

THE ASYMPTOTIC STABILITY OF STOCHASTIC KERNEL OPERATORS

by

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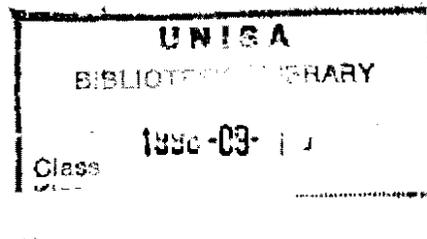
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ABSTRACT

A stochastic operator is a positive linear contraction, $P : L^1 \rightarrow L^1$, such that $\|Pf\|_1 = \|f\|_1$ for $f \geq 0$. It is called asymptotically stable if the iterates $P^n f$ of each density converge in the norm to a fixed density. $Pf(x) = \int K(x, y)f(y) dy$, where $K(\cdot, y)$ is a density, defines a stochastic kernel operator. A general probabilistic/deterministic model for biological systems is considered. This leads to the LMT operator

$$Pf(x) = \int_0^{\lambda(x)} -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) dy,$$

where $-H'(x) = h(x)$ is a density. Several particular examples of cell cycle models are examined. An operator overlaps supports if for all densities f, g , $P^n f \wedge P^n g \neq 0$ for some n . If the operator is partially kernel, has a positive invariant density and overlaps supports, it is asymptotically stable. It is found that if $h(x) > 0$ for $x \geq x_0 \geq 0$ and

$$\int_0^{\infty} x^\alpha h(x) dx < \liminf_{x \rightarrow \infty} (Q(\lambda(x))^\alpha - Q(x)^\alpha) \text{ for } \alpha \in (0, 1]$$

then P is asymptotically stable, and an opposite condition implies P is sweeping. Many known results for cell cycle models follow from this.

Keywords. Markov operator, stochastic operator, asymptotic stability, ergodic theory, biological models, cell cycle models, kernel operators, doubly stochastic operators, Harris operators, stochastic process.

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PREFACE

One could give a loose definition of ergodic theory as the mathematical study of the long term average behaviour of systems. The idea of a density has come to the foreground lately as attempts are made to provide unifying descriptions of phenomena that appear statistical in nature. The use of densities has increased dramatically in the study of biological, physical and economic systems. We have come to associate them with the appearance of large systems with inherent elements of uncertainty.

The term *stochastic process* is frequently used in connection with observations from a time-orientated physical process controlled by some random mechanism. In ergodic theory there is attached a precise meaning to a stochastic process called a Markov process. However, there is some confusion in the terminology, and that is why we have preferred to call a positive linear contraction on L^1 a *stochastic operator*, and then, its dual acting in L^∞ a *Markov operator*. Except for this we generally use the terminology of [LM1].

We are especially interested in integral stochastic operators, given by kernels, and their applications in biology. The asymptotic stability of such an operator is of great interest and means that our system eventually stabilises to some known equilibrium state. Specifically, we are giving a unified treatment of some cell cycle models. In this study some general conditions for asymptotic stability are given

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and proofs provided. The general conditions are then applied to kernel operators, and then in turn to the biological systems.

The tools used are functional analysis and measure theory, up to the level of Royden, [Roy].

We will briefly outline the structure of the study.

Chapter I gives a general background to ergodic theory. The emphasis is on definitions and important basic facts including those concerning Harris processes, doubly stochastic operators and Cesaro convergence. The notion of a sweeping operator is also discussed. Most of the proofs are omitted.

We study the convergence of the iterates $P^n f$ in the L^1 -norm in Chapter II. Here important proofs are included. The connection between constrictivity and asymptotic periodicity is examined first, and then the important lower bound function technique of Lasota and Mackey.

Chapter III sees the development of a general deterministic/probabilistic model specifically suited to the study of biological systems. The stochastic kernel operator which is derived is termed an LMT operator, in honour of A. Lasota, M. Mackey and J. Tyrcha who first collaborated in its derivation and study. It is then shown that some specific and well studied examples of cell cycle models are special cases, notably the models of Lasota and Mackey and also those of Tyson and Hannsgen.

To study this model further we need tools developed in Chapter IV which provide stability results for kernel operators. A proof for a general version of Krasnoselskii's theorem by A. Lasota is given. We then provide proofs for doubly stochastic oper-

ators. It is shown that a Harris operator with a strictly positive invariant density that overlaps supports is asymptotically stable, immediately providing the desired result for a kernel operator. General results of Bartoszek for strong Feller kernels are also provided. These can immediately be applied to LMT operators, since these are indeed strong Feller in the strict sense.

Chapter V starts with the derivation by Lasota and Baron of a sufficient condition that the LMT operator admits a stationary density. A simple extra condition implies asymptotic stability. Conditions are also stated that imply P is sweeping. These results are then used to unify results scattered in various papers on cell cycle models.

CHAPTER I

STOCHASTIC OPERATORS

Chapter I is of an informative nature and the necessary definitions are given together with the statements of theorems, without proofs. The intention is to make the work self-contained and place the rest of the text in perspective.

We will denote the space of Lebesgue-integrable real valued functions on the measure space (X, \mathcal{A}, μ) with $L^1(X, \mathcal{A}, \mu)$ and, its dual, the space of essentially bounded measurable functions with $L^\infty(X, \mathcal{A}, \mu)$, or L^1 and L^∞ respectively if the context is clear. We will also make the assumption throughout Chapter I that μ is σ -finite, and this will not be repeated. We will also make the convention that the word "measure" will always refer to a *positive* measure. The words "signed measure" will always be written out in full. Functions that agree almost everywhere will be identified, and all equalities will hold in the a.e. sense (unless otherwise stated). The norms are defined as usual i.e. $\|f\|_1 = \int |f| d\mu$ for $f \in L^1$ and $\|f\|_\infty = \text{ess sup}\{|f(x)| : x \in X\}$ when $f \in L^\infty$.

1.1 The Deterministic Case

Historically, the study of stochastic operators was preceded by the study of transformations on measure spaces. We will consider these in this section. We now give examples of L^1 spaces which we will have opportunity to use.

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Examples 1.1.1.

(i) The Lebesgue space on $[0, 1]$ is in a sense the most general probability space and for many applications it will suffice to consider this space. This is because of the definition and theorem we will now state. (See [Roy], p.409).

Definition 1.1.1. Two measure spaces (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) are said to be **isomorphic** if there is a bijective map $\phi : X \rightarrow Y$ such that $\phi(A) \in \mathcal{B}$ and $\nu(\phi[A]) = \mu(A)$ for every $A \in \mathcal{A}$ and, hence for all $B \in \mathcal{B}$ we have $\mu(\phi^{-1}[B]) = \nu(B)$.

Theorem 1.1.1 (Kuratowski). *Let μ be a Borel probability measure on a complete separable space X . If X is uncountable and μ has no atoms, then (X, \mathcal{A}, μ) is isomorphic to $[0, 1]$ with Lebesgue measure λ .*

(ii) The Banach spaces $L^1(\mathbf{R}_+)$ and $L^1(\mathbf{R}_+^n)$ are important in many applications. We have, for example, age distributions, distributions of cell size, etc.

(iii) We will also use Lebesgue spaces on discrete measure spaces, i.e. Euclidean spaces and spaces of real sequences. \square

The Frobenius-Perron operator which we will introduce after Example 1.1.3 is a special case of a stochastic operator. It arises in a natural way when we study non-singular measurable transformations of a measure space.

First we state what is meant by a stochastic operator on L^1 .

Definition 1.1.2. Let (X, \mathcal{A}, μ) be a measure space. A stochastic operator

is a linear operator $P : L^1(X, \mathcal{A}, \mu) \rightarrow L^1(X, \mathcal{A}, \mu)$ such that

$$Pf \geq 0 \text{ for } f \geq 0 \quad (1.1.1)$$

$$\|Pf\|_1 = \|f\|_1 \text{ if } f \geq 0. \quad (1.1.2)$$

Some authors use the term **stochastic process** for the quadruple (X, \mathcal{A}, μ, P) (See [Fog]). We will use this terminology when the measure space is not clear from the context. A lot of research has been done in this century on these operators and there exists a rich and varied theory.

The adjoint of P , P^* , acting in L^∞ will be called a **Markov operator**. It is also a positive linear contraction, but on L^∞ . We will also talk of a Markov process.

Example 1.1.2. A simple example of a stochastic operator is the **shift operator** $P : L^1 \rightarrow L^1$ where $X = \mathbf{R}$ or \mathbf{R}_+ , defined by $Pf(x) = f(x - a)$ and $a \in \mathbf{R}_+$ is fixed. It translates the graph of f to the right by a units. In the case where $X = \mathbf{R}_+$ it is not invertable. It is easy to check that the conditions for a stochastic operator are satisfied. \square

We know that a stochastic operator is **monotone** ($Pf(x) \geq Pg(x)$ whenever $f(x) \geq g(x)$), and is a **contraction** ($\|Pf\| \leq \|f\|$ for all $f \in L^1$).

We now introduce some terminology from the literature. By a **density** we will mean an $0 \leq f \in L^1$ such that $\|f\|_1 = 1$. The convex set of densities is denoted by $\mathcal{D}(X, \mathcal{A}, \mu)$ or \mathcal{D} .

A **fixed point** of P is an $f \in L^1$ such that $Pf = f$ and an **invariant density** is a density that is a fixed point.

The set of densities is mapped into itself by a stochastic operator and this is also a characterisation of such operators.

Now we consider the special case of an operator associated with a transformation. A transformation $S : X \rightarrow X$ is called **measurable** if $S^{-1}(A) \in \mathcal{A}$ for every $A \in \mathcal{A}$ and **non-singular** if $\mu(A) = 0$ implies that $\mu(S^{-1}(A)) = 0$. A **measure preserving transformation (m.p.t.)** is a measurable transformation such that $\mu(S^{-1}(A)) = \mu(A)$ whenever $A \in \mathcal{A}$. Note that a m.p.t. is always non-singular.

Examples 1.1.3.

(i) The **quadratic map**, $S : [0, 1] \rightarrow [0, 1]$ given by $S(x) = 4x(x-1)$ is a measurable transformation that was studied well by Ulam and Von Neumann and used by them to generate pseudorandom numbers on the first computers.

(ii) The **tent map** $S : [0, 1] \rightarrow [0, 1]$ given by

$$S(x) = \begin{cases} 2x, & \text{for } x < \frac{1}{2} \\ 2 - 2x, & \text{for } x \geq \frac{1}{2} \end{cases},$$

is related to (i).

(iii) The **dyadic transformation**, $S(x) = 2x \pmod{1}$, is an example of a transformation that preserves Lebesgue measure. \square

Note that the transformations given here are not invertible.

The operator

$$P\nu(A) = \nu(S^{-1}(A)) \tag{1.1.3}$$

on another measure ν on X describes the effect of applying S to the space (X, \mathcal{A}, ν) .

Thus the forward evolution of a measure under iterates of P are given by (1.1.3).

The Frobenius-Perron operator is a special case of (1.1.3) where ν is absolutely

continuous w.r.t. μ and the transformation is non-singular, which guarantees that $\nu \circ S^{-1} \ll \mu$. In this case ν can be identified with a function $f \in L^1(X, \mu, \mathcal{A})$ by the Radon-Nikodym theorem, i.e. $f = \frac{d\nu}{d\mu}$ and Pf is identified with $\nu \circ S^{-1}$. We can then in fact verify that P is stochastic. Thus we have the following:

Definition 1.1.3. The stochastic operator $P : L^1 \rightarrow L^1$ corresponding to a non-singular measurable transformation on X satisfying the equation

$$\int_A Pf(x)d\mu(x) = \int_{S^{-1}(A)} f(x)d\mu(x) \quad (1.1.4)$$

for all $f \in L^1$, $A \in \mathcal{A}$, is called the **Frobenius-Perron operator** corresponding to S .

From the way we have introduced this operator it is obvious that if P corresponds to the transformation S , then $P^n = P_n$ where P_n corresponds to the transformation

$$S_n = \underbrace{S \circ S \circ \dots \circ S}_{n \text{ times}}.$$

As has been indicated, iterates of the Frobenius-Perron operator describe the forward evolution in time of a density f on X under the action of the transformation.

The operator

$$Uf(x) = f(S(x)) \quad (1.1.5)$$

acting on L^∞ can be shown to be the adjoint of the Frobenius-Perron operator and is thus a Markov operator. We see that it describes the backward evolution in time of a density f . The operator U is called a **Koopman operator**.

The constant density is always a stationary density of P if S is measure preserving. For such a transformation the different strengths of convergence of an arbitrary

initial density to the constant stationary density under the action of the Frobenius Perron operator describe varying degrees of chaotic behavior of the transformation.

Definition 1.1.4. A transformation is called ergodic if the only invariant sets are trivial, i.e. if $A \in \mathcal{A}$ and $A = S^{-1}(A)$ imply that $A = X$ or $A = \emptyset$.

The following are simple examples of ergodic transformations.

Examples 1.1.4.

(i) Consider the transformation T on the integers given by $T(x) = x + 1$. Clearly there are no nontrivial invariant subsets and T is ergodic, but if we let $T(x) = x + 2$ then the even and odd numbers are two invariant subsets and T is not ergodic.

(ii) Let (X, μ) be the unit circle with Borel measure and S a rotation of the circle by an angle of ϕ radians. We can show that if $\phi/2\pi$ is rational then there are nontrivial invariant subsets and the transformation is not ergodic. However, if this number is irrational we can show that the transformation is ergodic, using some of the techniques we will later discuss. \square

We quote the following well known three theorems from [LM1], p.72,

Theorem 4.4.1. First we state a characterisation of ergodic measure preserving transformations.

Theorem 1.1.2. A m.p.t. S on a probability space is ergodic if and only if the iterates of f under the corresponding Frobenius-Perron operator, $\{P^n f\}$ are weakly Cesaro convergent to 1 for every density f , i.e.

$$\frac{1}{n} \sum_{k=0}^{n-1} P^k f \xrightarrow{n \rightarrow \infty} 1 \text{ weakly.}$$

The famous **Birkhoff individual ergodic theorem**, (see [Pet] p.30), which states that the iterates of the Koopman operator are Cesaro convergent a.e. if the transformation is measure preserving can be deduced from this.

There is an intermediate level of chaotic behavior which is defined as follows:

Definition 1.1.5. Let (X, \mathcal{A}, μ) be normalised and $S : X \rightarrow X$ a m.p.t. S is called **mixing** if

$$\lim_{n \rightarrow \infty} \mu(A \cap S^{-n}(B)) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{A}$.

We give the following characterisation.

Theorem 1.1.3. Let (X, \mathcal{A}, μ) be a probability space with S a m.p.t, and P the Frobenius-Perron operator corresponding to S . Then S is mixing if and only if $\{P^n f\}$ is weakly convergent to 1 for every density f .

We will generalise this property to arbitrary stochastic operators as follows:

Definition 1.1.6. Let P be a stochastic operator on (X, \mathcal{A}, μ) . P is called **weakly operator mixing (w.o.m.)** if there exists a unique density f_* such that $P^n f \xrightarrow[n \rightarrow \infty]{} f_*$ weakly in L^1 for every $f \in \mathcal{D}$.

The highest level of chaotic behaviour we consider is exactness.

Definition 1.1.7. A m.p.t. S on a probability space such that $S(A) \in \mathcal{A}$ if

$A \in \mathcal{A}$ is called **exact** if $\mu(S^n(A)) \rightarrow 1$ as $n \rightarrow \infty$ for every set $A \in \mathcal{A}$ with $\mu(A) > 0$.

Figuratively speaking we may say that the iterates of any set of positive measure under the transformation eventually "fill" the space. Obviously an invertible map T can never be exact because we have $\mu(T(A)) = \mu(A)$ for every $A \in \mathcal{A}$ when μ is preserved. Thus neither of the Examples 1.1.4 are exact.

Examples 1.1.5. All of the Examples 1.1.3 are exact when appropriate measures are defined, see Section 2.4. (It seems intuitively clear that the dyadic transformation is exact if one makes a few sketches.) \square

We finally give the following:

Theorem 1.1.4. *Let (X, \mathcal{A}, μ) be normalised and a m.p.t. S be given such that $S(A) \in \mathcal{A}$ whenever $A \in \mathcal{A}$. S is exact if and only if $P^n f \xrightarrow[n \rightarrow \infty]{} 1$ in L^1 -norm for all $f \in \mathcal{D}$ where P is the Frobenius-Perron operator corresponding to S .*

The rest of this work is mainly devoted to a generalisation of this property to arbitrary stochastic operators, the so-called asymptotic stability property.

Definition 1.1.8. Let P be a stochastic operator on (X, \mathcal{A}, μ) . P is called **asymptotically stable** if there exists a unique stationary density f_* such that $P^n f \xrightarrow[n \rightarrow \infty]{} f_*$ in L^1 -norm for every $f \in \mathcal{D}$.

If P is a Frobenius-Perron operator which is asymptotically stable then the corresponding transformation is exact with respect to an appropriately defined

measure (see [LY], Proposition 1). Definition 1.1.6 gives a very desirable property of stochastic operators since it means that any probability distribution will eventually converge to a fixed, known, distribution.

1.2 The General Case

The adjoint of a stochastic operator is a Markov operator, acting on L^∞ as defined in the previous section, i.e. we have $\langle Pu, f \rangle = \langle u, P^*f \rangle$ for all $u \in L^1$, $f \in L^\infty$ where $\langle u, f \rangle = \int u f d\mu$.

We may then consider P acting on the Banach lattice of all signed measures absolutely continuous w.r.t μ on (X, \mathcal{A}) by identifying $f \in L^1$ with $\nu \ll \mu$ by the Radon-Nikodym Theorem, i.e $f = d\nu/d\mu$. Then we have

$$(P\nu)(A) = \int P^*1_A(x) d\nu(x). \quad (1.2.1)$$

We may also extend P and P^* by monotone continuity, so that they are defined for all non-negative measurable functions (see [Fog], Ch.I).

We define a function $P : X \times \mathcal{A} \rightarrow \mathbf{R}$ by $P(x, A) = P^*1_A(x)$ where $A \in \mathcal{A}$. It is known (see [Fog] p.2), that $P(x, \cdot)$ is a probability measure on X for a.e. $x \in X$ and that $P(\cdot, A)$ is \mathcal{A} -measurable for each fixed $A \in \mathcal{A}$.

If these properties hold *everywhere* we call $P(x, A)$ a **transition probability**.

These properties of $P(x, A)$ are in fact equivalent to the definition of a stochastic operator if we let

$$P^*f(x) = \int f(y) P(x, dy) \quad (1.2.2)$$

for $f \in L^\infty$ and then as in (1.2.1) we have

$$(P\nu)(A) = \int_X P(x, A) d\nu(x) \quad (1.2.3)$$

for $\nu \ll \mu$ (see [Fog] p.2).

We give some examples of stochastic operators.

Example 1.2.1. We have already discussed the Frobenius-Perron operator corresponding to a non-singular measurable transformation. \square

The stochastic kernel operator is the most important for our purposes.

Definition 1.2.1. A stochastic kernel is a non-negative function

$$K : (X \times X, \mathcal{A} \otimes \mathcal{A}) \rightarrow \mathbf{R}$$

such that K is measurable and $\int K(x, y) d\mu(x) = 1$ for every $y \in X$, i.e $K(\cdot, y)$ is a density for every y .

Definition 1.2.2. If the transition probability of a stochastic operator is given by a stochastic kernel we call the operator a stochastic kernel operator .

