

(ii). A ring R is called T-noetherian provided that the direct sum of T-injective R-modules is T-injective. ∎

When the torsion theory T is hereditary we see in the next proposition that the class of noetherian rings is contained in the class of T-noetherian rings.

Proposition 3.2.

If T is hereditary every noetherian ring is T-noetherian.

Proof.

Let R be a noetherian ring and $\{C_i | i \in I\}$ a family of T-injective R-modules where I is some index set. We need to show that the direct sum $\Sigma_i C_i$ is T-injective. We will apply the T-injective Test Lemma (Lemma 2.14) to show this. To this end let K be a T-dense left ideal of R and let $f: K \to \Sigma_i C_i$ be a homomorphism. We need to find an element $x \in \Sigma_i C_i$ such that for each $k \in K$ we have $f(k) = kx$. For each $i \in I$ we consider the ith projection $\pi_i : \Sigma_i C_i \to C_i$ and we define f_i to be the composition $\pi_i \circ f$

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[rr, bend right, "f_i"'] & \Sigma_i C_i \arrow[r, "\pi_i"] & C_i \end{tikzcd}$$

Since each C_i is T-injective, we have for each $i \in I$ an $x_i \in C_i$ such that $f_i(k) = kx_i$ for every $k \in K$. Now K is finitely generated. Suppose that K is generated by $\{k_1, k_2, \ldots, k_n\}$, that is, each element of K can be expressed as a product of some powers of the elements in the set. If $c \in \Sigma_i C_i$, the element c will have only a finite number of non-zero components. Hence for each $j = 1,2, \ldots, n$, $f(k_j)$ (being an element of $\Sigma_i C_i$) will have only a finite number of non-zero components. So we see that there are only a finite number of non-zero $f_i\colon K \to C_i$ to consider. Let I' be a subset of the index set I consisting of all the $i \in I$ for which f_i is a nonzero homomorphism. For each $i \in I'$ there exists an $x_i \in C_i$ such that $f_i(k) = kx_i$ for every $k \in K$. Now let $x \in \Sigma_i C_i$ be the element whose ith component is :

$$\begin{cases} x_i & \text{if } i \in I' \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(k) = kx$ for each $k \in K$. To see this let $k \in K$. Then the ith component of the element kx will be :

$$\begin{cases} kx_i & \text{if } i \in I' \\ 0 & \text{otherwise.} \end{cases}$$

Now for any $k \in K$ and $i \in I'$ we have that $\pi_i \circ f(k) = f_i(k) = kx_i$. If $i \notin I'$ then $f_i(k) = 0$. So we can see that the ith component of the element $f(k)$ will be :

$$\begin{cases} kx_i & \text{if } i \in I' \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $f(k) = kx$. Then x is our required element and $\Sigma_i C_i$ is therefore T-injective. ∎

We know from Proposition 2.12 that an R-module C is T-injective if and only if $C = D(C)$. It therefore follows that every direct sum of T-injective R-modules is T-injective if and only if for every family $\{C_i | i \in I\}$ of T-injective R-modules we have that $D(\Sigma_i C_i) = \Sigma_i D(C_i)$: For suppose that every direct sum of T-injective R-modules is T-injective. Let $\{C_i\}$ be a family of T-injective R-modules. Then $D(\Sigma_i C_i) = (\Sigma_i C_i) = \Sigma_i D(C_i)$ because each C_i is T-injective. The reverse implication is clear.

We see that D(-) is an object function on **R-Mod**. Can D(-) be extended to act on homomorphisms of **R-Mod** as well? If $f: C \to A$ is a homomorphism between two R-modules, then we can define $D(f): D(C) \to D(A)$ as follows: Let us consider the following diagram:

$$\begin{tikzcd}
C \arrow[r, "i_1"] \arrow[d, "f"'] & D(C) \arrow[ddl, "D(f)"] \\
A \arrow[d, "i_2"'] & \\
D(A) &
\end{tikzcd}$$

where i_1 and i_2 are the inclusion homomorphisms. The homomorphism $D(f)$ exists because $D(C)/C$ is torsion and $D(A)$ is T-injective. Lambek, in [L_1], shows that in general the object function D(-) cannot be made into a functor such that $M \to D(M)$ is natural in M, for any R-module M. It is then shown that when D is restricted to the category **C** of torsion-free R-modules, then D(-) is a functor. In fact D(-) turns out to be the left adjoint of the inclusion functor $\mathbf{A} \to \mathbf{C}$ where **A** is the category of torsion-free divisible R-modules. We state some of the above mentioned results in the following proposition.

<u>Proposition 3.3.</u> (L_1, page 11)

Let T be a hereditary torsion theory. The object function D : **R-Mod** $\to$ **R-Mod** is not a functor on **R-Mod** in general. However when D(-) is restricted to the category, **X**, of torsion-free R-modules, then D(-)is a functor on **X**. ∎

Recall from Definition 2.17 that a ring R is called T-hereditary provided the epimorphic image of a T-injective R-module is T-injective. The next result shows that under certain conditions the class of T-injective R-modules forms the torsion class of a torsion theory. We denote this torsion theory by d_T.

