A MATHEMATICAL EXPLANATION OF THE TRANSITION BETWEEN LAMINAR AND TURBULENT FLOW IN NEWTONIAN FLUIDS, USING THE LIE GROUPS AND FINITE ELEMENT METHODS

by

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submitted in part fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in the subject

APPLIED MATHEMATICS

at the

UNIVERSITY OF SOUTH AFRICA

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AUGUST 2007

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Dedication

I dedicate this dissertation to the memory of my younger sister,

MARIANE TSAYEM GOUFO.
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Acknowledgements

This report was made possible by God, the Almighty Father, who helped us from the beginning until the accomplishment of this task.

I would like to express my gratitude to my supervisors: Dr J.M. Manale and Dr. R. Maritz, for their time, helpful discussions, critical evaluations, remarks, orientations, constructive criticism and availabilities. I most especially, acknowledge the unflinching contribution of Dr. R. Maritz, who has never stopped thinking of my condition and who helped me obtaining financial support for the accomplishment of this work.

I also wish to thank my whole family, especially my beloved mother, Jeannette Ngningaye Goufo, for being always there for me.

Thanks to Mrs. Stella Mugisha, Mr. Clovis Oukouomi Noutchie, Mr. Ervicks Ngatat and other friends for their advice.

Finally, I am grateful to Nélia Van Velden, for editing this dissertation and to the staff of the Department of Mathematical Sciences-University of South Africa, for offering me a kindly environment and the necessary material.
Abstract

In this scientific work, we use two effective methods: Lie groups theory and the finite element method, to explain why the transition from laminar flow to turbulence flow depends on the variation of the Reynolds number. We restrict ourselves to the case of incompressible viscous Newtonian fluid flows. Their governing equations, i.e. the continuity and Navier-Stokes equations are established and investigated. Their solutions are expressed explicitly thanks to Lie’s theory. The stability theory, which leads to an eigenvalue problem is used together with the finite element method, showing a way to compute the critical Reynolds number, for which the transition to turbulence occurs. The stationary flow is also studied and a finite element method, the Newton method, is used to prove the stability of its convergence, which is guaranteed for small variations of the Reynolds number.
Introduction

The fundamental laws of fluid flow can be expressed in mathematical form for a special type of controlled volume, the differential element. The differential equations of fluid flow provide a means of determining the variation of Newtonian fluid properties. These equations are the Navier-Stokes equations. The Navier-Stokes equations are the foundation of fluid mechanics and, strangely enough, are rarely recorded in their entirety. They describe motions of Newtonian fluid flows irrespective of whether they are laminar or turbulent. A laminar or turbulent flow depends on the relative importance of fluid friction (viscosity) and flow inertia. Turbulence is the most fundamental and, simultaneously, the most complex form of fluid flow.

More than a century after Reynolds’ paper, the understanding how turbulent regions grow (in a pipe flow, for example) and to bring laminar flow to fully developed turbulence, is not completely achieved. It has since been known (O. Reynolds [26]) that the transition to turbulence occurs in an intermittent fashion. As the Reynolds number increased beyond a critical value of about 2300 (although the precise value depends on the pipe used and on the experimental conditions at the inlet), intermittent flashes of turbulence can be seen in the pipe. Reynolds proved that the transition from laminar to turbulent flow in pipes, is a function of the fluid velocity.

Furthermore, the reason for this intermittency is well known, at least in a crude way: Laminar flow at a given flow has a lower drag than turbulent flow, and as the pressure drop driving the flow is increased, there arises a critical interval of flow rate within which laminar flow offers too low a resistance to the pressure drop, and turbulent flow
provides too high a resistance. In this intermediate case, the flow cycles between the two types of flow. This is manifested in the pipe through the regular occurrence of what Reynolds called turbulent "flashes", nowadays known as "slugs" or "puffs" depending on their provenance. The resultant flow oscillates, producing an oscillatory outlet flow. However, because understanding of the transition requires an understanding of laminar and turbulence flow, both are explored in this dissertation. With the assistance of existing experimental information, it is possible to develop a mathematical model of the transition between the two types of flow.