Then we have the following identities which are slight modifications of [Fog] p.5.

$$Pf(x) = \int_X K(x, y) f(y) d\mu(y) \text{ for } f \in L^1 \quad (1.2.4)$$

$$P^*h(y) = \int_X K(x, y) h(x) d\mu(x) \text{ for } h \in L^\infty \quad (1.2.5)$$

and

$$P(x, A) = \int_A K(x, y) d\mu(y) \text{ for } A \in \mathcal{A} \quad (1.2.6)$$

$$(P\nu)(A) = \int_X \int_X K(x, y) 1_A(x) d\mu(x) d\nu(y) \text{ if } A \in \mathcal{A} \text{ and } \nu \ll \mu. \quad (1.2.7)$$

We give the following examples of kernels:

Examples 1.2.2.

(i) If the space X is discrete, with μ counting measure, $K(x, y)$ is a matrix with $\sum_x K(x, y) = 1$, $K(x, y)$ is called a stochastic matrix and the stochastic process is called a Markov chain.

(ii) Consider (\mathbf{R}, λ) with

$$K(t, x, y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp \left[-\frac{(x-y)^2}{2\sigma^2 t} \right] \quad (1.2.8)$$

This is the kernel for the heat equation where

$$u(t, x) = P_t f(x) = \int_{-\infty}^{\infty} K(t, x, y) f(y) dy \quad (1.2.9)$$

gives the solution of the P.D.E

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} \text{ for } t > 0, x \in \mathbf{R} \quad (1.2.10)$$

with the initial condition $u(0, x) = f(x); f \in L^1(\mathbf{R})$. \square

We will later meet other examples that arise in the mathematical modelling of biological systems.

We give the following general examples of stochastic operators (see [Fog], Ch.I):

Examples 1.2.3.

(i) If for $B \in \mathcal{A}$ we define the **restriction operator** T_B by $T_B f(x) = 1_B(x) f(x)$, then $T_B \mu(A) = \mu(A \cap B)$.

(ii) Let \mathcal{B} be a σ -subalgebra of \mathcal{A} . Then $Pf = E(f | \mathcal{B})$ where $f \in L^1(\mathcal{A})$, the **conditional expectation** of f given \mathcal{B} , is a stochastic operator (See a text on probability theory e.g. [Bil]).

(iii) If P^* is a Markov operator and $B \in \mathcal{A}$ is given such that $\mu(B) > 0$ and $P^*1_{B^c} \leq 1_{B^c}$ then P and P^* may be restricted to B and P^* is a Markov operator on $L^\infty(B)$ and $P^*1_B \geq 1_B$. \square

1.3 The Geometric Structure of a Stochastic Process

In this section we summarise some results on the geometry of the space on which a stochastic operator acts. These properties were extensively investigated in the 1950s and 1960s by, amongst others, E.Hopf.

If (X, \mathcal{A}, μ, P) is a stochastic process, we define the **deterministic part** of \mathcal{A} , \mathcal{A}_d , as follows :

$$\mathcal{A}_d = \{A \in \mathcal{A} : P^{*n}1_A = 1_{B_A}, n = 1, 2, 3, \dots\}.$$

The process is called **deterministic** if $\mathcal{A} = \mathcal{A}_d$. Thus, a process is deterministic if all images under P^* of all iterates of characteristic functions are again characteristic functions. The size of \mathcal{A}_d gives an idea to the extent to which a stochastic operator acts as if given by a transformation, and an operator given by a transformation is purely deterministic.

We know that \mathcal{A}_d is a σ -algebra if P is stochastic and that $A \in \mathcal{A}_d$ implies that $B \in \mathcal{A}_d$ where $P^*1_A = 1_B$. The proofs of these statements can be found in [Fog], on p.7.

The **invariant sets**, $\mathcal{A}_i = \{A \in \mathcal{A} : P^*1_A = 1_A\}$ form a σ -algebra and P acting

on \mathcal{A}_i is simply the identity operator. From the definitions it follows that $\mathcal{A}_i \subseteq \mathcal{A}_d$. We may study the action of P on each element of \mathcal{A}_i separately, and thus we may always assume that \mathcal{A}_i is trivial if it is atomic. In the case that it is trivial we say the process is **ergodic**. We may show that a process is ergodic if the only solutions of the equation $P^*f = f$, $f \in L^\infty$ are constants (See [Fog], p.21). It is also not hard to establish that the terminology is consistent, i.e. ergodic transformations give rise to ergodic Frobenius-Perron operators.

We will now state the Hopf maximal ergodic theorem and then present some of its consequences. The proof of this result may be found in [Fog], Ch.II.

Theorem 1.3.1 (E.Hopf , 1954).

Let $u \in L^1$ and define

$$E = \{x : \sup_n \sum_{k=0}^n P^k u(x) > 0\}$$

Then $\int_E u(x) d\mu(x) \geq 0$ (Note that u may take on negative values).

Definition 1.3.1. Let $0 < u_0 \in L^1$ be arbitrary (u_0 exists because μ is σ -finite).

Set

$$C = \{x : \sum_{k=0}^{\infty} P^k u_0(x) = \infty\} \text{ and } D = X - C. \quad (1.3.1)$$

Hopf showed that this partitioning of X is independent of the choice of u_0 (see p.10 of [Fog]). The letters C and D , stand, respectively, for **conservative** and **dissipative**. A process is conservative if $X = C$ and dissipative if $X = D$. An

example of a dissipative operator is the translation operator, and there are many examples of conservative operators e.g. the identity operator.

We collect some well known results on C and D , for which elementary proofs may be found in [Fog], Ch.II.

Theorem 1.3.2.

Let $u \in L^1$, $f \in L^\infty$ with $u, f \geq 0$.

(i)

$$\sum_{k=0}^{\infty} P^k u(x) < \infty \quad \text{if } x \in D \quad (1.3.2)$$

and

$$\sum_{k=0}^{\infty} P^k u(x) = \infty \quad \text{or } 0 \quad \text{on } C, \quad (1.3.3)$$

(From this it follows that $\lim_{k \rightarrow \infty} P^k u(x) = 0$ if $x \in D$. Thus the process "escapes" from the dissipative part of X . For a transformation this means that after "many" iterations there are "no" points left in D .),

(ii)

$$\sum_{k=0}^{\infty} P^k f(x) = \infty \quad \text{or } 0 \quad \text{for } x \in C,$$

(iii) $P^* 1_D \leq 1_D$ and $P^* 1_C \geq 1_C$ so that the process may be restricted to C . In particular, if a process is conservative $P^* 1 = 1$,

(iv) If $0 < u < \infty$, u is measurable, and $Pu \leq u$ on C then $Pu = u$ on C . That is, a subinvariant measure on C is invariant.

If the process is both conservative and ergodic we have the following, (see [Fog], p.60):

Theorem 1.3.3.

Let P be conservative and ergodic, $u \in L^1$ and $f \in L^\infty$.

(i) $f \geq 0$, $P^*f \leq f \Rightarrow f$ is constant.

(ii) $f \geq 0$, $f \neq 0 \Rightarrow \sum_{n=0}^{\infty} P^{*n}f = \infty$.

(iii) $u \geq 0$, $u \neq 0 \Rightarrow \sum_{n=0}^{\infty} P^n u = \infty$.

(iv) There is at most one σ -finite invariant measure (up to a multiplicative constant).

Let (X, \mathcal{A}, P, μ) be a stochastic process with a subinvariant measure λ , i.e. $P\lambda \leq \lambda$ and $\lambda \sim \mu$.

Under these circumstances both the operators P and P^* may be extended to the Hilbert space $L^2(X, \mathcal{A}, \mu)$ (See Proposition 1.5.1.1), and in fact are both positive contractions on L^2 .

We may then consider the portion of L^2 on which all powers of both operators act as isometries, and we define :

$$K = \{f \in L^2 : \|P^n f\|_2 = \|P^{*n} f\|_2 = \|f\|_2, n = 1, 2, 3, \dots\}$$

We quote the following ([Fog], Ch.VIII, Theorem A), which gives a decomposition

of the space X which we will use later:

Theorem 1.3.4.

- (i) K is an invariant subspace of P and P^* ;
- (ii) on K , $PP^* = P^*P = I$ (where I is the identity operator);
- (iii) if $g \perp K$ then $P^n g$ and $P^{*n} g$ converge weakly to zero in L^2 .

We shall also need the notion of a non-disappearing operator. A stochastic operator is called **non-disappearing** if

$$P^*f = 0 \Rightarrow f = 0$$

Hence, (See Lemma 0 in [KL]), if $P^*g = 1_A$ with $0 \leq g \leq 1$ then there exists a unique $E \in \mathcal{A}$ such that $g = 1_E$. Note that if P is conservative, it is non-disappearing.

1.4 Harris Operators

Proofs of all the results mentioned here may be found in [Fog], Ch.V and Ch.VI.

We write $K \leq P$ if $Kf \leq Pf$ for all positive functions $f \in L^1$.

We have the following (See Theorem A and Theorem C, [Fog], Ch.V):

Theorem 1.4.1 (Harris decomposition). *Let P be stochastic. Then $P = Q + R$ where Q is a unique substochastic kernel operator, $R \geq 0$, and there is no*

kernel K with $R \geq K \geq 0$. Further, if P is given by a transition probability, so is R , and $R(x, \cdot)$ is singular w.r.t. μ for a.e. x .

Hence, if P is stochastic, then P^n has a Harris decomposition $P^n = Q_n + R_n$.

Definition 1.4.1. P is said to be **Harris**, provided:

- (i) P is conservative,
- (ii) $Q_j \neq 0$ for some integer j .

The fact that a process is Harris has many important consequences which follow from useful properties of kernels, such as the following, which is Theorem E in Ch.V of [Fog]:

Theorem 1.4.2. *If P is Harris, \mathcal{A}_d is atomic.*

This shows that a Frobenius-Perron operator is never Harris, and that a Harris process' deterministic part is trivial. It also follows that a nontrivial kernel operator cannot be given by a transformation (unless it is a finite or countable state chain, see Section 3.1.1). In this sense kernel operators and Frobenius-Perron operators are extremes of a stochastic process.

Since $\mathcal{A}_i \subseteq \mathcal{A}_d$ we may always restrict the study of P to the atoms of \mathcal{A}_d and we may assume that P is ergodic if it is Harris.

One of the most useful properties of Harris processes is given by the following, Theorem E in Ch.VI of [Fog], which states that such a process always has a unique non-trivial invariant measure. This is an important fact because many ergodic

theorems need the existence of an invariant measure.

Theorem 1.4.3. *If P is Harris, there exists a σ -finite invariant measure equivalent to μ . If Ω_n are the atoms of \mathcal{A}_i and λ_n is a invariant measure of Ω_n , then $\lambda = \sum a_n \lambda_n$, $0 \leq a_n < \infty$ is the most general invariant measure weaker than μ .*

1.5 Invariant Densities

As we have previously mentioned, the question whether an invariant measure exists for a stochastic process is very important in the study of asymptotic properties of the process. This is because it is a necessary precondition to many of the so-called ergodic theorems which describe the long-term behaviour of averages of the process, which is in turn important in the interpretation of models in real life. In this section we intend to introduce some general criteria which will guarantee the existence of such a measure. The ideal is to have a finite invariant measure equivalent to μ , since such a measure can always be normalised to give a probability measure.

One criterium that there exists an invariant measure, has already been given, namely that P be Harris.

Firstly we will consider some implications of the existence of an invariant measure.

If f_* is a stationary density for P then

$$\mu_{f_*}(A) = \int_A f_*(x) d\mu(x) \quad (1.5.1)$$

is an invariant probability measure absolutely continuous w.r.t. μ and vice versa.

If P is the Frobenius-Perron operator of the transformation S then S is μ_{f_*} preserving, as is easily verified.

1.5.1 Doubly stochastic operators.

Suppose μ_{f_*} is strictly positive, i.e. $f_* > 0$. If we replace our original stochastic process with $(X, \mathcal{A}, \mu_{f_*}, P)$ then we have a new stochastic process with $P1 = 1$ and $P^*1 = 1$. Thus P is doubly stochastic.

Definition 1.5.1.1. A stochastic operator is **doubly stochastic** if

$$P1 = P^*1 = 1. \quad (1.5.1.1)$$

Another way to introduce this is by defining

$$\bar{P} : \bar{P}(f) = \frac{P(ff_*)}{f_*} \quad (1.5.1.2)$$

for $f \in L^1$. This amounts to exactly the same thing, i.e. we have $\bar{P}1 = \bar{P}^*1 = 1$. Note that $f_* > 0$ on X is essential. It is easy to see that the asymptotic stability of \bar{P} implies that of P . We quote the simple and well known result (see [Bro], p.6):

Proposition 1.5.1.1. *If P is doubly stochastic it maps L^p into L^p for every $1 \leq p \leq \infty$ with $\|P\|_p \leq 1$ and $\|P\|_1 = \|P\|_\infty = 1$. Moreover P^* is doubly stochastic.*

1.5.2 General Criteria .

We now state a general condition which guarantees the existence of a finite invariant measure (see [Fog], Ch.IV, Theorem B).

Theorem 1.5.2.1. A necessary and sufficient condition that there exists a finite invariant measure equivalent to μ is:

$$\liminf_{n \rightarrow \infty} (P^n \mu)(A) > 0 \quad (1.5.2.1)$$

for every $A \in \mathcal{A}$, $\mu(A) > 0$.

We also state the following more general result. The rather hard proof can be found in [Fog], Ch.IV.

Theorem 1.5.2.2. X may be decomposed uniquely in the disjoint union $X = A' \cup A''$, where:

(i) $A' = \cup_n A_n$ for sets A_n with $A_n \subset A_{n+1}$, and

$$\frac{1}{k} \sum_{j=0}^{k-1} P^{*j} 1_{A_n} \rightarrow 0 \text{ as } k \rightarrow \infty$$

uniformly of a set of measure zero; $P 1_{A'} \leq 1_{A'}$ so that the restriction (A'', A, μ, P) is well defined, and $A'' \subset C$, where C is the conservative part of X ;

(ii) There exists a σ -finite measure μ_* with $P\mu_* = \mu_*$, μ_* is equivalent to the restriction of μ to A'' , and every invariant measure is weaker than μ_* .

Recall that a sequence $\{f_n\}$ is **weakly precompact** if it has a subsequence that converges weakly to an $f \in L^1$. We now state

Proposition 1.5.2.1. If $\{\frac{1}{n} \sum_{k=0}^{n-1} P^k f\}$ is weakly precompact for an $f \in L^1$ then it converges strongly to a fixed point f_* of P . Further, if $f \in \mathcal{D}$ then f_* is a stationary density.

Remark 1.5.2.1: This theorem is a special case of an abstract ergodic theorem due to Kakutani and Yosida. The proof may be found in [LM1], on p.89. \square

1.6 Cesaro Convergence (Ergodic Theorems)

This work is devoted to strong convergence, but we mention some results of Cesaro convergence, because of their historic importance and practical significance. The determination of long term averages are of obvious importance in many fields.

The theorems we present were anticipated by the (Weak) Law of Large Numbers, J. Bernoulli (1713) and the Strong Law of Large Numbers, E. Borel (1909) for Bernoulli sequences of large numbers. The results were published in the beginning of the 1930s for point transformations (Von Neumann, 1932, Birkhoff, 1931), and improvements and generalisations have been appearing ever since.

We define the operator A_n which gives the average of the first n iterates of the operator P , as

$$A_n = \frac{1}{n} \sum_{k=0}^{n-1} P^k \quad (1.6.1)$$

The original mean ergodic theorem for point transformations, due to Von Neumann, states that the Cesaro averages of the Koopman operator converge in L^2 -norm if the transformation preserves μ .

Firstly we state two generalisations of this theorem. The proof of the first may be found in [Bro] on p.11.

Theorem 1.6.1 (Yosida mean ergodic theorem). *Let P be doubly stochastic operator on a probability space with $f \in L^p$. Then there exists an f_* in L^p such that $A_n f \xrightarrow[n \rightarrow \infty]{} f_*$ in L^p -norm.*

We also mention an abstract theorem due to R.Sine, [Sin].

Let $(X, \|\cdot\|)$ be a Banach space and let X^* denote the dual space to X . We say that $A \subseteq X$ separates $B \subseteq X^*$ if for any distinct pair x_1, x_2 of elements from B there exists an $a \in A$ such that $\langle a, x_1 \rangle \neq \langle a, x_2 \rangle$.

Theorem 1.6.2 (Sine). *Let P be a linear contraction on a Banach space X . Then the Cesaro averages A_n converge in the strong operator topology if and only if the fixed points of P separate the fixed points of P^* .*

The proof of the next theorem for mean convergence in Hilbert spaces may be found in [Pet] on p.24.

Theorem 1.6.3. *Let P be a contraction on a Hilbert space \mathcal{H} . Let $\text{Fix}(P) = \{f \in \mathcal{H} : Pf = f\}$, a closed linear subspace, and let $S : \mathcal{H} \rightarrow \text{Fix}(P)$ be the surjective projection. Then for each $f \in \mathcal{H}$, $A_n f \xrightarrow[n \rightarrow \infty]{} S(f)$.*

As was stated in Section 1.1, Birkhoff's theorem guarantees pointwise convergence a.e. of the Cesaro means of a m.p.t.

A version of the Chacon-Ornstein theorem is given next (Theorem F, [Fog], Ch.III) and we show how a pointwise theorem can be deduced from it.

Theorem 1.6.4 (Chacon-Ornstein). *Let $P : L^1 \rightarrow L^1$ be conservative and ergodic, $0 \leq f, g \in L^1$, then*

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} P^k f}{\sum_{k=0}^{n-1} P^k g} = \frac{\int f d\mu}{\int g d\mu} \quad \text{a.e.} \quad (1.6.2)$$

on $\{x : \sum_{k=0}^{\infty} P^k g(x) > 0\}$.

A special case arises when we take $g = 1$.

Corollary 1.6.1. *If P is doubly stochastic and $f \in \mathcal{D}$, then $A_n f \xrightarrow[n \rightarrow \infty]{} f_*$ in L^1 -norm, where f_* is the unique stationary density.*

We also remind the reader of Proposition 1.5.2.1.

1.7 Sweeping and Cesaro Sweeping

In this section we consider properties reminiscent of dissipativeness that are, also, in a sense opposite to the notion of asymptotic stability. These new properties describe the situation where densities are dispersed under the action of a stochastic operator. These properties have been investigated recently in the context of biological systems and we give a simple proof of a result which will be used later.

Definitions 1.7.1. Let a family $\mathcal{A}_* \subset \mathcal{A}$ be given. A stochastic operator P is called **sweeping** w.r.t. \mathcal{A}_* if

$$\lim_{n \rightarrow \infty} \int_A P^n f d\mu = 0 \text{ for each } A \in \mathcal{A}_* \text{ and } f \in \mathcal{D}. \quad (1.7.1)$$

A stochastic operator P is called **Cesaro-sweeping** w.r.t. \mathcal{A}_* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_A P^k f d\mu = 0 \text{ for every } A \in \mathcal{A}_* \text{ and } f \in \mathcal{D}. \quad (1.7.2)$$

In the sequel we shall assume that \mathcal{A}_* satisfies the following properties:

$$\mu(A) < \infty \text{ for each } A \in \mathcal{A}_*, \quad (1.7.3)$$

$$A_1, A_2 \in \mathcal{A}_* \Rightarrow A_1 \cup A_2 \in \mathcal{A}_*, \quad (1.7.4)$$

$$\text{There exists a sequence } \{A_n\} \in \mathcal{A}_* \text{ s.t. } \cup_n A_n = X. \quad (1.7.5)$$

A family satisfying these conditions is called **admissible**.

Remark 1.7.1. We can see that if \mathcal{A}_* is admissible the condition $f \in \mathcal{D}$ in Definitions 1.7.1 may be replaced by $f \in L^1$. It can easily be deduced from the definitions that a sweeping operator is Cesaro sweeping and it follows from the definition of an admissible family that neither admits an invariant density. \square

Example 1.7.1. It is not hard to see that the shift operator on \mathbf{R}_+ is sweeping w.r.t. $\mathcal{A}_* = \{[0, c] : c \in \mathbf{R}_+\}$. \square

We now give a simple proof of a condition that verifies sweeping in more difficult situations, which we will need later (from [LM1], p128). A bounded Borel-measurable function $V : X \rightarrow \mathbf{R}$ is called a **Bielecki function** if it is non-negative and if $\inf_{x \in A} V(x) > 0$ for every $A \in \mathcal{A}_*$.