<u>Theorem 3.4.</u>

Let T be a hereditary torsion theory and let R be a T-hereditary and T-noetherian ring. Then the class of all T-injective R-modules forms a torsion class for a torsion theory, d_T.

<u>Proof.</u>

According to Definition 1.13 we need to show that the class of T-injectives is closed under the formation of direct sums, epimorphic images and extensions. The first two follow from the fact that R is T-noetherian and T-hereditary respectively. From Lemma 2.15 (iii) we have that whenever A is a submodule of an R-module B and both A and B/A are T-injective, then B is T-injective. Thus the class of T-injective R-modules is closed under extensions. ∎

We would like to prove a result similar to Theorem 3.4 for the class of all T-compact R-modules. The next theorem shows that under the same assumptions of Theorem 3.4 the class of T-compact R-modules also forms the torsion class for a torsion theory.

<u>Theorem 3.5.</u>

Let T be a hereditary torsion theory and let R be T-hereditary and T-noetherian. Then the class of all T-compact R-modules forms the torsion class for a torsion theory.

<u>Proof.</u>

We first show that the class of all T-compact R-modules is closed under direct sums. Let $\{C_i\}_I$ be a family of T-compact modules. We need to show that the direct sum $\Sigma_i C_i$ is T-compact. Now since R is T-hereditary, an R-module C is T-compact if and only if C/TC is T-injective (Theorem 2.18.). Therefore to show that $\Sigma_i C_i$ is T-compact it suffices to consider $(\Sigma_i C_i)/(T(\Sigma_i C_i))$. From Proposition 2.23, we know that T commutes with direct sums, so we have that $(\Sigma_i C_i)/T(\Sigma_i C_i)$ is isomorphic to $(\Sigma_i C_i)/(\Sigma_i T(C_i))$. But we find that $(\Sigma_i C_i)/(\Sigma_i T(C_i))$

is isomorphic to $\Sigma_i(C_i/T(C_i))$: For each i let $n_i : C_i \to C_i/T(C_i)$ be the canonical epimorphism. Then let $n : \Sigma_i C_i \to \Sigma_i(C_i/T(C_i))$ be defined by

$n(c_1 + c_2 + \ldots + c_i + \ldots) = n_1(c_1) + n_2(c_2) + \ldots + n_i(c_i) + \ldots$,

where $c_i \in C_i$ for each i. Then n is an epimorphism and

$Ker(n) = \Sigma_i(Ker(n_i) = \Sigma_i T(C_i)$. We know that

$n(\Sigma_i C_i) \cong (\Sigma_i C_i)/Ker(n)$ which means that $\Sigma_i(C_i/T(C_i)) \cong (\Sigma_i C_i)/(\Sigma_i T(C_i))$. Each C_i is T-compact, therefore each $C_i/T(C_i)$ is T-injective. Since R is T-noetherian, the R-module $\Sigma_i(C_i/T(C_i))$ will be T-injective. Hence $(\Sigma_i C_i)/T(\Sigma_i C_i)$ is T-injective.

We now show that the epimorphic image of a T-compact R-module is T-compact. So let K be a T-compact module and K/D a factor module of K where D is any submodule of K. We need to show that K/D is T-compact. Let B be a T-closed submodule of $(K/D) \times H$ where H is an arbitrary R-module. Let

$q_D: K \times H \to (K/D) \times H$ be the product of the canonical epimorphism and the identity homomorphism on H. Now let C be formed by the pullback of the inclusion $m_3: B \to (K/D) \times H$ along

$q_D: K \times H \to (K/D) \times H$ as shown below. The homomorphisms will be explained in the paragraph following Fig. 3.1.

$$\begin{tikzcd}
C \arrow[r, "q_D'"] \arrow[d, "m_2"'] \arrow[dr, phantom, "PB"] & B \arrow[d, "m_3"] \\
K \times H \arrow[r, "q_D"] & (K/D) \times H
\end{tikzcd}$$

By the construction of pullbacks in **R-Mod**, we see that $C = \{((k,h),b) \mid ((k,h),b) \in (K \times H) \times B \text{ and } q_D(k,h) = b\}$. Thus $C \cong \{(k,h) \mid q_D(k,h) \in B\}$. So C can be seen as an intersection of two R-modules which are isomorphic to B and $(K \times H)$ respectively and thus C can be looked at as a submodule of both $K \times H$ and B. We now consider the following commutative diagram

$$\begin{tikzcd}
\pi_2(C) \arrow[r, "\pi_2'"] \arrow[d, "m_1"'] & C \arrow[r, "q_D'"] \arrow[d, "m_2"'] & B \arrow[r, "p_2'"] \arrow[d, "m_3"'] & p_2B \arrow[d, "m_4"'] \\
H \arrow[r, "\pi_2"] & K \times H \arrow[r, "q_D"] & (K/D) \times H \arrow[r, "p_2"] & H
\end{tikzcd}$$

Fig. 3.1

where : π_2 and p_2 are the second projection homomorphisms of $K \times H$ to H and $(K/D) \times H$ to H respectively.

π_2' and p_2' are the restrictions of π_2 to C and p_2 to B

q_D' is the restriction of q_D to C, and

m_1, m_2, m_3 and m_4 are the inclusion homomorphisms.