Another example is the wake formation behind bluff bodies where Karniadakis and Triantafyllou [12] observed the existence of a transitional regime, depending on the Reynolds number. Over more than a century, it has received a great deal of attention from an experimental and a numerical point of view. Other researchers like Williamson [32] observed the existence of two modes of formation of streamwise vorticity in the near wake, each occurring at a different range of Reynolds numbers, and both being related to the three-dimensional transition between Reynolds numbers belonging to a specific real interval.

It always happens, as shown in the second part of Girault and Raviart [6], that the occurrence of transition is expressed in terms of the Reynolds number mainly, though there are other factors that are not of our interest in this work. Thus, when the Reynolds number, $Re$ is large (small viscosity) compared to the other parameters of the fluid, there arises a boundary layer in the neighborhood of the controlled domain, $\partial \Omega$, where the viscosity predominates while it is negligible in the interior of $\Omega$. At the same time the transition to turbulence occurs.

So why does the flow suddenly become unstable and break up into turbulent swirls at large enough values of the Reynolds number? There are certain standard ways to answer this question: One can simply try to solve analytically or numerically the Navier-Stokes equations governing the flow, so as to express explicitly the solutions in terms of the Reynolds number. Another way is to study the response of a fluid when it is subject to infinitesimally small disturbances, using a mathematical tool called hydrodynamic stability theory.
Hydrodynamic Stability Theory

Stability theory in general, according to Daniel D. Joseph [10], is the body of mathematical physics which enables one to deduce from first principles, the critical values which separate the different regimes of flow, as well as the forms of the fluid motions in these different regimes.

In the case of this dissertation, we seek the critical Reynolds number at which the transition from laminar flow to turbulent flow occurs.

Drazin & Reid [5] gave a more explicit definition of hydrodynamic instability, suitable for the scope of this dissertation, and which is defined as that branch of hydrodynamic concerned with "when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence". From this definition, we can propose the following general procedure for studying hydrodynamic stability mathematically:

1. Start with a laminar or non-perturbed solution of the Navier-Stokes equations,

2. Perturb this solution with small disturbances,

3. Substitute the disturbed solution into the Navier-Stokes equations to derive disturbance equations. This usually yields an eigenvalue problem.

4. Solve the eigenvalue problem to study the (in)stability from the obtained equations.

We will try to explain mathematically the transition to turbulence by investigating the equations governing the flows of incompressible Newtonian fluids. The complete modelling must include not only these equations, but also the physical boundary and initial conditions imposed on the fluid. At this level we will treat the model (problem) analytically and numerically, thanks to two chosen methods: The Lie Group analysis and finite elements method.

Lie Group Analysis is a method for solving linear or non-linear differential equations...
analytically. It augments intuition in understanding and using symmetry for formulation of mathematical models, and often discloses possible approaches to solving complex problems. For the Navier-Stokes problem, this method uses general symmetry groups to explicitly determine solutions, which are themselves invariant under some subgroups of the full symmetry group of the system. These group-invariant solutions are found by solving a reduced system of ordinary differential equations, involving fewer independent variables than the original system (which presumably makes it easier to solve).

The finite element method requires discretization of the domain into sub regions or cells. In each cell the sought function is approximated by a characteristic form which is often a linear function. The method is traditionally based on the Galerkin weighted residual and Crank-Nicolson methods. One manner to obtain a suitable framework for treating our Navier-Stokes problem is to pose it as a variational one. The numerical treatment of the system of the Navier-Stokes equations by the finite element method, consists of computing the primitive variables \( u \) (velocity), and \( p \) (pressure), using a special Galerkin method based on a variational formulation. The spatial and time discretizations of the Navier-Stokes problem are constructed in appropriate function spaces, and ”discrete” approximations will be determined in certain finite dimensional subspaces, consisting of piecewise polynomial functions.

Both methods are further explored in the second chapter. The third and fourth chapters deal with the resolution, and we end with the results and recommendations inspired by the works of great researchers.