Proposition 1.7.1. Let (X, \mathcal{A}, μ) be a measure space with $\mathcal{A}_* \subset \mathcal{A}$ given. Let P be a stochastic operator for which there exists a Bielecki function $V : X \rightarrow \mathbf{R}$ and a constant $\gamma < 1$ such that

$$\int_X V(x) P f(x) d\mu(x) < \gamma \int_X V(x) f(x) d\mu(x) \quad (1.7.6)$$

for every $f \in \mathcal{D}$. Then P is sweeping.

Proof. Fix an $f \in \mathcal{D}$ and $A \in \mathcal{A}_*$. Since the condition (1.7.6) holds for any density f , we may replace X by A and

$$\begin{aligned} \int_A P^n f(x) d\mu(x) &\leq \frac{1}{\inf_{x \in A} V(x)} \int_A V(x) P^n f(x) d\mu(x) \\ &\leq \frac{\gamma^n}{\inf_{x \in A} V(x)} \int_A V(x) f(x) d\mu(x) \end{aligned}$$

Since $\gamma < 1$ by assumption, the last expression converges to zero as $n \rightarrow \infty$ and condition (1.7.1) holds. Thus the proof is complete. ■

CHAPTER II

THE CONVERGENCE OF ITERATES

In this chapter we do not assume that μ is σ -finite. We investigate the strong convergence in L^1 of iterates of a stochastic operator.

2.1 Semigroups of Stochastic Operators

2.1.1 AL-spaces.

In much of the literature on stochastic operators the theorems are generalised to contractions on abstract L^1 spaces, the so-called AL-spaces. This is actually superficial as we will see from a theorem of Kakutani, Bohnenblust and Nakano which we will quote after a preliminary definition. For AL-spaces the result is due to Kakutani. We quote the theorems from [AB], p.192.

Definition 2.1.1.1. A Banach Lattice E is said to be an **abstract L^p -space**, for some $1 \leq p < \infty$, whenever its norm is p -additive, i.e. whenever $\|x + y\|^p = \|x\|^p + \|y\|^p$ holds for all $x, y \in E^+$ with $x \wedge y = 0$, where E^+ is the positive cone of E , i.e. $E^+ = \{x \in E : x \geq 0\}$.

We now state the well known result.

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Theorem 2.1.1.1. *A Banach lattice E is an abstract L^p -space for some $1 \leq p < \infty$ if and only if E is lattice isometric to some concrete $L^p(\mu)$ space.*

Thus all results proved for general $L^1(\mu)$ spaces are valid for AL-spaces.

We need to note that μ is not necessarily σ -finite and may exhibit other undesirable behaviour.

2.1.2 Semigroups.

Let T be a semigroup of real positive numbers (i.e. $\emptyset \neq T \subset (0, \infty)$, $t_1 + t_2 \in T$ for $t_1, t_2 \in T$) such that $t_1 - t_2 \in T$ for $t_1 > t_2$; $t_1, t_2 \in T$.

Definition 2.1.2.1. A family $\{P_t : t \in T\}$ of stochastic operators on L^1 will be called a **stochastic semigroup** (on L^1) if $P_{t_1+t_2} = P_{t_1}P_{t_2}$ for all $t \in T$.

The definition of asymptotic stability of a stochastic semigroup is similar to Definition 1.1.8 with n substituted by t . It is not difficult to verify the following, for which a simple proof may be found in [Pod].

Fact: If $\{P_t\}$ is asymptotically stable then the condition

$$P_t g = g$$

for a $t \in T$ and $g \in L^1$, $\|g\| = 1$, implies that $g = f_*$ or $g = -f_*$ where f_* is the unique stationary density. \square

We give the short proof out of [Pod] of the useful and well known

Proposition 2.1.2.1. *A stochastic semigroup $\{P_t : t \in T\}$ on L^1 is asymptotically stable if and only if there exists a $t_0 \in T$ such that P_{t_0} (as a stochastic operator) is asymptotically stable.*

Proof. The "only if" part is obvious (even for all $t_0 \in T$). For the proof of the "if" part assume that f_* is the stationary density of P_{t_0} , and observe that for a fixed $t \in T$

$$\|P_t f_* - f_*\|_1 = \|P_t(P_{t_0}^n f_*) - f_*\|_1 = \|P_{t_0}^n(P_t f_*) - f_*\|_1 \xrightarrow{n \rightarrow \infty} 0,$$

which implies that $P_t f_* = f_*$. Now, according to the fact that for each fixed $f \in \mathcal{D}$, the function $h(t) = \|P_t f - f_*\|_1 = \|P_t(f - f_*)\|_1$ is decreasing (each P_t is a contraction) and the fact that $h(nt_0) \xrightarrow{n \rightarrow \infty} 0$, we have $\|P_t f - f_*\|_1 \xrightarrow{t \rightarrow \infty} 0$ for every $f \in \mathcal{D}$, and the proof is complete. ■

2.2 Asymptotic Periodicity and Constrictivity

In this section we introduce some results for the property of asymptotic periodicity. See the paper [Ko2] for a unified exposition and for the proofs of the theorems mentioned here. The results followed the invited address of A. Lasota at the I.C.M. in 1982. Lasota introduced the concept of a constrictive operator.

Definition 2.2.1. A stochastic operator is called **weakly (strongly) constrictive** if there exists a weakly (strongly) compact/precompact set $\mathcal{F} \subset L^1$ such that the trajectories of all densities converge in L^1 -norm to \mathcal{F} . I.e. $d(\mathcal{F}, P^n f) \xrightarrow{n \rightarrow \infty} 0$ for every $f \in \mathcal{D}$, where $d(\mathcal{F}, f) = \inf\{\|g - f\|_1 : g \in \mathcal{F}\}$ for $f \in L^1$. The set \mathcal{F} is called a **constrictor** of P .

Lasota conjectured that weak and strong constrictivity are equivalent for stochastic operators on σ -finite spaces.

This was indeed proved by Komornik, [Ko3].

Theorem 2.2.1. *If a stochastic operator on a σ -finite space is weakly constrictive, it is strongly constrictive.*

Remark 2.2.1: This is a very useful result, as it is much easier to check for weak precompactness (several simple criteria are known). We will often talk of constrictive operators, leaving the prefix strongly or weakly. \square

Example 2.2.1. If a stochastic operator (on a σ -finite space X) has the property that there is an $f_0 \in L^1_+$ such that

$$\lim_{n \rightarrow \infty} \|(P^n f - f_0)^+\|_1 = 0 \quad (2.2.1)$$

for all $f \in \mathcal{D}$, then P is weakly and hence strongly constrictive. This is namely because the set \mathcal{F} of all functions $f \in L^1$ such that $|f(x)| \leq f_0(x)$ for a.e. $x \in X$ is weakly precompact and condition (2.2.1) guarantees that $\{P^n f\}$ converges to \mathcal{F} in the norm. \square

From the proof in [Ko3] it seemed that an even weaker sufficient condition for asymptotic periodicity (Definition 2.2.3) could be found. This was done in [KL] where the quasi-constrictive or smoothing operator was defined.

Definition 2.2.2. A stochastic operator P is quasi-constrictive (smoothing) if there exists a weakly compact set $\mathcal{F} \subset L^1$ and a constant $\theta < 1$ such

that

$$\limsup_{n \rightarrow \infty} d(P^n f, \mathcal{F}) \leq \theta$$

for every $f \in \mathcal{D}$.

There are several other characterisations which can be found in [Ko2].

Obviously constrictive operators are quasi-constrictive. These operators are so important because of the property of asymptotic periodicity that they possess.

Definition 2.2.3. A stochastic operator on a measure space (X, \mathcal{A}, μ) is called **asymptotically periodic** if there exist finitely many distinct densities g_1, \dots, g_r with disjoint supports, a permutation α of the set $\{1, \dots, r\}$ and positive continuous linear functionals $\lambda_1, \dots, \lambda_r$ on L^1 such that

$$\lim_{n \rightarrow \infty} \|P^n(f - \sum_{i=1}^r \lambda_i(f)g_i)\|_1 = 0 \quad (2.2.2)$$

for every $f \in \mathcal{D}$ and $Pg_i = g_{\alpha(i)}$, $i = 1, \dots, r$.

From the definition it immediately follows that an asymptotically periodic operator P may be written in the form

$$Pf(x) = \sum_{i=1}^r \lambda_i(f)g_i(x) + Qf(x), \quad (2.2.3)$$

where $\|P^n Qf\|_1 \xrightarrow{n \rightarrow \infty} 0$ for every $f \in L^1$, and that

$$P^{n+1}f(x) = \sum_{i=1}^r \lambda_i(f)g_{\alpha^n(i)}(x) + Q_n f(x), \quad (2.2.4)$$

where $Q_n = P^n Q$ and $\|Q_n f\|_1 \xrightarrow{n \rightarrow \infty} 0$ for every $f \in L^1$. The decomposition (2.2.3) is called the **Spectral Decomposition** of P .

We finally present (without proof):

Theorem 2.2.3 (Spectral Decomposition Theorem). *A quasi-constrictive stochastic operator on a σ -finite measure space is asymptotically periodic.*

The theorem is due to Lasota et al. [LLY] for strongly constrictive operators and due to Komornik and Lasota for quasi-constrictive operators [KoL].

Several further generalisations of the spectral decomposition theorem have been published. J. Komornik extended the results to positive power bounded linear operators acting on an $L^1(\mu)$ space where μ is σ -finite. Bartoszek showed in [Ba2] that a *strongly* constrictive operator on an AL-space is asymptotically periodic, extending the result to non σ -finite spaces. In a later paper he showed that all strongly constrictive positive contractions on Riesz spaces are asymptotically periodic [Ba6].

We give the short proof of the following from [LM1], p.105, which is a direct consequence of the spectral decomposition theorem and uses constrictivity to guarantee asymptotic stability.

Theorem 2.2.4. *Let P be a constrictive (asymptotically periodic) stochastic operator. Assume there exists $A \in \mathcal{A}$, $\mu(A) > 0$ such that for every $f \in \mathcal{D}$ there is a number $n_0(f)$ such that $P^n f(x) > 0$ for a.e $x \in A$ and $n \geq n_0(f)$. Then P is asymptotically stable.*

Proof. Since P is asymptotically periodic, representation (2.2.3) is valid. We will first show that $r = 1$.

Assume $r > 1$, and choose an integer i_0 such that A is not contained in the support of g_{i_0} . Take a density f of the form $f(x) = g_{i_0}(x)$ and let τ be the period of the

permutation α . Then we have

$$P^{n\tau} f(x) = g_{i_0}(x).$$

Clearly, $P^{n\tau} f(x)$ is not positive on the set A , since A is not contained in the support of g_{i_0} . This contradicts the assumption in the statement of the theorem and thus we must have $r = 1$.

Since $r = 1$, (2.2.4) reduces to

$$P^{n+1} f(x) = \lambda(f)g(x) + Q_n f(x),$$

so that

$$\lim_{n \rightarrow \infty} P^n f = \lambda(f)g \text{ in } L^1\text{-norm.}$$

Obviously $\lambda(f) = 1$, so that P is asymptotically stable, and the proof is complete. ■

2.3 Asymptotic Stability and Upper/Lower functions.

The technique developed by Lasota and Yorke in 1982 [LY] for proving the asymptotic stability of stochastic operators by the use of a lower-bound function has found many important applications.

The theorem has been generalised by Podhorodynski [Pod] to AL-spaces, but the proof is essentially the same. We note that the theorem thus also holds for non σ -finite spaces.

Definition 2.3.1. A function $h \in L^1$ is a **lower bound function** for a stochastic operator $P : L^1 \rightarrow L^1$ if

$$\lim_{n \rightarrow \infty} \|(P^n f - h)^-\|_1 = 0 \text{ for every } f \in \mathcal{D}. \quad (2.3.1)$$

Remark 2.3.1. Condition (2.3.1) may also be written as $(P^n f - h)^- = \epsilon_n$ where $\epsilon_n \xrightarrow{n \rightarrow \infty} 0$ or even more explicitly as $P^n f \geq h - \epsilon_n$. Thus, figuratively speaking, h is a function such that, for every density f , successive iterates of that density by P are eventually almost everywhere above h . \square

Since any non-positive function is a lower bound function, these functions are of no interest and we call a lower bound function **non-trivial** if $h \geq 0$ and $\|h\| > 0$.

Because condition (2.3.1) is framed in terms of the norm, it is sufficient that it is satisfied for all $f \in \mathcal{D}_0$ where \mathcal{D}_0 is dense in \mathcal{D} . This easily follows from the fact that all iterates P^n are contractions.

In the next theorem we will show that the existence of a lower bound function is indeed sufficient for asymptotic stability.

This gives an advantage of Theorem 2.3.1 above Theorem 2.2.4 because the latter theorem is not framed in terms of the norm, and we must check for each density. We also do not need to check for constrictivity. The disadvantage of Theorem 2.3.1 is that iterates must eventually be *uniformly* "above" zero on a set of positive measure.

We now prove the result from [LY] as in [LM1], on p.107.

Theorem 2.3.1. *Let $P : L^1 \rightarrow L^1$ be a stochastic operator, then P is asymptotically stable if and only if there is a non-trivial lower bound function for P .*

Proof. The "only if" part is obvious since the definition of asymptotic stability implies that f_* is a non-trivial lower bound function. The proof of the "if" part is

not so direct, and will be done in two steps. We first show that

$$\lim_{n \rightarrow \infty} \|P^n(f_1 - f_2)\|_1 = 0 \quad (2.3.2)$$

for every $f_1, f_2 \in \mathcal{D}$ and then proceed to construct the function f_* .

Step I. For every pair of densities $f_1, f_2 \in \mathcal{D}$, the $\|P^n(f_1 - f_2)\|_1$ is a decreasing function of n since a stochastic operator is a contraction on L_1 .

Now set $g = f_1 - f_2$ and note that, since $f_1, f_2 \in \mathcal{D}$, $c = \|g^+\|_1 = \|g^-\|_1 = \frac{1}{2}\|g\|_1$.

Assume $c > 0$. We have $g = g^+ - g^-$ and

$$\|P^n g\|_1 = c\|(P^n(g^+/c) - h) - (P^n(g^-/c) - h)\|_1. \quad (2.3.3)$$

Since g^+/c and g^-/c belong to \mathcal{D} , by equation (2.3.1), there must exist an integer n_1 , such that for all $n \geq n_1$

$$\|(P^n(g^+/c) - h)^-\|_1 \leq \frac{1}{4}\|h\|_1$$

and

$$\|(P^n(g^-/c) - h)^-\|_1 \leq \frac{1}{4}\|h\|_1.$$

Now we wish to establish upper bounds for $\|P^n(g^+/c) - h\|_1$ and $\|P^n(g^-/c) - h\|_1$.

To do this, first note that, for any pair of nonnegative and real numbers a and b ,

$$|a - b| = a - b + 2(a - b)^-.$$

Next write

$$\begin{aligned} \|P^n(g^+/c) - h\|_1 &= \int_X |P^n(g^+/c)(x) - h(x)| d\mu(x) \\ &= \int_X P^n(g^+/c)(x) d\mu(x) - \int_X h(x) d\mu(x) \\ &\quad + 2 \int_X (P^n(g^+/c)(x) - h(x))^- d\mu(x) \\ &= \|P^n(g^+/c)\|_1 - \|h\|_1 + 2\|(P^n(g^+/c) - h)^-\|_1 \\ &\leq 1 - \|h\|_1 + 2 \cdot \frac{1}{4}\|h\|_1 = 1 - \frac{1}{2}\|h\|_1 \text{ for } n \geq n_1. \end{aligned}$$

In the same way,

$$\|P^n(g^-/c) - h\|_1 \leq 1 - \frac{1}{2}\|h\|_1 \text{ for } n \geq n_1.$$

Thus (2.3.3) gives

$$\begin{aligned} \|P^n g\|_1 &\leq c\|P^n(g^+/c) - h\|_1 + c\|P^n(g^-/c) - h\|_1 \\ &\leq c(2 - \|h\|_1) = \|g\|_1(1 - \frac{1}{2}\|h\|_1) \text{ for } n \geq n_1. \end{aligned} \quad (2.3.4)$$

From (2.3.4), for all $f_1, f_2 \in \mathcal{D}$, we can find an integer n_1 such that

$$\|P^{n_1}(f_1 - f_2)\|_1 \leq \|f_1 - f_2\|(1 - \frac{1}{2}\|h\|_1).$$

By applying the same argument to $P^{n_1}f_1, P^{n_1}f_2$ we may find a second integer n_2 such that

$$\|P^{n_1+n_2}(f_1 - f_2)\|_1 \leq \|f_1 - f_2\|(1 - \frac{1}{2}\|h\|_1)^2.$$

Then, since $\|h\|_1 > 0$ we may repeat the procedure to conclude that (2.3.2) holds.

Step II. To complete the proof, we construct a maximal lower bound function for P . Thus, let

$$\rho = \sup\{\|h\|_1 : h \text{ is a lower bound function for } P\}.$$

Since by assumption there is a non-trivial h , we must have $0 < \rho \leq 1$.

Observe that for every two lower bound functions h_1, h_2 , the function $\max(h_1, h_2)$ is also a lower bound function. To see this, note that

$$\|(P^n f - h)^-\|_1 \leq \|(P^n f - h_1)^-\|_1 + \|(P^n f - h_2)^-\|_1.$$

Choose a sequence $\{h_j\}$ of lower bound functions such that $\|h_j\| \xrightarrow{j \rightarrow \infty} \rho$. Replacing, if necessary, h_j by $\max(h_1, \dots, h_j)$, we can construct an increasing sequence $\{h_j\}$ of lower bound functions, which will always have a limit (finite or infinite). This limiting function

$$h_* = \lim_{j \rightarrow \infty} h_j$$

is also a lower bound function since

$$\|P^n f - h_*\|_1 \leq \|P^n f - h_j\|_1 + \|h_j - h_*\|_1$$

and, by the Lebesgue monotone convergence theorem,

$$\|h_j - h_*\|_1 = \int_X h_*(x) d\mu(x) - \int_X h_j(x) d\mu(x) \xrightarrow{j \rightarrow \infty} 0.$$

Note that h_* is also the maximal lower bound function. This is because, for any other lower bound function, the function $\max(h, h_*)$ is also a lower bound function and

$$\|\max(h, h_*)\|_1 \leq \rho = \|h_*\|_1,$$

which implies $h \leq h_*$.

Observe that, since $(Pf)^- \leq Pf^-$, for every m, n , $n > m$,

$$\|(P^n f - P^m h_*)^-\|_1 \leq \|P^m (P^{(n-m)} f - h_*)^-\|_1 \leq \|(P^{(n-m)} f - h_*)^-\|_1,$$

which implies that, for every m , the function $P^m h_*$ is a lower bound function.

Thus, since h_* is the maximal lower bound function, $P^m h_* \leq h_*$ and since P^m preserves the integral, $P^m h_* = h_*$. Thus the function $f_* = h_*/\|h_*\|_1$ is a density satisfying $Pf_* = f_*$.

Finally, by (2.3.2), we have

$$\lim_{n \rightarrow \infty} \|P^n f - f_*\|_1 = \lim_{n \rightarrow \infty} \|P^n f - P^n f_*\|_1 = 0 \text{ for every } f \in \mathcal{D},$$

which completes the proof. ■

The result that follows is quoted from [BL], Theorem 1.3, and the proof uses Theorem 2.3.1.