Now T-closed embeddings are closed under formation of pullbacks (Proposition 1.4(iv)), therefore m_2 is a T-closed embedding. This means that $T((K \times H)/C) = 0$, that is, $(K \times H)/C$ is torsion-free. We note that $\pi_2 = p_2 \circ q_D$. So $\pi_2(C) = (p_2 \circ q_D)(C) = p_2(B)$. Since K is T-compact and $(K \times H)/C$ is torsion-free, it follows that $\pi_2(C)$ is a T-closed submodule of H. But $\pi_2(C) = p_2(B)$. Thus $p_2(B)$ is a T-closed submodule of H and K/D is T-compact.

We shall now show that the class of T-compact R-modules is closed under extensions. Let

$$0 \longrightarrow A \xrightarrow{m} B \xrightarrow{e} C \longrightarrow 0$$

be a short exact sequence with A and C T-compact. We must prove that B is T-compact. Note that m is a monomorphism, while e is an epimorphism, and C is isomorphic to B/A (because of the fact that the given sequence is a short exact one). Let D be formed by the indicated pullback in the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D \arrow[r, "m'"] \arrow[d, "i_1"'] \arrow[dr, phantom, "\mathrm{PB}"] & T(B) \arrow[r, "e'"] \arrow[d, "i_2"] & T(B)/D \arrow[r] \arrow[d, "i_3"] & 0 \\
0 \arrow[r] & A \arrow[r, "m"'] \arrow[d, "n_1"'] & B \arrow[r, "e"'] \arrow[d, "n_2"'] & C \arrow[r] \arrow[d, "n_3"] & 0 \\
0 \arrow[r] & A/D \arrow[r, "m''"'] \arrow[d] & B/T(B) \arrow[r, "e''"'] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The symbol i_2 denotes the inclusion of $T(B)$ into B. By construction of pullbacks in **R-Mod**, we see that $D = \{(a,b) \mid (a,b) \in A \times T(B) \text{ and } m(a) = i_2(b)\}$ and can therefore be seen as a submodule of both A and $T(B)$. The homomorphisms m' and e' are the restrictions of m to D and e to $T(B)$ respectively. The homomorphism i_1 can then be seen as the inclusion homomorphism. The R-module C is isomorphic to B/A and the homomorphism i_3 is induced by the action of i_1 and i_2. Then let Q be isomorphic to $C/(T(B)/D)$. The homomorphisms n_1, n_2 and n_3 are the canonical epimorphisms. We can see that all three columns are exact. From [B_2], page 55, Theorem 4.12, we find that the third row is also exact with m'' and e'' being the monomorphism and epimorphism respectively making that last row exact. The R-module Q will then also be isomorphic to $(B/T(B))/(A/D)$.

Now $B/T(B)$ is torsion -free and A/D is isomorphic to a submodule of $B/T(B)$. Therefore A/D is torsion-free because torsion-free R-modules are closed under submodules. Hence D is a T-closed submodule of A. Since A is T-compact, it follows by Theorem 2.18 that A/D is T-injective. Considering the short exact sequence

$$0 \longrightarrow A/D \xrightarrow{m''} B/T(B) \xrightarrow{e''} Q \longrightarrow 0$$

we have that A/D is T-injective and $B/T(B)$ is torsion-free. Therefore by Lemma 2.15 (i), the R-module Q will be torsion-free. Since Q is isomorphic to $C/(T(B)/D)$, the R-module $T(B)/D$ is a T-closed submodule of C. But C is T-compact, therefore by Theorem 2.18 $C/(T(B)/D)$ is T-injective. So we have that A/D and Q (which is isomorphic to $(B/T(B))/(A/D)$) are both T -injective. Since the class of T-injectives is closed under extensions (Lemma 2.15(iii)), this means that $B/T(B)$ must also be T-injective. Hence by Theorem 2.18, B will be T-compact. ∎

The torsion theory discussed in the preceding theorem will be denoted by the symbol s_T. Before we discuss the relationship between the torsion theories s_T, d_T, and T, we will first define what is meant by a "smaller" torsion theory.

<u>Definition 3.6.</u> ([L_1] , page 5)

Let $T = (\underline{B}, \underline{C})$ and $T' = (\underline{B}', \underline{C}')$ be two torsion theories. We say that T is <u>smaller</u> than T' if $\underline{B} \subseteq \underline{B}'$, or equivalently, $\underline{C}' \subseteq \underline{C}$. ∎

We will now show that given any arbitrary family of torsion theories, we can find a smallest torsion theory which contains every member of the family. The proof of the preceding statement is broken up into the following three results.

<u>Lemma 3.7.</u>

Given any family $\{(\underline{T}_i, \underline{F}_i) \mid i \in I\}$ of torsion theories on **R-Mod**, then $(\underline{\cap T_i}, \underline{(\cap T_i)^r})$ is a torsion theory, where

$\underline{(\cap T_i)^r} = \{A \in \textbf{R-Mod} \mid \text{Hom}(B,A) = 0\}$ for every $B \in \underline{\cap T_i}\}$.

<u>Proof.</u>

We use Definition 1.13 to prove this lemma. We firstly show that the class $\underline{\cap T_i}$ is closed under direct sums, epimorphic images and group extensions. Suppose that there is a family A_j, $j \in J$ with each $A_j \in \underline{\cap T_i}$. Then we need to show that the direct sum $\Sigma_j A_j \in \underline{\cap T_i}$. But $A_j \in \underline{T}_i$ for every $j \in J$ and $i \in I$. Thus $\Sigma_j A_j \in \underline{T}_i$ for each $i \in I$ because $(\underline{T}_i, \underline{F}_i)$ is a torsion theory. So we have $\Sigma_j A_j \in \underline{\cap T_i}$. So $\underline{\cap T_i}$ is closed under direct sums. If an R-module A belongs to $\underline{\cap T_i}$, then $A \in \underline{T}_i$ for every $i \in I$. Thus every epimorphic image of A belongs to each $\underline{T}_i$ since each $\underline{T}_i$ is closed under epimorphic images. Hence every epimorphic image of A belongs to $\underline{\cap T_i}$. Similar reasoning shows that $\underline{\cap T_i}$ is also closed under group extensions.

Secondly we show that the class $\underline{(\cap T_i)^r}$ is closed under products, submodules and group extensions. Now $B \in \underline{(\cap T_i)^r}$ implies that $\mathrm{Hom}(A,B) = 0$ for every $A \in \underline{\cap T_i}$. So suppose that a family $\{ B_j | \ j \in J \}$ is such that $B_j \in \underline{(\cap T_i)^r}$ for each $j \in J$. We must show that $\Pi B_j \in \underline{(\cap T_i)^r}$. To do that it suffices to show that $\mathrm{Hom}(A, \pi B_j) = 0$ for every $A \in \underline{\cap T_i}$. Now $A \in \underline{\cap T_i}$ implies that for each $j \in J$ $\mathrm{Hom}(A,B_j) = 0$ because $B_j \in \underline{(\cap T_i)^r}$. This implies that $\mathrm{Hom}(A,\Pi B_j) = 0$. For if $\mathrm{Hom}(A,\Pi B_j) \neq 0$, then there exists $f: A \to \Pi B_j$ such that $f \neq 0$. Then there would be an element j of J such that $f_j = \pi_j \circ f \neq 0$ and $f_j \in \mathrm{Hom}(A, B_j)$ contradicting the fact that $B_j \in \underline{(\cap T_i)^r}$. So $\underline{(\cap T_i)^r}$ is closed under products. Now suppose that an R-module $B_1 \in \underline{(\cap T_i)^r}$ and B_2 is a submodule of B_1. If $f: A \to B_2$ is a non-zero homomorphism from some R-module A of $\cap T_i$ to B_2, then f can also be seen as a non-zero homomorphism from A to B_1, contradicting the fact that $\mathrm{Hom}(A,B_1) = 0$ for every $A \in \underline{\cap T_i}$. Thus we must have $\mathrm{Hom}(A, B_2) = 0$ for every $A \in \underline{\cap T_i}$. Therefore $B_2 \in \underline{(\cap T_i)^r}$ and $\underline{(\cap T_i)^r}$ is closed under submodules. Now suppose that B_1 is a submodule of an R-module B_2 such that B_1 and B_2/B_1 both belong to $\underline{(\cap T_i)^r}$. We need to show that $B_2 \in \underline{(\cap T_i)^r}$. Suppose that $f: A \to B_2$ is a non-zero homomorphism from some R-module A of $\cap T_i$ to B_2. Now $f(A) \neq B_1$ otherwise f would be a non-zero homomorphism from A to B_1 contradicting the fact that $\mathrm{Hom}(A, B_1) = 0$ for every $A \in \underline{\cap T_i}$. So if $n: B_2 \to B_2/B_1$ is the canonical epimorphism, then $n \circ f: A \to B_2/B_1$ would be a non-zero homomorphism since we have shown above that $f(A) \neq B_1$. This contradicts the fact that $\mathrm{Hom}(A, B_2/B_1) = 0$. So we must have that $\mathrm{Hom}(A,B_2) = 0$ for every $A \in \underline{\cap T_i}$. Thus $B_2 \in \underline{(\cap T_i)^r}$. The third condition of Definition 1.13 follows from the definition of $\underline{(\cap T_i)^r}$. ■

Observation 3.8.

There is a torsion theory which contains every torsion theory.

Proof.

Let $(\underline{T}, \underline{F})$ be the torsion theory on **R-Mod** in which every R-module is torsion. Then every torsion theory is contained in $(\underline{T}, \underline{F})$. ■

Proposition 3.9.

Let $\{(\underline{T}_i, \underline{F}_i) \mid i \in I\}$ be a family of torsion theories. Then there exists a smallest torsion theory $(\underline{T}, \underline{F})$ such that $(\underline{T}_i, \underline{F}_i) \leq (\underline{T}, \underline{F})$ for each $i \in I$.

Proof.