We know that if f_* is a unique invariant density the operator P may be restricted to the support of f_* which we will call C' .

This property allows us to consider P on the space $L^1(C')$. (Note that this C' is contained in the conservative part of X .)

We will denote P restricted to C' by $P_{C'}$.

Theorem 2.3.2. *Let P be stochastic with invariant density f_* . Assume that the operator $P_{C'}$ with $C' = \text{supp } f_*$ is asymptotically stable. Assume further there is a $\delta > 0$ such that*

$$\sup_n \int_{C'} P^n f d\mu \geq \delta \text{ for each } f \in \mathcal{D}. \quad (2.3.5)$$

Then $P : L^1(X) \rightarrow L^1(X)$ is asymptotically stable.

Proof. According to Theorem 2.3.1 it is sufficient to find a non-trivial lower bound function for P .

Define $h = \frac{1}{2}\delta f_*$ and choose a fixed $f \in \mathcal{D}$. We will show that h satisfies (2.3.1). According to (2.3.5) there is an integer m such that

$$\eta := \int_{C'} P^m f d\mu \geq \frac{1}{2}\delta.$$

For $n \geq m$ we have

$$P^n f = P^{n-m}(1_{X \setminus C'} P^m f) + P_{C'}^{n-m}(1_{C'} P^m f). \quad (2.3.6)$$

Since $P_{C'}$ is asymptotically stable with invariant density f_* we also have

$$\lim_{n \rightarrow \infty} \|P_{C'}^{n-m}(1_{C'} P^m f) - \eta f_*\|_1 = 0.$$

From the inequality $h \leq \eta f_*$ it follows that

$$\|(P^n f - h)^-\|_1 \leq \|(P^n f - \eta f_*)^-\|_1 \leq \|P^n f - \eta f_*\|_1$$

and therefore (from (2.3.6)),

$$\begin{aligned} \|(P^n f - h)^-\|_1 &\leq \|P^{n-m}(1_{X \setminus C'} P^m f) + P_{C'}^{n-m}(1_{C'} P^m f) - \eta f_*\|_1 \\ &\leq \|P_{C'}^{n-m}(1_{C'} P^m f) - \eta f_*\|_1 \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

and since $\|h\|_1 > 0$ the proof is complete. ■

There is a dual result to Theorem 2.3.1 for upper functions.

Definition 2.3.2. A function $h \in L^1$ is an **upper bound function** for a stochastic operator $P : L^1 \rightarrow L^1$ if

$$\lim_{n \rightarrow \infty} \|(P^n f - h)^+\|_1 = 0$$

for every $f \in \mathcal{D}$.

The proof of the following is as in [LY], Theorem 3, and uses Theorem 2.3.1.

Theorem 2.3.3. Let $P : L^1 \rightarrow L^1$ be stochastic, then P is asymptotically stable if there exists an upper bound function $h \in L^1$ for P such that $\|h\|_1 < 2$.

Proof. By virtue of Theorem 2.3.1 it is sufficient to find a non-trivial lower bound function for P .

Starting with a given upper bound function h , we define a sequence

$$h_0 = h, h_1 = \inf(h_0, Ph_0), \dots, h_n = \inf(h_{n-1}, Ph_{n-1}), \dots$$

of upper bound functions.

The sequence $\{h_n\}$ is decreasing and bounded, because $0 \leq h_n \leq h$ and therefore convergent to a function $\tilde{h} \in L^1$. It is easy to verify that \tilde{h} is an upper bound function and that it is P -invariant (as in the proof of Theorem 2.3.1). Setting $\alpha = \|\tilde{h}\|_1$ we have $\alpha \leq \|h\|_1 < 2$. On the other hand, from the definition of an upper bound function, it follows easily that $\alpha \geq 1$. Now we consider two cases:

(i) $\alpha = 1$ and (ii) $1 < \alpha < 2$.

In case (i) the condition $\|\tilde{h}\|_1 = 1$ implies that

$$\|(P^n f - h)^+\|_1 = \|(h - P^n f)^+\|_1$$

for $f \in \mathcal{D}$ and therefore h is simultaneously an upper and lower bound function. In this case the proof is finished.

In case (ii) we are going to show that $(2 - \alpha)\tilde{h}$ is a lower bound function. For a given $f \in \mathcal{D}$ consider the sequence $q_n = (\alpha - 1)^{-1}P^n(\tilde{h} - f) = (\alpha - 1)^{-1}(\tilde{h} - P^n f)$.

Since \tilde{h} is an upper bound function we have

$$\lim_{n \rightarrow \infty} \|(-q_n)^+\|_1 = 0 \text{ and } \lim_{n \rightarrow \infty} \|q_n\|_1 = 1.$$

Thus, for a given $\epsilon > 0$ there exists an integer $m > 0$ and a function $r \in L^1$ such that $q_m + r \geq 0$, $\|q_m + r\|_1 = 1$, $\|r\|_1 \leq \epsilon/2$.

Again, since $q_m + r \in \mathcal{D}$ and \tilde{h} is an upper bound function

$$\|(P^n(q_m + r) - \tilde{h})^+\|_1 \leq \epsilon/2$$

for sufficiently large n , say $n \geq n_0$.

Multiplication by $\alpha - 1 < 1$ gives, according to the definition of q_m ,

$$\|(\tilde{h} - (\alpha - 1)\tilde{h} - P^{n+m}f + (\alpha - 1)P^n r)^+\|_1 \leq \epsilon/2.$$

Now from the inequality $\|P^n r\|_1 \leq \|r\|_1 \leq \epsilon/2$ it follows that

$$\|((2 - \alpha)\tilde{h} - P^{n+m}f)^+\|_1 \leq \epsilon$$

for $n \geq n_0$, which means that $(2 - \alpha)\tilde{h}$ is indeed a nontrivial lower bound function, and the proof is complete. ■

2.4 Transformations of an Interval

In this section we consider applications of the lower-bound function technique of Lasota and Yorke which was introduced in the previous section. This part is only informative and we do not include any proofs. This was the first application of the technique and the problems were, in a sense, completely solved.

The class of transformations we consider were first studied by Renyi (1957) and Rohlin (1964). Both were considering two classes of mappings, namely

$$S(x) = \tau(x) \pmod{1}, \quad 0 \leq x \leq 1,$$

where $\tau[0, 1] \rightarrow [0, \infty)$ is a C^2 function such that $\inf_x \tau' > 1$, $\tau(0) = 0$ and $\tau(1)$ is an integer, and the Renyi transformation

$$S(x) = rx \pmod{1}, \quad 0 \leq x \leq 1,$$

where $r > 1$ is constant. (The dyadic transformation is obviously a special case.)

Recall that a transformation is exact w.r.t. a suitable probability measure if and only if the Frobenius-Perron operator corresponding to the transformation is asymptotically stable. In this case we call the transformation **statistically stable**.

The following theorem which we quote from [LM1], p.145, completely solves the problem of piecewise linear expanding mappings of an interval. The proof there is based on Theorem 2.3.1.

Theorem 2.4.1. Consider a mapping $S : [0, 1] \rightarrow [0, 1]$ that satisfies the following conditions:

(i) There is a partition $0 = a_0 < a_1 < \dots < a_r = 1$ of $[0, 1]$ such that for each integer $i = 1, \dots, r$ the restriction of S to the interval $[a_{i-1}, a_i)$ is a C^2 function;

(ii) $S(a_{i-1}) = 0$ for $i = 1, \dots, r$;

(iii) There is a $\lambda > 1$ such that $S'(x) \geq \lambda$ for $0 \leq x < 1$ and

(iv) There is a real finite constant c such that $-S''(x)/[S'(x)]^2 \leq c$, $0 < x < 1$ (where $S'(a_i)$ and $S''(a_i)$ denote the right derivatives).

Then S is statistically stable.

Clearly this theorem guarantees that the Renyi transformation is statistically stable.

Theorem 2.4.1 obviously only applies to monotonically increasing mappings, but we have the following, which guarantees the exactness of the tent map (see [LM1], p.148).

Theorem 2.4.2. Let $S : [0, 1] \rightarrow [0, 1]$ satisfy:

(i) There is a partition $0 \leq a_0 < a_1 < \dots < a_r = 1$ of $[0, 1]$ such that for each integer $i = 1, \dots, r$ the restriction of S to (a_{i-1}, a_i) is a C^2 function;

(ii) $S((a_{i-1}, a_i)) = (0, 1)$ that is, S is onto on each subinterval;

(iii) There is a $\lambda > 1$ such that $|S'(x)| \geq \lambda$, for $x \neq a_i, i = 0, \dots, r$; and

(iv) There is a real constant c such that $|S''(x)|/[S'(x)]^2 \leq c$ for $x \neq a_i, i = 0, \dots, r$.

Then S is statistically stable.

We finally present the theorem of Lasota and Yorke [LY] which completely solves the problem of piecewise convex transformations with a strong repeller (see [LM1], p.154).

Theorem 2.4.3. Let $S : [0, 1] \rightarrow [0, 1]$ satisfy

(i) Condition (i) of Theorem 2.4.1 ;

(ii) $S'(x) > 0$ and $S''(x) \geq 0$ for all $x \in [0, 1]$;

(iii) For each integer $i = 1, \dots, r$, $S(a_{i-1}) = 0$; and

(iv) $S'(0) > 1$, and $S'(a_i), S''(a_i)$ denote the right derivatives.

Then S is statistically stable.

We also mention a theorem from [LM1] (see p.167) from which many other examples of exact transformations may be constructed.

Theorem 2.4.4. Let $T : [0, 1] \rightarrow [0, 1]$ be a measurable, non-singular trans-

formation and let $\phi \in \mathcal{D}((a, b))$ with $a, b \in [-\infty, \infty]$ be a given, strictly positive density. Let a second transformation $S : (a, b) \rightarrow (a, b)$ be given by $S = g^{-1} \circ T \circ g$ where $g(x) = \int_a^x \phi(y) dy$, $a < x < b$.

Then T is exact if and only if S is statistically stable and further ϕ is the density of the S -invariant measure.

Remark 2.4.1: We have already noted that the hat map T is exact. If we define $g(x) = \frac{1}{2} - \frac{1}{\pi} \sin^{-1}(1 - 2x)$ we may calculate $g^{-1}(x) = \frac{1}{2} - \frac{1}{2} \cos(\pi x)$ and find that $S(x) = g^{-1} \circ T \circ g(x)$ where S is the quadratic transformation. We also have $g(x) = \int_0^x \phi(y) dy$, where ϕ is the density given by

$$\phi(x) = \frac{1}{\pi \sqrt{x(1-x)}}.$$

We conclude that the quadratic transformation is statistically stable with the invariant density ϕ . \square

2.5 ω -limit sets and Ω

If P is asymptotically periodic and the permutation α in Definition 2.2.3 is cyclic, we say P is asymptotically cyclic. This is the same as the operator P being asymptotically periodic and ergodic.

Definition 2.5.1. The limit set $\omega(f)$ of the trajectory $\gamma(f) = \{P^n f : n \geq 0\}$ where $f \in L^1$ is the set

$$\omega(f) = \{g \in L^1 : \exists n_k \nearrow \infty, P^{n_k} f \rightarrow g \text{ in the norm } \},$$

the set of closure points of the trajectory.

We have the following from [Ba1], (Lemma 1), due to Dafermos and Slemrod concerning the behaviour of P on ω -limit sets.

Lemma 2.5.1. *Let P be stochastic and $f \in L^1$ with $\omega(f) \neq \emptyset$. Then*

(i) $g \in \omega(f) \Rightarrow \gamma(g) = \omega(f)$ ($\omega(f)$ is a P -invariant minimal subset of L^1)

(ii) $T|_{\omega(f)}$ is an invertible isometry.

We will denote by Ω the set of all limit points $\cup_{f \in L^1} \omega(f)$. We may now state without giving the proof the following result due to W.Bartoszek [Ba2], (Theorem 2),

Theorem 2.5.1. *Let P be stochastic on L^1 . If for every $f \in L^1$ the limit set $\omega(f) \neq \emptyset$ and*

there exists a natural k such that for all positive non-zero f_1, f_2 in L^1 there exist positive n, m with $|n - m| \leq k$ such that $P^n f_1 \wedge P^m f_2 \neq 0$ (where $g \wedge h$ denotes the ordinary minimum in L^1 of g and h),

then P is asymptotically cyclic and the length of the cycle $r \leq k + 1$.

The proof is rather technical and long, but Theorem 2.5.1 has an immediate implication for operators that overlap supports.

Definition 2.5.1. We say a stochastic operator P on L^1 **overlaps supports** if for all $f_1, f_2 \in \mathcal{D}$ there exists a positive number n such that $P^n f_1 \wedge P^n f_2 \neq 0$.

Corollary 2.5.1. *Let P be a stochastic operator that overlaps supports. If for every $f \in L^1$, $\omega(f) \neq \emptyset$ then P is asymptotically stable.*

Proof. It is enough to observe that if the parameter k is taken to be zero in Theorem 2.5.1, then $r = 1$ and P is asymptotically stable. ■

CHAPTER III

LMT OPERATORS

We devote this chapter to the study of applications of kernel operators, (defined in Example 1.2.1), in science, especially biology. The emphasis is on Section 3.2 which explains the work done by Lasota, Mackey, Tyrcha and others in an attempt to use a general stochastic model which has a combination of deterministic and probabilistic mechanisms. In this chapter all measure spaces are σ -finite.

3.1 Classical Models

We introduce some older models where the use of kernel operators have found success.

3.1.1 Markov Chains.

These examples first studied by Markov are actually very simple probabilistic models, (defined in Example 1.2.1), and have found many applications in science and engineering (see [Hin], p.544). The study of general stochastic processes are, to a great extent, an attempt to extend some powerful results to a much greater variety of problems.

We are considering a discrete time physical process which at any time n can be in

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one of m mutually exclusive and exhaustive states, $X_n \in \{1, 2, \dots, m\}$. We say the phase space is discrete. Furthermore we assume that the probability of the system being in state j at time $n + 1$ is only dependent on the state at time n (the process has the Markovian property). If the state at time n is i then this probability is p_{ji} . That is, the so-called transition probabilities are:

$$p_{ji} = \text{prob}\{X_{n+1} = j \mid X_n = i\}.$$

We assume the transition probabilities are stationary, i.e. they do not change with time.

The model described is called a finite state Markov chain.

We may define

$$\begin{aligned} p_{ji}^{(n)} &= \text{prob}\{X_n = j \mid X_0 = i\} \\ &= \text{prob}\{X_{n+k} = j \mid X_k = i\} \quad k = 1, 2, 3, \dots \end{aligned}$$

The Chapman-Kolmogorov equations now simply state that $\mathbf{P}^{(n)} = \mathbf{P}^n$ (in the sense of matrix multiplication) where $\mathbf{P} = (p_{ji})$ and $\mathbf{P}^{(n)} = (p_{ji}^{(n)})$.

The matrix \mathbf{P} is the kernel of a stochastic operator P because if an initial density \mathbf{f}_0 is given we can see that the Markov operator P which gives the next density is

$$P\mathbf{f}_0(j) = \sum_i p_{ji} \mathbf{f}_0(i)$$

(as in (1.2.4)), where

$$\mathbf{f}_0(i) = \text{prob}\{X_0 = i\}$$

and

$$\mathbf{f}_1(j) = P\mathbf{f}_0(j) = \text{prob}\{X_1 = j\}.$$

3.1.2 The Heat equation.

The kernel for the solution of the heat equation was given in Section 1.2.1. The derivation of the heat equation is well known and can be found in almost any elementary textbook on P.D.E's. It is simple to check that the equation is indeed satisfied by this solution.

3.1.3 Independent Random Variables.

An application of kernel operators to probability theory is given, as in [LMB], p.304.

We recall some notions from probability theory. A function $f \in \mathcal{D}(\mathbf{R})$ is called the density of a random variable ξ if

$$\text{prob}\{\xi \in B\} = \int_B f(x) dx$$

for every Borel set $B \subseteq \mathbf{R}$.

We say ξ and η are independent random variables if, for all Borel sets B and C

$$\text{prob}\{\xi \in B, \eta \in C\} = \text{prob}\{\xi \in B\} \text{prob}\{\eta \in C\}.$$

A simple consequence of this is the following: If ξ and η are independent random variables that have densities f and g respectively, then the joint density function for the random vector (ξ, η) is given by $h(u, v) = f(u)g(v)$.

For an arbitrary Borel set $B \subseteq \mathbf{R}$,

$$\text{prob}\{\xi + \eta \in B\} = \iint_{u+v \in B} f(u)g(v) dudv$$

or, setting $x = u + v$, $y = v$,

$$\begin{aligned} \text{prob}\{\xi + \eta \in B\} &= \iint_{B \times \mathbf{R}} f(x - y)g(y) dx dy \\ &= \int_B \left\{ \int_{-\infty}^{\infty} f(x - y)g(y) dy \right\} dx. \end{aligned}$$

From the definition of a density it now follows that

$$\begin{aligned} P_g f(x) &= \int_{-\infty}^{\infty} f(x - y)g(y) dy \\ &= \int_{-\infty}^{\infty} g(x - y)f(y) dy, \end{aligned}$$

the convolution of f and g , is the density of $\xi + \eta$ and it is easy to recognise $g(x - y)$ as a stochastic kernel.

3.2 Biological Models

This section follows [LMT] very closely and we make no claim of originality in either the content or the presentation.

A general modelling framework is presented within which many models for systems which produce events at irregular times through a combination of probabilistic and deterministic dynamics can be comprehended. The results are applied to some published models of the cell division cycle in Section 3.2.3. Though we clearly have biological systems in mind, the development here is also applicable to other systems.

In Section 3.2.1 the model is formulated in terms of a discrete time dynamical system with stochastic perturbations. From this an integral recurrence relation is derived for densities describing the statistical behaviour of trajectories.

In this theory, a concept called the internal or physiological time of the system plays an important role. With respect to this time the model behaves the same way in each period between consecutive events, but not with respect to the physical time. The use of the internal time significantly simplifies the theory.

In Section 3.2.2 the nature of this internal time is explicitly considered, since in the cell cycle models presented the physiological time is hidden in the description of the system. In Section 3.3 it is shown how many of the old cell cycle models may be encompassed within the general framework presented.

3.2.1 The general modelling framework.

Firstly we explain the basic system, as in [LMT]. The description is the following. A (biological) system which produces events is considered. In addition to the usual laboratory time the system is also assumed to have an internal or physiological time. This internal time is denoted by τ to distinguish it from the laboratory (or clock) time t . When an event appears the physiological time resets from the value $\tau = \tau_{\max}$ to $\tau = 0$. It is assumed that the rate of maturation $\frac{d\tau}{dt}$ depends on the amount of an activator (or maturation factor) which we denote by a . Thus

$$\frac{d\tau}{dt} = \phi(a), \quad \phi \geq 0. \quad (3.2.1.1)$$

It is further assumed that the activator is produced by a dynamics described by the solution to the differential equation

$$\frac{da}{dt} = g(a), \quad g \geq 0. \quad (3.2.1.2)$$

The solution of (3.2.1.2) satisfying the initial condition $a(0) = r$ will be denoted by

$$a(t) = \Pi(t, r),$$

and it is assumed that it is defined for all $t \geq 0$. When an event is produced at a time $\tau = \tau_{\max}$ and activator level a_{\max} , then a portion $\rho = \rho(a_{\max})$ of a_{\max} is consumed in the production of the event. Thus, after the event the activator resets to the level

$$a = a_{\max} - \rho(a_{\max}). \quad (3.2.1.3)$$

The function $y - \rho(y)$ is called the **reset function**, and it is assumed that it is invertible. The inverse of $y - \rho(y)$ is denoted by λ .