Let $K = \{(\underline{S}_j, \underline{G}_j) \mid j \in J\}$ be the family of all torsion theories $(\underline{S}_j, \underline{G}_j)$ such that $(\underline{T}_i, \underline{F}_i) \leq (\underline{S}_j, \underline{G}_j)$ for each $i \in I$, $j \in J$. By Observation 3.8 this class is not empty. Now let $\underline{S} = \cap \underline{S}_j$ and $\underline{G} = (\cap \underline{S}_j)^r$. Then from Lemma 3.7 we know that $(\underline{S}, \underline{G})$ is a torsion theory. Note that $\underline{T}_i \subseteq \underline{S}_j$ for each $i \in I$, $j \in J$ and consequently $\underline{T}_i \subseteq \cap \underline{S}_j$ for all $i \in I$ and $j \in J$. Therefore $(\underline{T}_i, \underline{F}_i) \leq (\underline{S}, \underline{G})$. Suppose that $(\underline{S'}, \underline{G'})$ is a smaller torsion theory than $(\underline{S}, \underline{G})$ which contains $(\underline{T}_i, \underline{F}_i)$ for every $i \in I$. Now $(\underline{T}_i, \underline{F}_i) \leq (\underline{S'}, \underline{G'})$ for each $i \in I$ implies that $(\underline{S'}, \underline{G'})$ is one of the torsion theories belonging to K. Now $\underline{S} = \cap \underline{S}_i$, therefore $\underline{S} \subseteq \underline{S'}$. But $\underline{S'} \subseteq \underline{S}$ because $(\underline{S'}, \underline{G'}) \leq (\underline{S}, \underline{G})$. So we must have that $(\underline{S'}, \underline{G'}) = (\underline{S}, \underline{G})$. ■

We are now in a position to discuss the relationship between an arbitrary hereditary torsion theory T and the two torsion theories, s_T and d_T, that T induces, namely, the torsion theories determined by the classes of T-compact and T-injective R-modules respectively.

Corollary 3.10.

Let T be a hereditary torsion theory and let R be T-hereditary and T-noetherian. Then the torsion theory, s_T, determined by the class of T-compact R-modules is the smallest torsion theory containing both d_T and T.

Proof.

Recall that d_T denotes the torsion theory determined by the class of T-injective R-modules. Suppose that $s_T(C) = C$, that is, C is T-compact. Then C/T(C) is T-injective by Theorem 2.18. Now

$$0 \longrightarrow T(C) \longrightarrow C \longrightarrow C/T(C) \longrightarrow 0$$

is a short exact sequence. So we can see that C is an extension of a torsion R-module by a T-injective R-module. Then C must belong to the torsion class of every torsion theory that contains both T and d_T. Thus s_T is contained in the smallest torsion theory containing both d_T and T.

On the other hand, if an R-module C is a torsion module, then C = T(C). Hence C/T(C) will be T-injective and C will be T-compact by Theorem 2.18. Also, if an R-module C is T-injective, then C/T(C) will be T-injective by Definition 2.17

and therefore C will be T-compact (by Theorem 2.18) in this instance as well. ∎

We will now look at another characterisation of T-compactness. It was pointed out in a discussion with S.Joubert, that an additional assumption needed to be made about the isomorphism mentioned in the statement of Corollary 3.4 of [F_1]. This assumption is now mentioned in the statement of the following corollary.

<u>Corollary 3.11.</u>

Let T be hereditary and R be both T-hereditary and T-noetherian. An R-module C is T-compact if and only if, for every torsion R -module A, $Ext^1(A,T(C)) \cong Ext^1(A,C)$, where the isomorphism is assumed to be the natural one, that is, it is the homomorphism $Ext^1(A,T(C)) \to Ext^1(A,C)$ which appears in the long exact sequence

$$0 \longrightarrow Hom(A, T(C)) \longrightarrow Hom(A, C) \longrightarrow$$
$$Hom(A, C/T(C)) \longrightarrow Ext^1(A, T(C)) \longrightarrow$$
$$Ext^1(A, C) \longrightarrow Ext^1(A, C/T(C)) \longrightarrow Ext^2(A, T(C))$$
$$\longrightarrow \ldots.$$

which is obtained from the short exact sequence

$$0 \longrightarrow T(C) \longrightarrow C \longrightarrow C/T(C) \longrightarrow 0$$

(see [L_2], page 159)

Proof.

Assume that C is a T-compact R-module and let A be any torsion R-module. Furthermore let f denote the homomorphism between $Ext^1(A, T(C))$ and $Ext^1(A, C)$ mentioned in the statement of the corollary. From Theorem 2.18 we find that C/T(C) is T-injective. Now applying Proposition 2.13 we find that $Ext^1(A, C/T(C))$ is equal to zero. Now A is torsion and C/T(C) is torsion -free. Hence Hom (A, C/T(C)) is equal to zero. So we have the exact sequence

$$0 \longrightarrow Ext^1(A, T(C)) \xrightarrow{f} Ext^1(A, C) \longrightarrow 0$$

obtained from the long exact one above. We can thus see that the homomorphism f is an isomorphism.