The main assumption is related to the physiological time. Namely it is assumed that the survival function of τ_{\max} is independent of the initial value of the activator. This survival function is denoted by H . Thus, using the notion of conditional probability we may write

$$\text{prob}(\tau_{\max} \geq x \mid a(\tau = 0) = r) = H(x) \quad (3.2.1.4)$$

for every $r > 0$. It is felt that this assumption corresponds to the intuitive meaning of physiological time, and a mathematical argument for it is offered in Section 3.2.2. In the terminology of population dynamics it could be said that the lifespan of an organism will be shorter when its rate of maturation is increased.

With these assumptions, a recurrence relation will be derived for the values of the activator when the events occur. Assume that the events appear at the times

$$t_0 < t_1 < t_2 \dots$$

Let a_n be the amount of the activator at the beginning of the interval (t_n, t_{n+1}) . According to Eq. (3.2.1.2), the amount at time $t \in (t_n, t_{n+1})$ is given by

$$a = \Pi(t - t_n, a_n).$$

Now using (3.2.1.1) the physiological time τ corresponding to t may be calculated, namely

$$\tau = \int_{t_n}^t \phi(\Pi(s - t_n, a_n)) ds. \quad (3.2.1.5)$$

Substitute $z = \Pi(s - t_n, a_n)$, $dz = g(\Pi(s - t_n, a_n)) ds$ and observe that $z = a_n$ for $s = t_n$ and $z = a$ for $s = t$. Then (3.2.1.5) becomes

$$\tau = \int_{a_n}^a q(z) dz = Q(a) - Q(a_n), \quad (3.2.1.6)$$

where

$$q(z) = \frac{\phi(z)}{g(z)} \text{ and } Q(z) = \int_0^z q(y) dy. \quad (3.2.1.7)$$

The function q has a simple biological interpretation, since it gives the rate of change of the physiological time relative to the activator.

When t approaches t_{n+1} , the physiological time τ and the amount of the activator a take their maximal values which is denoted by τ_n and $a_{\max, n}$ respectively. In this case (3.2.1.6) gives

$$\tau_n = Q(a_{\max, n}) - Q(a_n). \quad (3.2.1.8)$$

Further, from the definition of the reset function we have $a_{n+1} = \lambda^{-1}(a_{\max, n})$, and consequently

$$a_{n+1} = \lambda^{-1}(Q^{-1}(Q(a_n) + \tau_n)) \text{ for } n = 0, 1, \dots \quad (3.2.1.9)$$

This is desired recurrence relation between successive activator levels at event occurrence. By assumption, the variables a_n and τ_n are independent, see (3.2.1.4), and thus (3.2.1.9) may be considered as a discrete time dynamical system with stochastic perturbations by the τ_n .

The behaviour of this system from a statistical point of view may be described by the sequence of distributions

$$F_n(x) = \text{prob}(a_n < x) \text{ for } n = 0, 1, \dots$$

A recurrence formula for the densities $f_n = dF_n/dx$ is derived in the next paragraph, and then the convergence properties of the densities f_n are considered.

Set $H_1 = 1 - H$ and denote by $h = H_1'$ the density of the distribution of τ_n (assuming that this density exists). If a_n has a distribution F_n then $Q(a_n)$ has the distribution function $G_n(x) = F_n(Q^{-1}(x))$. Further, since a_n and τ_n are independent, the variable $u_n = Q(a_n) + \tau_n$ has a distribution function given by the convolution (see Section 3.1.3),

$$\int_0^x h(x-y) dG_n(y) = \int_0^{Q^{-1}(x)} h(x-Q(y)) dF_n(y). \quad (3.2.1.10)$$

Finally, $\lambda^{-1}(Q^{-1}(u_n))$ has the distribution function

$$\int_0^{\lambda(x)} H(Q(\lambda(x)) - Q(y)) dF_n(y).$$

From this and the definition of the density, it follows that $a_{n+1} = \lambda^{-1}(Q^{-1}(u_n))$ has the density

$$f_{n+1}(x) = \lambda'(x)q(\lambda(x)) \int_0^{\lambda(x)} h(Q(\lambda(x)) - Q(y)) f_n(y) dy. \quad (3.2.1.11)$$

Introducing the operator P , which we will call the **LMT operator** after Lasota, Mackey and Tyrcha, who have contributed greatly to this theory, defined by

$$Pf(x) = \int_0^{\lambda(x)} \left[-\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) \right] f(y) dy, \quad (3.2.1.12)$$

these relations may be written in the more abbreviated forms $f_{n+1} = Pf_n$ and $f_n = P^n f_0$. Under some simple regularity conditions concerning λ , Q and H , (3.2.1.12) defines a stochastic operator on the space $L^1(\mathbf{R}_+)$ of all integrable functions defined on the half line $\mathbf{R}_+ = [0, \infty)$. These assumptions will be formulated in (3.2.1.16) and (3.2.1.17).

At this point it is worth noting the explicit use of the inverse function $Q^{-1}(x)$ in the derivations of (3.2.1.9) and (3.2.1.11). In some applications it may happen that the functions $\phi(x)$ and $q(x)$ vanish on an interval $0 \leq x \leq x_0$ and are only positive for $x > x_0$. In this case it is clear that $Q(x)$ as given by (3.2.1.7) also vanishes for

$0 \leq x \leq x_0$ and is thus not invertable. However, as is shown in the Appendix of [LMT], (3.2.1.9) and (3.2.1.11) are still valid.

If the densities f_n are given then it is easy to find the density of the distribution of the interevent intervals, i.e., the time intervals $\Delta t_n = t_{n+1} - t_n$ between the n^{th} and $(n+1)^{\text{st}}$ events. In fact (3.2.1.5) with $t = t_{n+1}$ gives

$$\tau_n = \int_{t_n}^{t_{n+1}} \phi(\Pi(s - t_n, a_n)) ds = \int_0^{\Delta t_n} \phi(\Pi(s, a_n)) ds.$$

Therefore

$$\begin{aligned} \text{prob}(\Delta t_n \geq x) &= \text{prob}\left(t_n \geq \int_0^x \phi(\Pi(s, a_n)) ds\right) \\ &= \int_0^\infty \text{prob}\left(t_n \geq \int_0^x \phi(\Pi(s, t)) ds \mid a_n = r\right) f_n(r) dr. \end{aligned}$$

From this and (3.2.1.4) it follows immediately that

$$\text{prob}(\Delta t_n \geq x) = \int_0^\infty H\left(\int_0^x \phi(\Pi(s, r)) ds\right) f_n(r) dr.$$

By differentiation we can find the density distribution function of Δt_n which is denoted by $\alpha_n(x)$. Namely, the density of the interevent intervals is

$$\alpha_n(x) = \int_0^\infty h\left(\int_0^x \phi(\Pi(s, r)) ds\right) \phi(\Pi(x, r)) f_n(r) dr. \quad (3.2.1.13)$$

In the particular case when $f_n = f_*$, ($n = 0, 1, \dots$) is a time independent stationary sequence the α_n has the same property.

The study of the asymptotic properties of the LMT operator

$$Pf(x) = \int_0^{\lambda(x)} K(x, y) f(y) dy, \quad (3.2.1.14)$$

is commenced, where

$$K(x, y) = -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)). \quad (3.2.1.15)$$

It will always be assumed that Q , λ and H satisfy the following conditions:

1. The functions $Q : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ and $\lambda : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ are non-decreasing and absolutely continuous on each subinterval $[0, c]$ of the half-line \mathbf{R}_+ . Moreover

$$Q(0) = \lambda(0) = 0 \text{ and } \lim_{x \rightarrow \infty} Q(x) = \lim_{x \rightarrow \infty} \lambda(x) = \infty. \quad (3.2.1.16)$$

2. The function $H : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is non-increasing, absolutely continuous on each interval $[0, c]$, and

$$H(0) = 1, \quad \lim_{x \rightarrow \infty} H(x) = 0. \quad (3.2.1.17)$$

Obviously (3.2.1.14) defines a stochastic kernel operator (see Definition 1.2.1), since for each $y \in \mathbf{R}_+$ we have that $\int_0^{\lambda(x)} K(x, y) dx = 1$, because $H(0) = 1$. Thus the results which are presented in Chapter IV are valid.

In studying the asymptotic properties of LMT operators the following equation will prove very helpful:

$$\int_0^\infty V(Q(\lambda(x)))Pf(x) dx = \int_0^\infty f(y) dy \int_0^\infty V(x + Q(y))h(x) dx, \quad (3.2.1.18)$$

where $f \in L^1$ is non-negative and $V : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is an arbitrary Borel measurable function. The proof is as in [LMT], p.780.

To verify (3.2.1.18), note that from (3.2.1.12)

$$\begin{aligned} I &= \int_0^\infty V(Q(\lambda(x)))Pf(x) dx \\ &= \int_0^\infty \lambda'(x)q(\lambda(x))V(Q(\lambda(x))) dx \int_0^{\lambda(x)} h(Q(\lambda(x)) - Q(y))f(y) dy. \end{aligned}$$

Setting $z = \lambda(y)$ we have

$$\begin{aligned} I &= \int_0^\infty V(Q(z))q(z) dz \int_0^z h(Q(z) - Q(y))f(y) dy \\ &= \int_0^\infty f(y) dy \int_0^\infty V(Q(z))h(Q(z) - Q(y))q(z) dz. \end{aligned}$$

Now substituting $Q(z) - Q(y) = x$, the following is immediately obtained,

$$I = \int_0^\infty f(y) dy \int_0^\infty V(x + Q(y))h(x) dx,$$

which completes the derivation of (3.2.1.18).

3.2.2 The Exponential model.

The assumed independence of a_n and τ_n plays a crucial role in the theory as developed to this point. It is obvious that this assumption is not easily justified even if one accepts the intuitive interpretation of biological time which has been used to support the independence assumption. In this section a mathematical argument is presented to strengthen the plausibility of the independence assumption concerning a_n and τ_n . We will need the following ([LMT], Lemma 2):

Lemma 3.2.2.1. *Assume that X and Y are random variables such that $Y \geq X \geq 0$ with probability 1, and that the conditional probability of Y with respect to X satisfies*

$$\text{prob}(Y \geq y \mid X = r) = H(Q(y) - Q(r)) \text{ for } y \geq r \geq 0, \quad (3.2.2.1)$$

where $Q : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is strictly increasing and onto, and $H : \mathbf{R}_+ \rightarrow [0, 1]$ is a decreasing function. Then H is the survival function for the random variable $Q(Y) - Q(X)$ and the variables $Q(Y) - Q(X)$ and X are independent.

Proof. Denote by F_X the cumulative distribution function for X . Then it follows that

$$\begin{aligned} \text{prob}(Q(Y) - Q(X) \geq u, X \geq v) &= \int_v^\infty \text{prob}(Q(Y) - Q(X) \geq u \mid X = r) F_X(dr) \\ &= \int_v^\infty \text{prob}(Y \geq Q^{-1}(u + Q(r)) \mid X = r) F_X(dr) \\ &\text{for } u \geq 0, v \geq 0. \end{aligned}$$

From this and (3.2.2.1) we have

$$\begin{aligned} \text{prob}(Q(Y) - Q(X) \geq u, X \geq v) &= \int_v^\infty H(Q(Q^{-1}(u + Q(r)) - Q(r))) F_X(dr) \\ &= \int_v^\infty H(u) F_X(dr) \\ &= H(u)(1 - F_X(v)) \end{aligned}$$

which completes the proof. ■

Lemma 3.2.2.1 will also be used in situations where $Q(x)$ vanishes for $x \leq x_0$ (see the remarks following our derivation of (3.2.1.12) and the Appendix in [LMT]). It is straightforward to show that Lemma 3.2.2.1 also holds in the case that $Q(x)$ is invertible for $x \geq x_0$ and $Y \geq x_0$ with probability one.

To illustrate the usefulness of Lemma 3.2.2.1 in understanding the independence assumption, return to the considerations of Section 3.2.1. However, the existence of an internal (biological) time is not now assumed, nor is the assumption embodied in (3.2.1.4) made. Rather, it is assumed that the activator substance is produced according to (3.2.1.2) as before, and the following condition:

The probability that an event occurs in the time interval $[t, t + \Delta t]$ given that it has not occurred up to time t , is equal to

$$\phi(a(t))\Delta t + o(\Delta t), \quad (3.2.2.2)$$

where $a(t)$ is the activator level at time t . As before, it is assumed that after the event occurs, the activator level is reset to the level $\lambda^{-1}(a_{\max})$.

Now consider the situation in which the system starts at time $t = 0$, when the previous event occurred, with an activator level $a(0) = r$. By (3.2.1.2), the activator level at time t is simply

$$a(t) = \Pi(t, r).$$

Furthermore, using (3.2.2.2) it is easy to calculate the probability that the next event appears at a time $t_1 > t$. Namely,

$$\text{prob}(t_1 \geq t \mid a(0) = r) = \exp \left\{ - \int_0^\infty \phi(\Pi(s, r)) ds \right\}.$$

Making, as before, the change of variables $z = \Pi(x, r)$ we have

$$\begin{aligned} \text{prob}(t_1 \geq t \mid a(0) = r) &= \exp \left\{ - \int_r^{a(t)} q(z) dz \right\} \\ &= \exp \{ -Q(a(t)) + Q(r) \}. \end{aligned}$$

Clearly, the condition $t_1 \geq t$ is equivalent to $a_{\max} \geq y$ where $y = a(t)$. Thus,

$$\text{prob}(a_{\max} \geq y \mid a(0) = r) = \exp \{ -Q(y) + Q(r) \}$$

By Lemma 3.2.2.1, this shows that the variables $Q(a_{\max}) - Q(a(0))$ and $a(0)$ are independent, and furthermore that $Q(a_{\max}) - Q(a(0))$ has an exponential survival function e^{-x} .

Now define

$$\tau = Q(a(t)) - Q(a(0)) \quad (3.2.2.3)$$

so, in particular

$$\tau_{\max} = Q(a_{\max}) - Q(a(0))$$

Then, since

$$\frac{d\tau}{dt} = q(a(t))a'(t) = q(a(t))g(a(t)) = \phi(a(t))$$

we know that the function τ satisfies (3.2.1.1). Furthermore, τ_{\max} is independent of $a(0)$ and has the exponential survival function $H(x) = e^{-x}$.

Thus, through the use of Lemma 3.2.2.1 the existence of a function having all of the characteristics that were originally postulated for the internal (biological) time has been demonstrated. As a consequence, the activator levels a_n satisfy

the recurrence relation (3.2.1.9) with exponentially distributed τ_n and the density distribution functions of f_n of a_n satisfy the operator equation $f_{n+1} = Pf_n$ with P defined by

$$Pf(x) = \lambda'(x)q(\lambda(x)) \int_0^{\lambda(x)} \exp \left\{ - \int_y^{\lambda(x)} q(z) dz \right\} f(y) dy. \quad (3.2.2.4)$$

In the sequel, the system 3.2.2.4 will be referred to as an *exponential model* with transition probability given by (3.2.2.2).

3.3 Cell Cycle Models

In this section we offer concrete examples of the application of the general formulation in the previous section by considering several mathematical models of the cell cycle.

In interpreting the cell division cycle within the context of the general model, the occurrence of an event is associated with the triggering of the process which ultimately leads to mitoses and cytokinesis, and the activator is associated with an (as yet) hypothetical substance called **mitogen** (in some models, the cell size), that is necessary but not sufficient for cell division to occur.

3.3.1 Tyrcha models.

The class of models proposed by Lasota and Mackey (1984), Tyson and Hannsgen (1986) and Tyrcha (1988) is now considered. Within our framework these models may be described as follows.

During the lifetime of the cell it must traverse two phases denoted by A and B. The end of phase B corresponds with cell division. The duration of phase B is

constant, and is denoted by t_B , while the length of phase A is considered to be a random variable. The transition of phase A to phase B is taken to be coincident with the occurrence of an event, and the probability that this event occurs during the interval $[t, t + \Delta t]$ is given by (3.2.2.2) where $a(t)$ is the mitogen level. The production of mitogen is governed by (3.2.1.1) with $g(x) > 0$ for $x > 0$ which means the activator is an increasing function of the clock time t . Within the context of the general framework developed earlier, the transition between phases A and B, i.e. when the event occurs, corresponds to the moment when the activator has reached the level $a_{\max, n}$. Since the production of mitogen during B is still governed by (3.2.1.2) at cell division (the end of B) the activator has level $\Pi(t_B, a_{\max, n})$. Finally, in these models the mitogen is assumed to be divided equally between both daughter cells at cell division, so

$$\frac{1}{2}\Pi(t_B, a_{\max, n}) = \lambda^{-1}(a_{\max, n}) = a_{n+1} \quad (3.3.1.1)$$

$$\text{or } \lambda(x) = \Pi(-t_B, 2x). \quad (3.3.1.2)$$

The class of cell cycle models satisfies all the conditions of the exponential model, which corresponds to the fact that the random variable T_A has intensity function $\phi(s, (t, r))$, and has an internal time defined by (3.2.2.3).

Furthermore the quantities of mitogen in consecutive generations of newly born cells satisfy the recurrence relation (3.2.1.9) with $\tau(x)$ having survival function e^{-x} .

Lastly the transition operator for the evolution of mitogen density is given by (3.2.2.4). There are two specific features of these models we need to mention. Firstly, the reset function is not arbitrary but is explicitly defined by (3.3.1.2). Secondly (3.2.1.13) gives the distribution of the lengths of phase A of the cell cycle,

with the density of the duration of the entire cycle given by

$$\bar{\alpha}_n(t) = \begin{cases} \alpha_n(t - t_B) & \text{for } t \geq t_B, \\ 0 & \text{for } t < t_B. \end{cases}$$

This description was first proposed by Tyrcha [Tyr]. It reduces to the Lasota-Mackey model [LM2] if $t_B = 0$, i.e. the cell division occurs with the critical event. It reduces to the Tyson-Hannsgen model [TH1] if we assume $g(x) = kx$ i.e. the cells grow (mitogen increases) exponentially with time and,

$$\phi(x) = \begin{cases} p & \text{for } x \geq 1, \\ 0 & \text{for } x < 1, \end{cases} \quad (3.3.1.3)$$

which means that the probability of cell division is zero for $x < 1$ and constant for $x \geq 1$.

These models will be examined in the light of some stability results we will prove in Chapter V.

3.3.2 Tyson-Hannsgen models.

Extensions proposed by Tyson and Hannsgen et al. (1988) of the well known cell cycle models of Smith and Martin (1973) and Shields (1977) also fall within the general modelling framework.

In these situations it is also assumed that the cell goes through phases A and B, t_B is constant, and the end of B corresponds with cell division.

The difference is that t_A is considered to be a random variable with a density distribution function ψ so that,

$$\text{prob}(t_A \geq x) = \int_x^{\infty} \psi(z) d(z).$$

The activator (mitogen) produced by dynamics described by (3.2.1.2) is assumed not to affect t_A and divides equally between mother and daughter cells at division. Thus, by assumption t_A and $a(0)$ are independent.

To show this model may be described by the general framework assume that the event occurs at the transition from A to B.

Furthermore, set

$$\phi(x) \equiv 1, \quad H(x) = \int_x^\infty \psi(z) d(z),$$

and define λ by (3.3.1.2).

The condition $\phi \equiv 1$ simply means that the internal (biological) time τ and the laboratory time t are either identical during any given cell cycle or differ by a constant amount.

The special form of the function $q = \frac{1}{g}$ is used to simplify the recurrence relation (3.2.1.9) considerably. Thus, solving (3.2.1.2) with $a(0) = r$ we have

$$\int_r^{a(t)} \frac{dx}{g(x)} = t \text{ or}$$

$$Q(a(t)) - Q(r) = t, \text{ from (3.2.1.7).}$$

This, in turn implies

$$a(t) = \Pi(t, r) = Q^{-1}(Q(r) + t)$$

and, in particular

$$\lambda(x) = \Pi(-t_B, 2x) = Q^{-1}(Q(2x) - t_B)$$

Finally, out of (3.2.1.9)

$$\begin{aligned} a_{n+1} &= \lambda^{-1}(Q^{-1}(Q(a_n) + \tau_n)) \\ &= \frac{1}{2}Q^{-1}(Q(a_n) + \tau_n + t_B) \end{aligned} \quad (3.3.2.1)$$

where $\tau_n = t_{A_n}$ denotes the length of the A-phase during the n 'th generation.