Conversely suppose that C is an R-module such that $Ext^1(A,T(C)) \cong Ext^1(A, C)$ for every torsion R-module A, where the indicated isomorphism is as explained in the statement of the corollary. We need to show that C is T-compact. We first show that if C is any R-module, then $Ext^2(A,C) = 0$ for all torsion R-modules A : Consider the R-module C. We know that C can be embedded in an injective R-module I, say (page 138 of [L_2]). Then by the remarks following Definition 2.3 we see that I is T-injective. We can obtain the following short exact sequence

$$0 \longrightarrow C \longrightarrow I \longrightarrow I/C \longrightarrow 0.$$

Since I is T-injective and R is T-hereditary, we find that I/C is T-injective. If A is any torsion R-module, we can now obtain the long exact sequence

$$0 \longrightarrow \text{Hom }(A,C) \longrightarrow \text{Hom }(A,I) \longrightarrow \text{Hom }(A,I/C) \longrightarrow$$
$$\text{Ext}^1 (A,C) \longrightarrow \text{Ext}^1 (A,I) \longrightarrow \text{Ext}^1 (A,I/C) \longrightarrow \text{Ext}^2 (A,C)$$
$$\longrightarrow \text{Ext}^2 (A,I) \longrightarrow \text{Ext}^2 (A,I/C) \longrightarrow ..$$

Now $\text{Ext}^1 (A,I/C)$ is equal to zero by Proposition 2.13. By Theorem 6.1(v), page 142 of [L_2], we find also that $\text{Ext}^2 (A,I)$ is zero. So we obtain the following exact sequence from the above long exact sequence

$$0 \longrightarrow \text{Ext}^2 (A,C) \longrightarrow 0$$

which proves that $\text{Ext}^2 (A,C) = 0$.

Now let A be any torsion R-module. From the short exact sequence

$$0 \longrightarrow T(C) \longrightarrow C \longrightarrow C/T(C) \longrightarrow 0$$

we again obtain the long exact sequence

$$0 \longrightarrow \text{Hom }(A,T(C)) \longrightarrow \text{Hom }(A,C) \longrightarrow$$
$$\text{Hom }(A, C/T(C)) \longrightarrow \text{Ext}^1(A, T(C)) \longrightarrow \text{Ext}^1(A,C) \longrightarrow$$
$$\text{Ext}^1(A, C/T(C)) \longrightarrow \text{Ext}^2(A, T(C)) \longrightarrow \text{Ext}^2(A,C) \longrightarrow \ldots$$

Now $\text{Hom }(A, C/T(C))$ is zero because A is torsion and $C/T(C)$ is torsion-free. We have already shown that $\text{Ext}^2(A, C) = 0$ for any R-module C, hence $\text{Ext}^2(A, T(C)) = 0$. We therefore obtain the following exact sequence

$$0 \xrightarrow{f_1} \operatorname{Ext}^1(A, T(C)) \overset{f_2}{\cong} \operatorname{Ext}^1(A, C) \xrightarrow{f_3} \operatorname{Ext}^1(A, C/T(C)) \xrightarrow{f_4} 0$$

where f_1, f_2, f_3, and f_4 are the corresponding homomorphisms as indicated above.

Now $\operatorname{Ker}(f_3) = \operatorname{Im}(f_2) = \operatorname{Ext}^1(A, C)$ because f_2 is assumed to be an isomorphism and is hence surjective. This means that $f_3(\operatorname{Ext}^1(A, C)) = 0$. But f_3 is an epimorphism. Hence $\operatorname{Ext}^1(A, C/T(C))$ is zero. Since this holds for all torsion R-modules A, we know from Proposition 2.13 that $C/T(C)$ will be T-injective. We also have R to be T-hereditary. So applying Theorem 2.18, we see that C is T-compact. ∎

CHAPTER FOUR

TOPOLOGICAL COMPARISONS.

Due to the categorical nature of the definition of compactness, a number of topological results carry over to this algebraic setting where the analogues to the topological notion of closed subspaces, Hausdorff spaces and compact spaces will be the T-closed submodules, torsion-free R-modules and T-compact R-modules respectively. In Section 4 of [HSS] many topological results are generalised to obtain corresponding categorical results. However the results appearing in Section 4 of [HSS] apply only to hereditary constructs. We have already pointed out in an earlier discussion in the introduction to this dissertation that **R-Mod** is not such a category. However we will demonstrate in this chapter that many of these results can hold in the category of R-modules. We list some of these topological results .

In this algebraic setting of R-modules and R-module homomorphisms we find that the analogue to the notion of a Hausdorff space is given by the torsion-free R-module. Here is a characterisation of torsion-free R-modules in terms of the factorisation system discussed in Proposition 1.31. We find that it is the analogue to the theorem which states that "A topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in $X \times X$". (See Theorem 13.7 of $[W_2]$)

Proposition 4.1.

An R-module X is torsion-free if and only if

$$X \xrightarrow{\Delta_X} X \times X$$

is a T -closed embedding.

Proof.