A comparison of (3.2.1.9) and (3.3.2.1) suggests the following correspondence.

Introduce a new variable $\bar{\tau}_n = t_B + \tau_n$ with density distribution function

$$h(x) = \begin{cases} \psi(x - t_B) & \text{for } x \geq t_B, \\ 0 & \text{for } x < t_B, \end{cases}$$

and a new reset function $\bar{\lambda}(x) = \frac{1}{2}x$.

This corresponds to shifting events to the division points $t_{A_n} + t_B$.

With these new functions (3.3.3.1) is again a special case of (3.1.1.9) and the recurrence relation for densities is given by

$$Pf(x) = 2q(2x) \int_0^{2x} h(Q(2x) - Q(y)) f(y) dy.$$

CHAPTER IV
KERNEL OPERATORS

This chapter is the backbone of the thesis and we develop the theory of the asymptotic stability of kernel operators.

4.1 Krasnoselskii's Theorem

This theorem is an essential part of the theory and the well known proof is due to A. Krasnoselskii, [Kra]. However, his proof is based on the assumption that the space $L^1(\mu)$ is separable. We now present the following beautiful elementary proof due to A. Lasota, ([Las], Theorem 4.1), which does not require that μ is σ -finite. Note that we do not assume that the kernel is stochastic, i.e. we only assume it is non-negative, $\mathcal{A} \otimes \mathcal{A}$ -measurable, and maps L^1 into L^1 .

Theorem 4.1.1. *Let (X, \mathcal{A}, μ) be a measure space with P an operator on $L^1(X)$, given by the kernel K , that maps L^1 into L^1 . Then every sequence $\{f_n\}$ which is weakly convergent in L^1 is mapped into a strongly convergent sequence $\{Pf_n\}$.*

Proof. Note that P is a non-negative operator and therefore bounded (see [Kra], Ch.I, 2.2). Consequently every weakly convergent sequence $\{f_n\}$ is mapped into a weakly convergent sequence $\{Pf_n\}$. It remains to verify that $\{Pf_n\}$ is strongly

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convergent. This will be done in three steps.

Step I. Assume in addition that $\mu(X) < \infty$ and

$$M = \sup\{K(x, y) : (x, y) \in X \times X\} < \infty.$$

In this case for almost every $x \in X$ the mapping $f \rightarrow Pf(x)$ defines a bounded linear functional on L^1 . Denote by f_* the weak limit of $\{f_n\}$. We therefore have

$$\lim_{n \rightarrow \infty} Pf_n(x) = Pf_*(x) \quad \text{a.e.}$$

This pointwise convergence and the inequality

$$|Pf_n(x) - Pf_*(x)| \leq M\|f_n - f_*\|_1 \leq M(\|f_*\|_1 + \sup_n \|f_n\|_1)$$

imply the strong convergence of $\{Pf_n\}$ to f_* .

Step II. Consider a more general situation where $\mu(X) < \infty$ but K is not necessarily bounded. We consider the weakly convergent sequence $\{f_n\}$ again, and fix an $\epsilon > 0$. Since the functions $\{f_n\}$ are uniformly integrable, there is a $c > 0$ such that

$$\|P\| \int_{A_n} |f_n(x)| d\mu(x) \leq \frac{\epsilon}{2} \quad \text{for every } n \in \mathbf{N} \quad (4.1.1)$$

where $A_n = \{x : |f_n(x)| > c\}$ and $\|P\|$ denotes the norm of the operator $P : L^1 \rightarrow L^1$. Write f_n in the form $f_n = f_{n0} + f_{n1}$ where

$$f_{n0}(x) = \begin{cases} f_n(x), & \text{if } |f_n(x)| \leq c \\ cf_n(x)/|f_n(x)|, & \text{if } |f_n(x)| > c. \end{cases}$$

Evidently $|f_{n1}| \leq |f_n|$ and $f_{n1}(x) = 0$ for $x \notin A_n$. Therefore (4.1.1) implies that

$$\|P\| \|f_{n1}\|_1 \leq \frac{\epsilon}{2} \quad \text{for every } n \in \mathbf{N}. \quad (4.1.2)$$

From the equality

$$\iint_{X \times X} K(x, y) d\mu(x) d\mu(y) = \|P1_X\|_1 < \infty$$

it follows that K is integrable on $X \times X$. Write K in the form $K = K_0 + K_1$ where

$$K_0(x, y) = \begin{cases} K(x, y), & \text{if } K(x, y) \leq r \\ r, & \text{if } K(x, y) > r, \end{cases}$$

and r is a positive number. Denote by P_0 and P_1 the integral operators corresponding to the kernels K_0 and K_1 respectively. For sufficiently large r we have

$$\|P_1 1_X\|_1 = \iint_{X \times X} K_1(x, y) d\mu(x) d\mu(y) \leq \frac{\epsilon}{2c}. \quad (4.1.3)$$

Now, using (4.1.2) and (4.1.3) we may evaluate the difference $Pf_n - P_0f_n$, namely

$$\begin{aligned} \|Pf_n - P_0f_n\|_1 &= \|P_1f_n\|_1 \leq \|P_1f_{n0}\|_1 + \|P_1f_{n1}\|_1 \\ &\leq c\|P_1 1_X\|_1 + \|P\| \|f_{n1}\|_1 \leq c\frac{\epsilon}{2c} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \quad (4.1.4)$$

By Step I the sequence $\{P_0f_n\}$ is strongly convergent. Since $\epsilon > 0$ was arbitrary, (4.1.4) implies that $\{Pf_n\}$ satisfies the Cauchy condition and is also strongly convergent.

Step III. Consider the general case, without any additional assumptions. As usual, we assume that $\{f_n\}$ is weakly convergent. Then $\{Pf_n\}$ has the same property and for every $\epsilon > 0$ there is a set A of finite measure (See [DS], Ch.IV, 13), such that

$$\int_{X \setminus A} |Pf_n(x)| d\mu(x) \leq \frac{\epsilon}{2}, \quad \|P\| \int_{X \setminus A} |f_n(x)| d\mu(x) \leq \frac{\epsilon}{2} \quad \text{for } n \in \mathbb{N}. \quad (4.1.5)$$

We write $K_A(x, y) = 1_A(x)1_A(y)K(x, y)$ and denote by P_A the kernel operator corresponding to K_A . Now, using (4.1.5), we have

$$\begin{aligned} \|Pf_n - P_Af_n\|_1 &\leq \int_{X \setminus A} |Pf_n(x)| d\mu(x) + \int_A |Pf_n(x) - P_Af_n(x)| d\mu(x) \\ &\leq \frac{\epsilon}{2} + \int_A |P(1_{X \setminus A}f_n)(x)| d\mu(x) \\ &\leq \frac{\epsilon}{2} + \|P\| \int_{X \setminus A} |f_n(x)| d\mu(x) \leq \epsilon. \end{aligned} \quad (4.1.6)$$

The restriction of an $f \in L^1(X)$ to the set A will be denoted by \hat{f} . We claim that the sequence $\{\hat{f}_n\}$ is weakly convergent in $L^1(A)$. In fact, to every linear functional $\hat{\varphi} : L^1(A) \rightarrow \mathbf{R}$ there corresponds a linear functional $\varphi : L^1(X) \rightarrow \mathbf{R}$ given by $\varphi(f) = \hat{\varphi}(\hat{f})$. This functional is evidently bounded, since

$$|\varphi(f)| \leq \|\hat{\varphi}\|_{L^1(A)} \|f\|_{L^1(X)}.$$

Consequently

$$\lim_{n \rightarrow \infty} \varphi(f_n) = \varphi(f_*)$$

where f_* is the weak limit of $\{f_n\}$. The last condition may be written in the form

$$\lim_{n \rightarrow \infty} \hat{\varphi}(\hat{f}_n) = \hat{\varphi}(\hat{f}_*)$$

and the claim is proved.

We now denote by \hat{P} the restriction of P_A to $L^1(A)$. According to Step II the sequence $\{\hat{P}\hat{f}_n\}$ is strongly convergent in $L^1(A)$. Thus, since $P_A f_n$ are the trivial extensions of $\hat{P}\hat{f}_n$ to the space $L^1(X)$, the sequence $\{P_A f_n\}$ is strongly convergent in $L^1(X)$. Returning to the inequality (4.1.6), observe that it is valid for every $\epsilon > 0$ and sufficiently large set A . This implies that $\{P f_n\}$ satisfies the Cauchy condition for the strong convergence in $L^1(X)$ and the proof is complete. ■

This proof is quite long, because in the general situation we cannot use the identification between the space of all bounded linear functionals on $L^1(X)$ and the space $L^\infty(X)$.

The theorem can also be formulated in terms of compact sets, not necessarily sequentially compact. We have, namely, [Las], Corollary 4.2,

Corollary 4.1.1. *If a kernel operator $P : L^1(X, \mathcal{A}, \mu) \rightarrow L^1(X, \mathcal{A}, \mu)$ is given, then for each weakly compact set $\mathcal{F} \subset L^1$ the image $P(\mathcal{F})$ is strongly compact.*

Proof. Let $\mathcal{F} \subset L^1$ be weakly compact. According to the Eberlein-Smulian theorem the set \mathcal{F} is sequentially compact, i.e. every sequence $f_n \in \mathcal{F}$ contains a subsequence $\{f_{\alpha_n}\}$ which converges weakly to an $f_* \in L^1$. Since \mathcal{F} is weakly closed, we have $f_* \in \mathcal{F}$. By Theorem 4.1.1 this implies that $P(\mathcal{F})$ is sequentially compact and closed. Since L^1 is a metric space, $P(\mathcal{F})$ is compact. ■

Remark 4.1.1: It should be noted that this theorem is not related to the weak and strong compactness of integral operators. See [Las] for a stochastic kernel operator which is neither strongly nor weakly compact. □

This theorem has far reaching consequences we will exploit later in this chapter. As an example of its usefulness we can easily prove two simple corollaries, using Krasnoselskii's theorem instead of the general form of the Spectral Decomposition Theorem. We follow the reasoning in [LM3], p.107.

We will apply the theorem to stochastic kernel operators where the kernel is bounded by a L^1 -function g , i.e.

$$K(x, y) \leq g(x), \text{ where } g \in L^1.$$

For every density f we have

$$\begin{aligned} Pf(x) &= \int_X K(x, y)f(y) dy \\ &\leq g(x) \int_X f(y) dy = g(x). \end{aligned}$$

Thus, for this kernel, the set $P(\mathcal{D})$ is weakly precompact. Further, since $P^k(\mathcal{D}) \subseteq \mathcal{D}$ we have $P^n f \in P(\mathcal{D})$ for each $f \in \mathcal{D}$ and $n \in \mathbb{N}$. Combining this with Krasnoselskii's theorem we see that $\{P^n f\}$ is strongly precompact, and we may apply the original form of the Spectral Decomposition Theorem of Lasota to conclude that P is asymptotically periodic.

We also note that these results remain valid if for some iterate P^m the corresponding kernel K_m is bounded by an L^1 -function. (It is easy to show that if P is given by a kernel, so is every iterate).

We have proved the following two results, p.107 of [LM3],

Corollary 4.1.2. *If the kernel K is bounded by an integrable function, i.e.*

$$K(x, y) \leq g(x)$$

for $g \in L^1$, then the integral operator P is asymptotically periodic.

and, in conjunction with Theorem 2.2.4,

Corollary 4.1.3. *If there exist an integer m and a $g \in L^1$ such that*

$$K_m(x, y) \leq g(x),$$

where K_m corresponds to P^m and there exists a set $S \subset X, \mu(S) > 0$, such that $K_m(x, y) > 0$ for $x \in S, y \in X$, then P is asymptotically stable.

4.2 Doubly Stochastic Operators that Overlap Supports

As explained in Chapter I a stochastic operator with a positive invariant density can always be replaced by its doubly stochastic counterpart. It is therefore sufficient to consider operators for which $P1 = 1$.

We recall from Chapter I that conservative operators are non-disappearing and the result from [KL] regarding non-disappearing operators and also Theorem 1.3.4.

We give as in [BaB]

Proposition 4.2.1. *If P is doubly stochastic and overlaps supports, it is weakly operator mixing.*

Proof. We consider P and P^* as linear contractions on $L^2(\mu)$ and we denote the linear subspace of $L^2(\mu)$ on which P acts as an invertible isometry by K , as in Theorem 1.3.4.

It is now shown that K consists of constant functions. We have that $P^{*n}P^n = P^nP^{*n} = \text{Id}$ on K . Consider f such that $P^{*n}P^n f = f$ for all natural n . Since the $P^{*n}P^n$ are conservative, we have that for all $a \in \mathbb{R}$ the sets

$$F_a = \{x \in X : f(x) > a\}$$

are invariant (i.e. $P^{*n}P^n 1_{F_a} = 1_{F_a}$). If f is not constant, we can find a such that F_a is non-trivial. Since $P^{*n} = P^n$ and the operators we are considering are non-disappearing $P^n 1_{F_a} = 1_{E_{n,a}}$ for some $E_{n,a} \in \mathcal{A}$. Because P^n preserves 1 we get $P^n 1_{F_a^c} = 1_{E_{n,a}^c}$. Therefore $P^n 1_{F_a} \wedge P^n 1_{F_a^c} = 0$ for all n , which contradicts that P overlaps supports.

Now, according to Theorem 1.3.4, $P^n(f-1) \xrightarrow{n \rightarrow \infty} 0$ weakly in L^1 for every $f \in \mathcal{D}$ and the proof is complete. ■

To the end of Section 4.2 we will deal with stochastic operators which are defined by a family of *transition probabilities*, i.e. we assume $P(x, A)$ is a probability measure for each x . Clearly if P has transition probabilities so has every iterate P^n . These are denoted by $P^n(x, \cdot)$ and satisfy the following Kolmogorov type

equation

$$P^{n+m}(x, A) = \int P^n(y, A) P^m(x, dy).$$

The formula (1.2.3) extends P on the Banach lattice $M(X)$ of all signed and bounded measures on (X, \mathcal{A}) . We denote by $\nu = \nu_{ac} + \nu_s$ the Lebesgue decomposition of ν into the absolutely continuous and singular parts w.r.t. μ and $\frac{d\nu_{ac}}{d\mu} \in L^1(\mu)$ stands for the Radon-Nikodym derivative.

We quote the "0-2 Law" which is due to Ornstein and Sucheston [OS]. This version comes from [Fo2], (Theorem 2).

Theorem 4.2.1 (0-2 Law). *Let P be a stochastic operator on $L^1(X, \mathcal{A}, \mu)$ such that P^k is ergodic and conservative for every positive k .*

Then, either

$$(0) \quad \lim_{n \rightarrow \infty} \|P^{n+k} - P^n\| = 0 \quad \text{for all } k \geq 1$$

$$\text{or } (2) \quad P^{n+k}(x, \cdot) \perp P^n(x, \cdot) \text{ for } \mu \text{ almost all } x \in X \text{ and all } n, k \geq 1.$$

As in [BaB] we provide an alternative proof for the following fact first noticed in [OS]. We note that it holds for general stochastic operators.

Corollary 4.2.1. *Let P be in the 0-class and doubly stochastic. Then P is asymptotically stable.*

Proof. Let $f \in \mathcal{D}$ be arbitrary. The Mean Ergodic Theorem holds for doubly stochastic operators, hence

$$A_N f = \frac{1}{N} \sum_{k=0}^{N-1} P^k f \rightarrow 1 \text{ in the norm as } N \rightarrow \infty.$$

Given $\epsilon > 0$ we choose $N_0 \leq N$ such that $\|A_{N_0}f - 1\|_1 \leq \epsilon$ and $\|P^{n+k} - P^n\| \leq \epsilon$ for all $0 \leq k \leq N_0$ and $n \in \mathbb{N}$. Now for each $f \in \mathcal{D}$ we have

$$\begin{aligned} \|P^n f - 1\|_1 &\leq \|P^n f - P^n A_{N_0} f\|_1 + \|P^n A_{N_0} f - 1\|_1 \\ &\leq \left\| \frac{1}{N_0} \sum_{k=0}^{N_0-1} (P^n - P^{n+k}) f \right\|_1 + \|P^n (A_{N_0} f - 1)\|_1 \\ &\leq \frac{1}{N_0} \sum_{k=0}^{N_0-1} \|P^n - P^{n+k}\| + \|A_{N_0} f - 1\|_1 \leq 2\epsilon \end{aligned}$$

for all $n \in \mathbb{N}$, and P is asymptotically stable, which completes the proof. ■

Modifying the terminology of [Rud] slightly we say P is **non-singular** if there exists a set of $Y \subseteq X$ of positive measure such that for every $x \in Y$ there exists $m(x)$ such that $P^{m(x)}$ has a non-trivial absolutely continuous part w.r.t. μ .

The following result from [BaB] corresponds to Theorem 1 from [Rud] but uses different techniques and the proof is much simpler.

Theorem 4.2.2. *A doubly stochastic non-singular operator P that overlaps supports is asymptotically stable.*

Proof. By Proposition 4.2.1 P is w.o.m. This implies that P^k is ergodic and conservative for all k . Now it suffices to exclude alternative (2) of the 0-2 Law.

For this take $x \in Y$ and consider $P_{ac}^m(x, \cdot)$. If $m \geq m(x)$ then

$$f_m(x, \cdot) = \frac{dP_{ac}^m(x, \cdot)}{d\mu} \text{ and } f_{m+1}(x, \cdot) = \frac{dP_{ac}^{m+1}(x, \cdot)}{d\mu}$$

are non-negative nonzero $L^1(\mu)$ functions. Since P overlaps supports we have

$$0 < P^n f_m(x, \cdot) \wedge P^n f_{m+1}(x, \cdot) \leq \frac{d(P^{m+n}(x, \cdot) \wedge P^{m+n+1}(x, \cdot))}{d\mu}$$

if n is large enough. Therefore (2) fails and by Corollary 4.2.1 P is asymptotically stable and the proof is complete. ■

From the definitions we notice that a Harris operator is non-singular and thus we immediately have,

Corollary 4.2.2. *A Harris operator given by transition probabilities with a positive invariant density that overlaps supports is asymptotically stable.*

We also have the following corollary similar to Theorem 2.2.4 which does not assume constrictivity but that P is Harris instead. It generalises Corollary 1.1 in the paper [BL].

Corollary 4.2.3. *Let $P : L^1(\mu) \rightarrow L^1(\mu)$ be a Harris operator with a positive stationary density. Assume there exists $A \in \mathcal{A}$, $\mu(A) > 0$ such that for every $f \in D$ there exists $n_0 = n_0(f)$ such that $P^{n_0} f(x) > 0$ for a.e. $x \in A$. Then P is asymptotically stable.*

We will need the following which is Corollary 1.2 [BL] and uses Theorem 2.3.2.