We recall that Δ_X is the homomorphism which sends $x \in X$ to $(x,x) \in X \times X$. Now $\Delta_X(X) \cong X \cong X \times \{0\}$. Also $G \cong (G \times G)/(G \times \{0\})$ for all R-modules G. Hence $X \cong (X \times X)/(\Delta_X(X))$. So X is torsion-free if and only if $(X \times X)/\Delta_X(X)$ is torsion -free which holds if and only if $\Delta_X: X \to X \times X$ is a T-closed embedding. ∎

Earlier on in this dissertation, in Proposition 1.5, a class E-Sep relative to a factorisation structure, (E , M) on a given category **X** , was mentioned. We now look at this class in the category **R-Mod** and in relation to the (T-dense , T-closed embedding) factorisation structure and we find that E-Sep consists of all the torsion-free R-modules. The latter can thus be viewed as the analogue of Hausdorff spaces in topology.

Proposition 4.2.

Let (E , M) be the (T-dense, T-closed embedding) -factorisation system on **R- Mod**. Then an R-module Y belongs to the class

E-Sep if and only if Y is torsion -free.

Proof.

Let Y be an arbitrary R -module. Then $Y \in$ E-Sep if and only if $\Delta_X: Y \to Y \times Y$ belongs to the class M (Proposition 1.7) which in turn holds if and only if Y is torsion-free (Proposition 4.1). ■

The following result is analagous to the topological result which states that the compact subset of a Hausdorff space is closed (Theorem 17.5 of [W_2]). This result also corresponds with Proposition 4.14 of [HSS].

Proposition 4.3.

If A is a T-compact submodule of a torsion-free R-module B, then A is T-closed.

Proof.

Let $m: A \to B$ be the inclusion homomorphism. By Proposition 4.2, the R-module B belongs to the class E-Sep and from Proposition 1.7 we see that the graph of m, that is $\langle id_A, m\rangle$, is a T -closed embedding. Let us consider the following commutative diagram, where π_B is the second projection and id_A is the identity homomorphism acting on A.

$$\begin{tikzcd} A \times B \arrow[r, "\pi_B"] & B \\ A \arrow[u, "<id_A, m>"] \arrow[r, "id_A"'] & A \arrow[u, "m"'] \end{tikzcd}$$

Now $(\pi_B \circ <id_A , m>)(A)$ is equal to $(m \circ id_A)(A)$ which in turn is equal to $m(A)$. Since A is T-compact, the projection π_B will preserve T-closed submodules. Hence $m(A) = \pi_B (<id_A ,m>(A))$ is a T-closed submodule of B. Thus A is T-closed in B. ■

The following proposition echoes the topological result which states that a closed subspace of a compact space is compact. (See Theorem 17.5 of [W_2])

Proposition 4.4.

Let C be a T-closed submodule of a T -compact R-module A. Then C is T-compact.

Proof.

Suppose that C is a T-closed submodule of a T-compact R-module A with corresponding embedding m. We need to show that C is T-compact. So let Z be an arbitrary R-module and N any T-closed submodule of $C \times Z$ with corresponding embedding n. Let us consider the following commutative diagram

$$\begin{tikzcd}
A \times Z \arrow[r, "\pi_2"] & Z \\
C \times Z \arrow[u, "m \times id_Z"] & \\
N \arrow[u, "n"] \arrow[uur, "p_2"'] &
\end{tikzcd}$$

where π_2 and p_2 are the second projection homomorphisms from $A \times Z$ to Z and N to Z respectively. We need to show that $p_2(N)$ is a T-closed submodule of Z. Now $m \times id_Z$ is a T-closed embedding by Proposition 1.4(iv). It then follows from Proposition 1.4(i) that $(m \times id_Z)\circ n$ is a T-closed embedding as well. Therefore N is a T-closed submodule of $A \times Z$. Then $p_2(N) = (p_2 \circ n)(N) = (\pi_2 \circ (m \times id_Z))(N) = \pi_2(N)$. By the T-compactness of A we see that $\pi_2(N) = p_2(N)$ is a T-closed submodule of Z. ∎

The topological result which states that the product of two compact spaces is compact can also be reflected in this setting. The following result shows this. We also find that Proposition 4.10 of [HSS] contains the categorical equivalent of the following proposition.

<u>Proposition 4.5.</u>

The finite product of T-compact R-modules is T-compact.

Proof.

Clearly it suffices to show that the product of two T-compact R-modules is T-compact. To this end let A and B be two T-compact R-modules, H any arbitrary R-module, and C a T-closed submodule of $A \times B \times H$. Now we consider the following diagram

$$\begin{tikzcd}
C \arrow[r, "p_2'"] \arrow[d, "m_1"'] & p_2(C) \arrow[r, "\pi_2'"] \arrow[d, "m_2"'] & (\pi_2 \circ p_2)C \arrow[d, "m_3"'] \\
A \times B \times H \arrow[r, "p_2"] \arrow[d, "n_1"'] & B \times H \arrow[r, "\pi_2"] \arrow[d, "n_2"'] & H \arrow[d, "n_3"'] \\
(A \times B \times H)/C \arrow[r, "p_2''"] & (B \times H)/p_2(C) \arrow[r, "\pi_2''"] & H/(\pi_2 \circ p_2)(C)
\end{tikzcd}$$

where : $p_2 : A \times (B \times H) \to B \times H$ and $\pi_2 : B \times H \to H$ are the usual second projections,

p_2' and π_2' are the restrictions of p_2 to C and π_2 to $p_2(C)$ respectively,

m_1, m_2, and m_3 are the inclusion homomorphisms because Im $(p_2') = p_2\ (C)$ which is a submodule of $B \times H$ and $(\pi_2 \circ p_2)(C)$ is a submodule of H,

n_1, n_2, and n_3 are the canonical epimorphisms, and

p_2'' and π_2'' are induced by the homomorphisms p_2 and π_2 respectively.