Corollary 4.2.4. *Let $P : L^1(\mu) \rightarrow L^1(\mu)$ be a kernel operator that overlaps supports with invariant density f_* (we do not assume $f_* > 0$). Set $C' = \text{supp } f_*$. If there exists a $\delta > 0$ such that condition (2.3.5) is satisfied, then P is asymptotically stable.*

Proof. According to Theorem 2.3.2 it is sufficient to show that the operator $P_{C'}$

is asymptotically stable. Evidently,

$$P_{C'} f(x) = \int_{C'} K(x, y) f(y) d\mu(y)$$

for every $f \in L^1(C')$ and

$$\begin{aligned} 0 &= \int_{C'} f_*(y) d\mu(y) - \int_{C'} P_{C'} f_*(x) d\mu(x) \\ &= \int_{C'} (1 - K(x, y) d\mu(x)) f_*(y) d\mu(y), \end{aligned}$$

from which we get

$$\int_{C'} K(x, y) d\mu(x) = 1 \text{ for a.e. } y \in C'.$$

This shows that $P_{C'}$ is a kernel operator, and thus non-singular and we may apply Theorem 4.2.2. Therefore $P_{C'}$ is asymptotically stable. ■

4.3 Strong Feller Kernels

We prove asymptotic stability results for kernels which are Strong Feller. We will notice in Chapter V that LMT kernels are Strong Feller (under weaker assumptions than those in [BL]), and thus these results are applicable.

Let (X, d) be a locally compact, metric, Polish space and let \mathcal{B} denote the Borel σ -algebra of subsets of X . Let μ be a σ -finite measure on (X, \mathcal{B}) .

Note that each stochastic kernel operator may be extended to the Banach lattice $M(X)$ of all bounded and signed Borel measures on (X, \mathcal{B}) by (1.2.3).

Definition 4.3.1. We say a kernel operator on $L^1(\mu)$ is **Strong Feller in the strict sense** if its kernel satisfies

$$X \ni y \rightarrow K(\cdot, y) \in \mathcal{D}_\mu \text{ is } L^1\text{-norm continuous.} \quad (4.3.1)$$

Note that (4.3.1) implies the continuity of P^*h , where $h \in L^\infty(\mu)$.

It is also well known that if X is compact then kernel operators satisfying (4.3.1) are compact. The behavior of iterates of compact, linear and positive operators on Banach lattices is well understood (see [Ba1]). But if X is not compact then the asymptotic regularity of iterates may be lost. To restore them we need some extra conditions. Given a stochastic operator satisfying (4.3.1) we will identify an invariant sublattice on $L^1(\mu)$ on which P is asymptotically periodic. This sublattice appears to be trivial exactly when for each compact $K \subseteq X$, there exists an $f \in \mathcal{D}_\mu$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_K P^j f d\mu = 0.$$

By $C_0(X)$ we denote the Banach lattice of all continuous functions (endowed with the ordinary sup-norm $\|\cdot\|_{\text{sup}}$ or $\|\cdot\|_\infty$), h on X , such that for every $\epsilon > 0$ there exists a compact set $E_\epsilon \subseteq X$ such that $|h(x)| \leq \epsilon$ for all $x \notin E_\epsilon$.

Given a stochastic operator P we denote by F the minimal measurable set which carries the supports of all P -invariant densities. (The existence follows from the separability of $L^1(\mu)$). Obviously $L^1(F)$ is P -invariant.

We quote the following theorem due to W. Bartoszek, ([Ba3], Theorem 1), and present the proof.

Theorem 4.3.1. *Let P be a stochastic kernel operator on $L^1(\mu)$ which is Strong Feller in the strict sense and such that P^* preserves $C_0(X)$. If:*

there exists a compact set $K \subseteq X$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \int_K \sum_{j=0}^{n-1} P^j f d\mu > 0 \tag{4.3.2}$$

for all $f \in \mathcal{D}_\mu$, then F is non-trivial and P is asymptotically periodic on $L^1(F)$. In particular, there are only finitely many P -invariant ergodic densities.

Proof. We first show that P admits a stationary density.

The set of all subprobabilistic measures on X is a compact, convex set w.r.t. the vague topology (we say that a variation norm bounded sequence of measures ν_n is vaguely convergent to ν if $\lim_{n \rightarrow \infty} \int_X h d\nu_n = \int_X h d\nu$ for all $h \in \mathcal{C}_0(X)$). Given $f \in \mathcal{D}_\mu$ we may choose a sequence $n_k \nearrow \infty$ so that the measures with densities

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} P^j f = A_{n_k} f$$

are vaguely convergent. By (4.3.2) the limit ν is nonzero and PA_{n_k} tends to $P\nu$ vaguely. Since

$$\|A_{n_k} f - PA_{n_k} f\|_1 = \left\| \frac{P^{n_k} f - f}{n_k} \right\|_1 \xrightarrow{k \rightarrow \infty} 0$$

we conclude that $\nu = P\nu \in L^1(\nu)$ is a fixed point of P . Normalising ν if necessary, we obtain a P -invariant density.

We now show that the linear subspace (sublattice) $\text{Fix}(P)$ of all P -invariant functions is finite dimensional.

Assume we are given pairwise orthogonal P -invariant densities f_1, \dots, f_k . By (4.3.2) we have $\int_K f_j d\mu > 0$. Now consider the following family of (restricted to K) continuous functions

$$g_j = (P^* 1_{F_j})|_K \quad \text{where} \quad F_j = \text{supp}(f_j).$$

Clearly

$$g_j(x) = 1 \quad \text{for all} \quad x \in \overline{F_j} \cap K,$$

and

$$g_j(x) = 0 \quad \text{if} \quad x \in \bigcup_{i \neq j} \overline{F_i} \cap K.$$

As a result

$$\|g_j - g_l\|_\infty = 1 \quad \text{for } j \neq l.$$

The condition (4.3.1) combined with the Arzeli theorem now easily give $\|\cdot\|_\infty$ compactness of $\overline{P^*B_1}|_K$, where B_1 stands for the unit ball of $L^\infty(\mu)$. Hence k is bounded and there are only finitely many ergodic P -invariant densities $f_1 \dots f_r$.

For a fixed $1 \leq j \leq r$ we will now show that P is asymptotically periodic on $L^1(F_j)$.

First we see that every trajectory

$$\gamma(f) = \{P^n f\}_{n \geq 0}, \quad \text{where } f \in L^1(F_j),$$

is L^1 -norm relatively compact. We may confine discussion to $0 \leq f \leq f_j$. Clearly $\gamma(f)$ is weakly compact (it follows from the invariance and weak compactness of the order interval $[0, f_j] = \{f \in L^1(F_j) : 0 \leq f \leq f_j\}$, see [AB], Theorem 12.9). It now immediately follows from Krasnoselskii's theorem that $\gamma(f)$ is relatively compact.

We denote the subspace of all L^1 -norm recurrent $f \in L^1(F_j)$ by Ω_j . It is well known that Ω_j consists of all limit vectors in $L^1(F_j)$ (see Lemma 2.5.1). Given a sequence $\underline{n} = n_k \nearrow \infty$ we denote the closed sublattice of Ω_j consisting of all vectors f which are recurrent along the sequence n_k (i.e. $\|P^{n_k} f - f\|_1 \rightarrow 0$ as $n_k \rightarrow \infty$) by $\Omega_{\underline{n}}$.

We notice that regardless of the dimension of $\Omega_{\underline{n}}$, for every compact $C \subseteq X$ the restricted sublattice $\Omega_{\underline{n}}|_C$ is finite dimensional. In fact, we see that $\dim \Omega_{\underline{n}}|_C \leq r_C$ where r_C denotes the largest j such that there are

$$0 \leq h_1, \dots, h_j \leq 1, \quad h_l \in P^*B_1$$

with

$$\sup_{x \in C} |h_l(x) - h_{\bar{l}}(x)| \geq \frac{1}{2}$$

for distinct l, \bar{l} (It follows from (4.3.1) that r_C is finite).

Let

$$\tilde{g}_1 = \beta_1 g_1|_C, \dots, \tilde{g}_{r_C} = \beta_{r_C} g_{r_C}|_C$$

form a normalised, positive and orthogonal basis in $\Omega_{\underline{n}}|_C$ (for some $\beta_l \geq 1$ and $g_l \in \Omega_{\underline{n}}$). Given $\epsilon > 0$ we find a compact set $C = C_\epsilon \subseteq X$ such that

$$\int_C f_j d\mu > 1 - \epsilon.$$

For each density $g \in \Omega_{\underline{n}}$ there exists n such that

$$\int_C P^n g d\mu > 1 - \epsilon.$$

We have

$$P^n g|_C = \sum_{l=1}^{r_C} \alpha_l \tilde{g}_l, \text{ where } \alpha_l > 0, \text{ and } \sum_{l=1}^{r_C} \alpha_l > 1 - \epsilon.$$

Equivalently, for each $g \in \Omega_{\underline{n}}$ there is a natural n such that

$$\text{dist}(P^n g, \text{conv}\{\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_{r_C}, 0\}) < \epsilon.$$

Therefore

$$\text{dist}(P^{n+k} g, \mathcal{F}_{\epsilon,j}) \leq \epsilon \text{ for all } k \geq 0,$$

where $\mathcal{F}_{\epsilon,j}$ denotes the L^1 -norm closure of the set

$$\left\{ \sum_{l=1}^{r_C} \alpha_l P^k \tilde{g}_l : k = 0, 1, 2, \dots, \sum_{l=1}^{r_C} \alpha_l \leq 1, \alpha_l \geq 0 \right\}.$$

Because all trajectories in $L^1(F_j)$ are norm relatively compact the set $\mathcal{F}_{\epsilon,j}$ is compact. Clearly it is P -invariant. Hence by recurrence of $P^n g$ we obtain

$$\text{dist}(g, \mathcal{F}_{\epsilon,j}) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary this implies that the set of all densities from $\Omega_{\underline{n}}$ is relatively compact, and $\Omega_{\underline{n}}$ is finite dimensional with $\dim \Omega_{\underline{n}} \leq r_C$. P has a positive inverse on $\Omega_{\underline{n}}$, so from the general theory of stochastic operators P permutes vectors of a unique, positive, normalised and orthogonal basis in $\Omega_{\underline{n}}$. In particular, P is periodic (i.e. $P^d = Id$ where $d = d(\underline{n})$ depends on \underline{n}) on $\Omega_{\underline{n}}$.

For arbitrary $\Omega_{\underline{n}}, \Omega_{\underline{m}}$ we may find d (for instance $d = d(\underline{n}) \cdot d(\underline{m})$) such that $\Omega_{\underline{n}}, \Omega_{\underline{m}} \subseteq \Omega_{kd}$. Hence

$$\dim \Omega_j|_C = \dim \{f|_C : f \in \Omega_j\} \leq r_C.$$

Repeating the arguments we applied to $\Omega_n|_C$, we construct a compact set \mathcal{F}_ϵ such that

$$\text{dist}(g, \mathcal{F}_\epsilon) \leq \epsilon \quad \text{for all densities } g \in \Omega_j.$$

Finally, this implies that Ω_j is finite dimensional and for each density $f \in L^1(F_j)$ the iterates $P^n f$ are attracted to the set $\mathcal{D}_\mu \cap \Omega_j$, which is obviously norm compact. By Theorem 2.2.3 P is asymptotically periodic on $L^1(F_j)$.

We easily extend this property to $L^1(F)$ where $F = \cup_{j=1}^r F_j$. ■

We also present the next result which we will need in Chapter V, as in [Ba3], (Corollary 1), which investigates the case where inequality (4.3.2) holds uniformly (condition (ii) in the following).

Corollary 4.3.1. *Let P be a stochastic kernel operator on $L^1(\mu)$ satisfying (4.3.1) and such that P^* preserves $C_0(X)$. Then the following conditions are equivalent:*

(i) P is asymptotically periodic on $L^1(\mu)$.

(ii) There exist a compact set $K \subseteq X$ and $\delta > 0$ such that

$$\limsup_{n \rightarrow \infty} \int_K \frac{f + Pf + \cdots + P^{n-1}f}{n} d\mu > \delta$$

for all $f \in \mathcal{D}_\mu$.

Proof. Only (ii) implies (i) needs to be proved. By Theorem (4.3.1) it is enough to show that for each $f \in \mathcal{D}_\mu$ we have

$$\lim_{n \rightarrow \infty} \int_F P^n f d\mu = 1.$$

Choosing a subsequence if necessary we may insure that

$$\left(\frac{1}{n_k} \sum_{j=0}^{n_k-1} P^j f \right) |_K \xrightarrow{k \rightarrow \infty} f_* |_K$$

in the L^1 -norm where f_* is P -invariant. By (ii) we easily get

$$\delta < \|f_* |_K\|_1.$$

As a result for every $f \in \mathcal{D}_\mu$ there is a natural n such that

$$\int_F P^n f d\mu > \delta.$$

Suppose that there exists $f \in \mathcal{D}_\mu$ with

$$\delta(f) = \lim_{n \rightarrow \infty} \int_F P^n f d\mu < 1$$

and m is large enough so that

$$\int_F P^m f d\mu > \delta(f) - \frac{(1 - \delta(f))\delta}{2}.$$

Consider

$$f_1 = \frac{1_{F^c} P^m f}{\int_{F^c} P^m f d\mu}.$$

There is further an n such that

$$\int_F P^n f_1 d\mu = \frac{1}{\int_{F^c} P^m f d\mu} \int_F P^n (1_{F^c} P^m f) d\mu > \delta.$$

We now have

$$\begin{aligned} \int_F P^{n+m} f d\mu &= \int_F P^n (1_F P^m f + 1_{F^c} P^m f) d\mu \\ &= \int_F P^n (1_F P^m f) d\mu + \int_F P^n (1_{F^c} P^m f) d\mu \\ &> \int_F P^m f d\mu + \delta \int_{F^c} P^m f d\mu \\ &\geq \delta(f) - \frac{(1 - \delta(f))\delta}{2} + (1 - \delta(f))\delta \\ &= \delta(f) + \frac{(1 - \delta(f))\delta}{2} > \delta(f). \end{aligned}$$

contradicting the definition of $\delta(f)$. ■

Remark 4.3.1. Both the results remain valid for P being Strongly Feller (i.e. P^*h is continuous for all $f \in L^\infty(\mu)$). In fact, it is well known (See Theorem 5.9, p. 37 in [Rev]), that Strong Feller implies (4.3.1) for P^2 . □

CHAPTER V

CONVERGENCE OF LMT OPERATORS

In this chapter we apply the general theory of kernel operators developed in Chapter IV to the special case of the modelling framework that was described in Chapter III, i.e. we use the results obtained to study the asymptotic stability of LMT operators. Of course, this is only one of the many possible applications of the theory in Chapter IV.

Firstly we must notice that the results of Section 5.3 are applicable to LMT operators.

Using [Loj] Theorem 7.4.8, we can easily check (as in [Ba3]), that, if $y_n \xrightarrow[n \rightarrow \infty]{} y$ then

$$\begin{aligned} & \int_0^\infty \left| \frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) - \frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y_n)) \right| dx \\ &= \int_0^\infty |h(Q(\lambda(x)) - Q(y)) - h(Q(\lambda(x)) - Q(y_n))| (Q \circ \lambda)'(x) dx \\ &= \int_0^\infty |h(t - Q(y)) - h(t - Q(y_n))| dt \xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

using our convention that $h(x) \equiv 0$ if $x \leq 0$. Hence LMT operators are Strong Feller in the strict sense.

Furthermore,

$$K(x, y) = -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) = 0$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-T}\mathcal{E}\mathcal{X}$

if $x \leq \lambda^{-1}(y) \equiv \inf\{0 \leq z : \lambda(z) = y\}$ and $\lambda^{-1}(y) \rightarrow \infty$ as $y \rightarrow \infty$, thus P^* preserves $\mathcal{C}_0(\mathbb{R}_+)$.

5.1 Bartoszek Stability Results

We first prove the following as in [BL] (Theorem 2.1), but we may simplify the proof because of our Theorem 4.3.1.

Let h be the density which corresponds to the probability distribution H , i.e.

$$h(x) = -H'(x).$$

Theorem 5.1.1. *If there exists an $\alpha \in (0, 1]$ such that*

$$\int_0^\infty x^\alpha h(x) dx < \liminf_{x \rightarrow \infty} (Q(\lambda(x))^\alpha - Q(x)^\alpha), \quad (5.1.1)$$

then the corresponding LMT operator has a stationary density and is asymptotically periodic.

Proof. We define

$$\sigma = \int_0^\infty x^\alpha h(x) dx.$$

Using (5.1.1) we can find positive numbers ϵ , ρ and x_0 such that

$$\sigma + \epsilon < \rho < Q(\lambda(x))^\alpha - Q(x)^\alpha \text{ for } x \geq x_0. \quad (5.1.2)$$

We will show that for every $f \in \mathcal{D}$ there exists an integer $n_0(f)$ such that

$$\int_0^{x_0} \frac{1}{n} \sum_{k=1}^n P^k f(x) dx \geq \frac{\epsilon}{2M} \text{ for } n \geq n_0(f), \quad (5.1.3)$$

where

$$M := \sup\{|Q(\lambda(x))^\alpha - Q(x)^\alpha - \rho| : 0 \leq x \leq x_0\}, \quad (5.1.4)$$

which will imply that the conditions for Corollary 4.3.1 are satisfied, and thus P is asymptotically periodic and admits a stationary density.

Using (3.2.1.18) with $V(x) = x^\alpha$ and $f \in \mathcal{D}$ we have

$$\begin{aligned} \int_0^\infty (Q(\lambda(x)))^\alpha P f(x) dx &= \int_0^\infty f(y) dy \int_0^\infty (x + Q(y))^\alpha h(x) dx \\ &\leq \int_0^\infty f(y) dy \int_0^\infty (x^\alpha + Q(y)^\alpha) h(x) dx \\ &= \sigma + \int_0^\infty f(y) Q(y)^\alpha dy. \end{aligned} \quad (5.1.5)$$

Fix $f \in \mathcal{D}$ such that

$$\int_0^\infty Q(x)^\alpha f(x) dx < \infty \quad (5.1.6)$$

and define

$$f_n = \frac{1}{n} \sum_{k=1}^n P^k f \text{ for } n = 1, 2, \dots \quad (5.1.7)$$

From (5.1.2), (5.1.5) and (5.1.6) it follows that

$$\int_0^\infty (Q(\lambda(x)))^\alpha P f_n(x) dx \leq \sigma + \int_0^\infty Q(x)^\alpha f_n(x) dx$$

and that the integral on the right hand side is finite for every n . Hence

$$\int_0^\infty (Q(\lambda(x))^\alpha - Q(x)^\alpha) f_n(x) dx \leq \sigma + \frac{1}{n} \int_0^\infty Q(\lambda(x))^\alpha P f(x) dx.$$

Since $\sigma < \rho - \epsilon$, there exists a positive integer $n_0(f)$ such that

$$\int_0^\infty (Q(\lambda(x))^\alpha - Q(x)^\alpha) f_n(x) dx \leq \rho - \epsilon \text{ for } n \geq n_0(f).$$

On the other hand, taking (5.1.2) into account, we have

$$\begin{aligned} &\int_0^\infty (Q(\lambda(x))^\alpha - Q(x)^\alpha) f_n(x) dx \\ &\geq \int_0^{x_0} (Q(\lambda(x))^\alpha - Q(x)^\alpha) f_n(x) dx + \rho \int_{x_0}^\infty f_n(x) dx. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_0^{x_0} (Q(\lambda(x))^\alpha - Q(x)^\alpha) f_n(x) dx &\leq \rho - \epsilon - \rho \int_{x_0}^{\infty} f_n(x) dx \\ &= \rho \int_0^{x_0} f_n(x) dx - \epsilon \end{aligned}$$

for $n \geq n_0(f)$, which together with (5.1.4) gives

$$-M \int_0^{x_0} f_n(x) dx \leq \int_0^{x_0} (Q(\lambda(x))^\alpha - Q(x)^\alpha - \rho) f_n(x) dx \leq -\epsilon$$

for $n \geq n_0(f)$. This implies (5.1.3) and even a stronger inequality with the right hand side ϵ/M . The above argument was valid for f satisfying (5.1.6). To get (5.1.3) for every density it is enough to observe that the set of all $f \in \mathcal{D}$ such that (5.1.6) holds is dense in \mathcal{D} . This completes the proof. ■

We give the proof of the rather technical result from [Ba3] (Theorem 2), which gives some more general information on LMT operators, using Corollary 4.3.1.

Theorem 5.1.2. *Let P be the LMT operator associated with H , Q and λ .*

Assume that there exists numbers $a > 0$ and $\delta > 0$ so that

$$\limsup_{n \rightarrow \infty} \int_0^a \frac{f + Pf + \dots + P^{n-1}f}{n} > \delta$$

for all $f \in \mathcal{D}$. Then

- (a) $a_* := \sup\{x \geq 0 : \lambda(x) \leq x\} < a$
- (b) $\text{Fix}(P)$ is finite dimensional and $\lim_{n \rightarrow \infty} \|P^n f - S f\| = 0$ for all $f \in L^1(\mathbb{R}_+)$ where S is the stochastic projection on $\text{Fix}(P)$.

- (c) $\dim(\text{Fix}(P)) \leq \left\lceil \frac{a}{T(P, a)} \right\rceil$ where

$$T(P, r) = \sup\{t > 0 : \text{if } 0 \leq y, \tilde{y} \leq r \text{ and } |y - \tilde{y}| \leq t \text{ then } \|K(\cdot, y) - K(\cdot, \tilde{y})\| < 2\}$$

and $\lceil z \rceil \in \mathbb{N}$ is the smallest number such that $\lceil z \rceil \geq z$.

In particular, if $T(P, a) \geq a$ then P is asymptotically stable.

Proof. By Corollary 4.3.1 the operator P is asymptotically periodic. If $\lambda(x) \leq x$ then the space $L^1([x, \infty))$ is P -invariant. It easily follows that

$$P^*1_{[c,d)}(y) = \begin{cases} H(Q(\lambda(c)) - Q(y)) - H(Q(\lambda(d)) - Q(y)), & \text{if } 0 \leq y < \lambda(c) \\ 1 - H(Q(\lambda(d)) - Q(y)), & \text{if } \lambda(c) \leq y < \lambda(d), \\ 0 & \text{if } \lambda(d) \leq y. \end{cases}$$

If $\lambda(c) \leq c$ then substituting $d = \infty$ we get

$$P^*1_{[c,\infty)}(y) \geq 1_{[c,\infty)}(y) \text{ for all } y.$$

Hence the set $\{x : \lambda(x) \leq x\}$ must be bounded and a_* is finite. Now it is clear that

$$\lambda(a_*) = a_* \text{ and } a_* < a.$$

Let g_1, \dots, g_r be a basis of positive, normalised and pairwise orthogonal functions in the space Ω of all recurrent elements, and let g_1, \dots, g_l be a cycle (i.e. $Pg_j = g_{j+1}$ for $1 \leq j < l$ where $j+1$ is understood modulo l). Denote

$$D_j = \text{supp } g_j \text{ and } c_j = \text{essinf } D_j.$$

Then we have

$$\begin{aligned} P^*1_{D_j}(y) &= 1 \text{ if } y \in \overline{D_{j-1}}, \text{ and} \\ P^*1_{D_j}(y) &= 0 \text{ for all } y \in \overline{D_s} \text{ if } s \neq j-1. \end{aligned} \quad (5.1.8)$$

We may assume that $\max\{c_1, \dots, c_l\} = c_l$. Thus,

$$P^*1_{[c_l, \infty)} \geq P^*1_{D_l} \geq 1_{D_{l-1}}.$$

By continuity we have

$$P^*1_{[c_l, \infty)}(c_{l-1}) = P^*1_{D_{c_l}}(c_{l-1}) = 1.$$

Since

$$P^* 1_{[c_l, \infty)}(y) = \begin{cases} H(Q(\lambda(c_l)) - Q(y)) & \text{if } 0 \leq y \leq \lambda(c_l), \\ 1 & \text{otherwise,} \end{cases}$$

we may conclude that

$$H(Q(\lambda(c_l)) - Q(y)) = 1 \quad \text{for all } c_{l-1} \leq y \leq \lambda(c_l),$$

and therefore

$$P^* 1_{[c_l, \infty)} \geq 1_{[c_{l-1}, \infty)} \geq 1_{[c_l, \infty)}.$$

This implies that $L^1([c_l, \infty))$ is P -invariant, and since g_1, \dots, g_l form a cycle it is possible only if $c_1 = c_2 = \dots = c_l$. Hence $l = 1$, because by (5.1.8) the continuous functions $P^* 1_{D_j}$ take values 0 and 1 arbitrary close to c_l . Repeating this discussion for other cycles, one obtains the result that each of them is reduced to a singleton and the convergence

$$\lim_{n \rightarrow \infty} \|P^n f - S f\| = 0$$

follows. It is clear that S is a finite dimensional stochastic projection onto $\Omega = \text{Fix}(P)$. Let F_1, \dots, F_r be supports of ergodic densities. We have

$$\|k(\cdot, y) - k(\cdot, \tilde{y})\| = 2$$

if y, \tilde{y} are taken from distinct sets $F_j \cap [0, a]$. This yields the estimation

$$\dim(S) \leq \left\lceil \frac{a}{T(P, a)} \right\rceil$$

and the result is proved. ■

Combining Theorem 5.1.1 and Theorem 5.1.2 we immediately get:

Corollary 5.1.1. *Let P be an LMT operator which satisfies condition (5.1.1).*

Then there exists a finite dimensional projection S such that $\lim_{n \rightarrow \infty} \|P^n f - S f\| = 0$ for all $f \in L^1(\mathbf{R}_+)$. Moreover $\dim(S) \leq \left\lceil \frac{a}{T(P, a)} \right\rceil$.

Proof. Equation (5.1.3) holds because the requirements of Theorem 5.1.1 are satisfied, hence we may apply Theorem 5.1.2. ■

Remark 5.1.1. By a slight abuse of terminology we say P is "stable". This means that we know that $P^n f$ converges in L^1 -norm for every density f , but the invariant density f_* to which it will converge depends on f .

5.2 Asymptotically Stable LMT operators

We give a result from [BL] (Theorem 2.2), which guarantees asymptotic stability in the case where h is positive on an infinite interval and (5.1.1) is satisfied. These are quite general conditions which hold for many of the cell cycle models.

Theorem 5.2.1. *If there exists a positive number $\alpha \leq 1$ such that (5.1.1) holds and a non-negative number c such that $h(x) > 0$ for a.e. $x \geq c$, then the corresponding LMT operator is asymptotically stable.*

Proof. According to Theorem 5.1.1 the operator P has a stationary density f_* . We now define $C = \text{supp } f_*$ and we fix positive numbers ϵ, ρ and x_0 such that (5.1.2) holds. Further, we choose a positive number a such that

$$\lambda(a) > x_0, \quad Q(\lambda(a)) \geq c + Q(x_0)$$

and we define

$$A = \{x \geq a : (Q \circ \lambda)'(x) > 0\}.$$

Since $Q \circ \lambda$ is absolutely continuous and $\lim_{x \rightarrow \infty} Q(\lambda(x)) = \infty$ the set A is unbounded ($\text{ess sup } A = \infty$). We finally define the number M by (5.1.4).

If $x \in A$, then

$$\begin{aligned} f_*(x) &= Pf_*(x) = (Q \circ \lambda)'(x) \int_0^{\lambda(x)} h(Q(\lambda(x) - Q(y))) f_*(y) dy \\ &\geq (Q \circ \lambda)'(x) \int_0^{x_0} h(Q(\lambda(x) - Q(y))) f_*(y) dy \end{aligned}$$

and

$$(Q \circ \lambda)'(x) > 0, \quad h(Q(\lambda(x) - Q(y))) \quad \text{for } y \in [0, x_0].$$

From (5.1.3) with $f = f_*$ it follows that

$$\int_0^{x_0} f_*(y) dy > 0.$$

This shows that $f_*(x) > 0$ for $x \in A$ and that $A \subset C$.

Using (5.1.3) it is also easy to show that

$$\sup_n \int_C P^n f(x) dx \geq \frac{\epsilon}{2M} \int_{Q(\lambda(a))}^{\infty} h(u) du \quad \text{for } f \in \mathcal{D}. \quad (5.2.1)$$

In fact, according to (5.1.3) for every density f there is a positive integer k such that

$$\int_0^{x_0} P^k f(x) dx \geq \frac{\epsilon}{2M}$$

and consequently,

$$\begin{aligned} \int_C P^{k+1} f(x) dx &\geq \int_A P^{k+1} f(x) dx \\ &= \int_A (Q \circ \lambda)'(x) dx \int_0^{\lambda(x)} h(Q(\lambda(x) - Q(y))) P^k f(y) dy \\ &\geq \int_A (Q \circ \lambda)'(x) dx \int_0^{x_0} h(Q(\lambda(x) - Q(y))) P^k f(y) dy \\ &= \int_0^{x_0} P^k f(y) dy \int_a^{\infty} (Q \circ \lambda)'(x) h(Q(\lambda(x) - Q(y))) dx \\ &\geq \int_0^{x_0} P^k f(y) dy \int_{Q(\lambda(a))}^{\infty} h(u) du \\ &\geq \frac{\epsilon}{2M} \int_{Q(\lambda(a))}^{\infty} h(u) du. \end{aligned}$$

We finally observe that, for every density f there exists a positive number $b = b(f)$ such that

$$Pf(x) > 0 \quad \text{for } x \in [b, \infty) \cap A.$$

To show this we choose $b_0 > 0$ such that $\int_0^{b_0} f(y) dy > 0$, and $b > 0$ such that

$$\lambda(b) \geq b_0, \quad Q(\lambda(b)) \geq c + Q(b_0).$$

For $x \in [b, \infty) \cap A$ we then have

$$Pf(x) \geq (Q \circ \lambda)'(x) \int_0^{b_0} h(Q(\lambda(x)) - Q(y)) f(y) dy > 0.$$

Setting $d = d(f, g) = \max(b(f), b(g))$ we obtain

$$\mu(\text{supp } Pf \cap \text{supp } Pg) \geq \mu([d, \infty) \cap A) > 0 \quad \text{for } f, g \in \mathcal{D}.$$

Thus the requirements of Corollary 4.2.4 are satisfied and the proof is complete. ■

We provide the reader with an example of an LMT operator as in [BL] (Example 2.1), which is "stable" but not asymptotically stable, showing that the assumption in Theorem 5.2.1 is indeed essential.

Example 5.2.1. Let h be a density on $[0, \infty)$ such that $h(x) = 0$ for $x \geq \sqrt{c} - c$ where $c \in (0, 1)$ is constant. Consider the operator $P : L^1 \rightarrow L^1$ given by the formula

$$Pf(x) = \begin{cases} \frac{1}{2\sqrt{x}} \int_0^{\sqrt{x}} h(\sqrt{x} - y) f(y) dy & \text{for } x \in (0, 1), \\ \frac{1}{2} \int_0^{2x-1} h(2x - y - 1) f(y) dy & \text{for } x \geq 1. \end{cases} \quad (5.2.2)$$

In this case $Q(x) = x$,

$$\lambda(x) = \begin{cases} \sqrt{x} & \text{for } x \in [0, 1], \\ 2x - 1 & \text{for } x > 1, \end{cases} \quad H(x) = 1 - \int_0^x h(t) dt,$$

and evidently assumptions (3.2.1.16) and (3.2.1.17) are satisfied. Moreover, for every $\alpha \in (0, 1]$,

$$\int_0^{\infty} x^{\alpha} h(x) dx < 1 < \infty = \lim_{x \rightarrow \infty} ((Q(\lambda(x)))^{\alpha} - Q(x)^{\alpha}).$$

According to Corollary 5.1.1 the operator P is "stable".

However, using (5.2.2) it is easy to verify the following property of P .

If $\text{supp } f \subset [1, \infty)$ then $\text{supp } Pf \subset [1, \infty)$ and if $\text{supp } f \subset (0, c)$ then $\text{supp } Pf \subset (0, c)$. Since $c < 1$ it follows from the definition that P cannot be asymptotically stable. \square

5.3 Sweeping LMT operators

In the previous results concerning LMT operators inequality (5.1.1) played a crucial role, thus the question arises: "What can we say when (5.1.1) is not satisfied?". A partial answer is obtained when it is shown that an opposite condition implies that P is sweeping, as in [BL], Theorem 2.3.

Theorem 5.3.1. Assume that

$$\sup_{x \geq x_0} (Q(\lambda(x))^{\beta} - Q(x)^{\beta}) < \int_0^{\infty} x^{\beta} h(x) dx < \infty, \quad (5.3.1)$$

for an $x_0 \geq 0$ and $\beta \geq 1$ and that

$$\int_{Q(\lambda(x_0))}^{\infty} h(x) dx > 0.$$

Then P is sweeping.

Proof. Define

$$z_0 = Q(\lambda(x_0))^{\beta}, \quad w(z) = \begin{cases} e^{-\epsilon z_0} & \text{for } z \in [0, z_0], \\ e^{-\epsilon z} & \text{for } z > z_0, \end{cases}$$

and

$$V(x) = w(Q(\lambda(x)))^\beta$$

where $\epsilon > 0$ will be chosen later. We will show that there exists a nonnegative constant $\gamma < 1$ such that

$$\int_0^\infty V(x)Pf(x) dx \leq \gamma \int_0^\infty V(x)f(x) dx \quad \text{for each } f \in \mathcal{D}. \quad (5.3.2)$$

Since $V(x)$ admits a positive minimum on every compact set this inequality implies that P is sweeping (Proposition 1.7.1).

According to (5.3.1) there exists a number ρ such that

$$\sup_{x \geq x_0} (Q(\lambda(x))^\beta - Q(x)^\beta) < \rho < \int_0^\infty x^\beta h(x) dx.$$

We now define

$$I(y) = \int_0^\infty \frac{w((x + Q(y))^\beta)}{V(y)} h(x) dx \quad \text{for } y \geq 0.$$

If $y \leq x_0$ then $V(y) = w(z_0)$ and

$$\begin{aligned} I(y) &\leq \int_0^\infty \frac{w(x^\beta)}{V(y)} h(x) dx = \int_0^{Q(\lambda(x_0))} \frac{w(x^\beta)}{w(z_0)} h(x) dx + \int_{Q(\lambda(x_0))}^\infty \frac{w(x^\beta)}{w(z_0)} h(x) dx \\ &= \int_0^{Q(\lambda(x_0))} h(x) dx + \int_{Q(\lambda(x_0))}^\infty h(x) e^{-\epsilon(x^\beta - z_0)} dx \\ &= 1 - \int_{Q(\lambda(x_0))}^\infty h(x)(1 - e^{-\epsilon(x^\beta - z_0)}) dx =: \gamma_1(\epsilon) < 1. \end{aligned}$$

If $y > x_0$, then $(Q(\lambda(y))^\beta - Q(y)^\beta) < \rho$ and, $w(z) \leq e^{-\epsilon z}$ for $z \geq 0$,

$$\frac{w((x + Q(y))^\beta)}{V(y)} \leq \frac{e^{-\epsilon(x + Q(y))^\beta}}{e^{-\epsilon(Q(\lambda(y))^\beta)}} \leq \frac{e^{-\epsilon(x + Q(y))^\beta}}{e^{-\epsilon(\rho + Q(y)^\beta)}} \leq e^{-\epsilon(x^\beta - \rho)},$$

consequently,

$$I(y) \leq \int_0^\infty h(x) e^{-\epsilon(x^\beta - \rho)} dx =: \gamma_2(\epsilon),$$

From (3.2.1.18) it follows that

$$\begin{aligned} \int_0^\infty V(x)Pf(x) dx &= \int_0^\infty f(y) dy \int_0^\infty w((x+Q(y))^\beta)h(x) dx \\ &= \int_0^\infty f(y)V(y)I(y) dy \\ &\leq \gamma_1(\epsilon) \int_0^{x_0} V(y)f(y) dy + \gamma_2(\epsilon) \int_{x_0}^\infty V(y)f(y) dy \end{aligned}$$

for every $f \in \mathcal{D}$. Now, since $\gamma_1(\epsilon) < 1$ for every $\epsilon > 0$, in order to show (5.3.2) with a constant $\gamma < 1$ it suffices to prove that there exists an $\epsilon > 0$ such that $\gamma_2(\epsilon) < 1$. But the function γ_2 is differentiable on $[0, \infty)$ and

$$\gamma_2'(\epsilon) = - \int_0^\infty h(x)(x^\beta - \rho)e^{-\epsilon(x^\beta - \rho)} dx,$$

and thus

$$\gamma_2'(0) = \rho - \int_0^\infty x^\beta h(x) dx < 0.$$

Consequently, for sufficiently small $\epsilon > 0$ we have $\gamma_2(\epsilon) < \gamma_2(0) = 1$, which completes the proof. ■

5.4 Applications to Cell Cycle Models

The results of Bartoszek, and especially those of Baron and Lasota, for which proofs have been given in this chapter, unifies and extends many previous results on specific examples of cell cycle models scattered through the literature. See [GL], [LM2], [LMT], [LR], [Tyr], [TH1] and [TH2]. We proceed to apply these results to the models described in Section 3.3.

Consider first the *Lasota Mackey model*. In this case we have

$$\lambda(x) = 2x \quad \text{and} \quad h(x) = e^{-x} > 0.$$

We shall examine the generalised Lasota Mackey model with arbitrary λ . Thus (5.1.1) with $\alpha = 1$ becomes

$$\int_0^{\infty} x e^{-x} dx < \liminf_{x \rightarrow \infty} (Q(\lambda(x)) - Q(x))$$

or

$$\liminf_{x \rightarrow \infty} (Q(\lambda(x)) - Q(x)) > 1. \quad (5.4.1)$$

Thus (5.4.1) is a sufficient condition for asymptotic stability, on the other hand, if

$$\liminf_{x \rightarrow \infty} (Q(\lambda(x)) - Q(x)) < 1$$

then P is sweeping, according to Theorem 5.3.1. These results are more general than the earlier published ones of Lasota and Mackey and, later, Tyrcha, in [LM], [Tyr], and [LMT].

Many results have been published for the Tyson Hannsngen model, using a variety of techniques, but they all seem to follow easily from the theory we have presented.

It can be checked, that, in this model

$$h(x) = e^{-x}, \quad \lambda(x) = \frac{x}{\sigma} \quad \text{where} \quad \sigma = e^{k\tau_B}$$

and

$$Q(x) = \left(\frac{p}{k}\right) \ln^+ x \quad \text{where} \quad \ln^+ x := \max(0, \ln x).$$

After a simple calculation with $\alpha = 1$, (5.1.1) reduces to

$$\frac{-p}{k} \ln \sigma > 1, \quad \text{or} \quad \frac{k}{p} < -\ln \sigma.$$

This gives

$$\tau > \frac{1}{p} + \tau_B \quad \text{where} \quad \tau = \frac{\ln 2}{k},$$

is the size doubling time. This has the following simple interpretation: If the average sojourn time in phases A and B is less than the size doubling time, then P

is asymptotically stable. From Theorem 5.3.1 it follows that the opposite condition implies that P is sweeping. These results were hypothesised by Tyson and Hannsgen and were first proved by Tyrcha in [Tyr] using other methods.

We finally turn to the extensions of Tyson and Hannsgen to the *Smith, Shields* and *Martin* models. As in [LMT] we consider the special case where $g(x) = kx^a$. Then

$$Q(x) = \frac{x^{1-a}}{k(1-a)}, \quad a \neq 1, \quad \lambda(x) = 2x.$$

By a simple application of Theorem 5.2.1 we see that the condition

$$\int_0^{\infty} x^a h(x) dx < \infty, \quad \alpha \leq 1, \quad a < 1,$$

is sufficient for P to be "stable". If, in addition, ψ is positive on an infinite interval, then so is h , and P is asymptotically stable. Theorem 5.3.1 shows that P is sweeping if $a > 1$.

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