Because A is T-compact we have that $(B \times H)/p_2(C)$ is torsion-free, that is, $p_2(C)$ is a T-closed submodule of $B \times H$. Since B is T-compact it is true that $H/(\pi_2 \circ p_2(C))$ is torsion-free. So $\pi_2 \circ p_2(C)$ is a T-closed submodule of H. Hence $A \times B$ is T-compact. ∎

We find that the Tychonoff theorem which states that an arbitrary product of compact spaces is compact does not have a corresponding result in this setting here. The following result from page 77 of [DG_2] demonstrates this.

Example 4.6.

Let p be a fixed prime number. For each $n \in N$ let $M_n = Z/p^nZ$, where Z is the ring of integers. Then $M = \Pi \{M_n : n \in N\}$ is not t-compact, where t is the radical of Example 1.17 with A taken to be Z.

Proof.

We first note that M/t(M) is not equal to zero, because the element of M which has each of its infinite number of components equal to 1 cannot have finite order. Let us denote the element of M described above as 1* and (1* +t(M)) will be the corresponding element belonging to M/t(M). We will show that (1* +t(M)) is not p-divisible. Suppose that there is an element (a + t(M)) of M/t(M) such that (1* + M) = p(a + t(M)), where $a = (a_1, a_2, \ldots a_k, \ldots)$ and each a_n belongs to M_n. Then we deduce that the element $(1-pa_1, 1-pa_2, \ldots 1-pa_k, \ldots)$ of M will then belong to t(M). For this element to belong to t(M) it must have at most a finite number of non-zero components. This then implies that $1 - pa_i = 0$ for some $i \in N$. This implies that $1 = pa_i$

for some a_i in the group Z_{p^i} which is a contradiction because the order of 1 in Z_{p^i} is p^i while the order of pa_i in Z_{p^i} is certainly less than p^i. Thus M/t(M) is not p-divisible which implies that M/t(M) is not divisible. By Corollary 2.26 we see that M is not t-compact. ■

The following can be seen as analgous to the topological result which states that if a Hausdorff space B contains a dense subspace A, that is compact, then B is compact. In fact our analogue is even stronger than the original result because in this case B does not have to be Hausdorff.

Proposition 4.7.

If T is hereditary and A is a T -dense submodule of an R - module B, and if A is T -compact, then so is B .

Proof.

Let A be a T -dense and T -compact submodule of the R -module B and C any T -closed submodule of B. We consider the following commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A\cap C \arrow[r, "i_1"] \arrow[d, "i_4"'] & C \arrow[r, "n_1"] \arrow[d, "i_5"'] & C/A\cap C \arrow[r] \arrow[d, "i_6"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_2"] \arrow[d, "n_4"'] & B \arrow[r, "n_2"] \arrow[d, "n_5"'] & B/A \arrow[r] \arrow[d, "n_6"] & 0 \\
0 \arrow[r] & A/(A\cap C) \arrow[r, "i_3"'] \arrow[d] & B/C \arrow[r, "n_3"'] \arrow[d, no head] & B/(A+C) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

where : i_1 , i_2, i_4 and i_5 are the inclusion homomorphisms,

i_3 is induced by the action of i_2,

n_1, n_2, n_4, and n_5 are the canonical epimorphisms

A+C is the direct sum of the R-modules A and C.

We can see that the first and second rows and columns are exact. By the proof of Theorem 4.6 of [B1], page 39, the third row and column are also exact. Hence n_3 and n_6 can also be seen as the canonical epimorphisms while i_6 can be seen as the inclusion homomorphism.

Since $B/(A+C)$ is an epimorphic image of the torsion R-module B/A, the R-module $B/(A+C)$ will also be torsion (by Definition 1.13). Now $A/(A\cap C)$ can be seen as a submodule of the torsion-free R-module B/C and will therefore be torsion-free. So $A\cap C$ is a T-closed submodule of the R -module A. But A is T-compact and T is hereditary. Therefore by Theorem 2.18 the R-module $A/(A\cap C)$ will be T-injective. Looking at the exact sequence formed by the third row of our diagram

$$0 \longrightarrow A/(A\cap C) \xrightarrow{i_3} B/C \xrightarrow{n_3} B/(A+C) \longrightarrow 0$$

we find that the first R-module is T-injective while the second is torsion -free. By Lemma 2.15 (i), the R-module $B/(A+C)$ will be torsion-free. But it was shown earlier that $B/(A+C)$ was torsion as well. Hence $B/(A+C)$ must be zero. So the third row now becomes

$$0 \longrightarrow A/(A\cap C) \xrightarrow{i_3} B/C \longrightarrow 0$$

We therefore see that $A/(A\cap C)$ is isomorphic to B/C. Hence B/C is T-injective. From Theorem 2.18 we find that B is T-compact. ∎

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