First we need to establish the Navier-Stokes equations for incompressible Newtonian fluids and therefore set the model. This is done in the first chapter. We will also outline the basics of laminar, the transition and turbulent flow. More precisely, we will comment on the Reynolds number and the meaning of the non-dimensionalization.
Chapter 1

Basic Considerations

The goal of this chapter is to provide a brief presentation of the equations governing an incompressible Newtonian fluid flow. We will use more simple ways to obtain the continuity and Navier-Stokes equations by utilizing the equations of conservation of mass as well as momentum in some domain of $\mathbb{R}^3$. Before doing that, let us outline some useful notations and concepts that characterize flows of Newtonian fluids.

**Notations**
The following notations will be considered throughout this dissertation:

For simplicity, we keep vectors represented in **bold** character.

- $\Omega$, open and bounded domain of $\mathbb{R}^3$
- $\mathbf{x} = (x, y, z)$, point in $\Omega$
- $t$, time over the time interval $[0;T]$
- $\partial \Omega$ or $\Gamma$, boundary of $\Omega$
- $\mathbf{n}$, outward normal to $\Gamma$
- $\mathbf{s}$, $\mathbf{t}$, tangents to $\Gamma$
- $\mathbf{u} = (u, v, w) = (u(t, x, y, z), v(t, x, y, z), w(t, x, y, z))$, fluid velocity vector field with components $u$, $v$ and $w$ at the point $(x,y,z)$ and time $t$.
- Note: $x = x(t)$, $y = y(t)$, $z = z(t)$, $u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$, $w = \frac{dz}{dt}$
- $p = p(t,x,y,z)$, pressure
\( \rho \), constant density (assumed)
\( \mu \), constant viscosity (assumed)
\( \nu \), kinematic viscosity
\( u^i \), the \( i^{th} \) component (or coordinate) of \( u \)
\( u_x, \frac{\partial u}{\partial x_i} \) or \( \partial_i u \), partial derivative of \( u \) with respect to the \( i^{th} \) coordinate
\( u_t, \frac{\partial u}{\partial t} \) or \( \partial_t u \), partial derivative in \( t \) of \( u \)
\( u_{tt} \), second time derivative
\( u_{x_i x_i} \), second derivative of \( u \) with respect to the \( i^{th} \) coordinate
\( \nabla p = (p_x, p_y, p_z) \), vector gradient of \( p \)
\( \nabla \cdot u = u_x + v_y + w_z \), divergence of \( u \)
\( \nabla u = \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \), second order tensor (velocity gradient)
\( u^T \), transpose of \( u \).
New notations will be defined as we go along.

### 1.1 Concepts of Fluids and Properties

The following concepts are defined according to the book by Yuan [30].

Incompressible material referred to as fluids may be liquids or gasses. To understand the meaning of fluids, we must define a shearing stress. A force that acts on an area can be decomposed into a normal component and a tangential component. The force divided by the area upon which it acts is called stress. The force vector divided by the area is a stress vector, and the normal component of the force divided by the area is a normal stress. The tangential force divided by the area is a shear stress.

The fluids considered in this dissertation are those liquids or gasses that move under the action of a shear stress, no matter how small that shear stress may be: this means that even a very small shear stress results in motion in the fluid. Therefore a liquid is a state of matter in which the molecules are relatively free to change their positions with
respect to each other, but restricted by cohesive forces so as to maintain a relatively fixed volume. In our study, it is convenient to assume that fluids are continuously distributed throughout a region of interest, that is, the fluid is treated as a *continuum*.

The primary property used to determine if the continuum assumption is appropriate, is the *density* of the fluid defined as the mass per unit volume. The density may vary significantly throughout the fluid. The concept of density at a mathematical point is defined as

\[ \rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} , \]

where \( \Delta m \) is the incremental mass contained in the incremental volume \( \Delta V \).

In fluid mechanics, other fluid quantities are the velocity vector field, and the pressure. They are both functions of time and space coordinates. The velocity, \( \mathbf{u} \), at any point of a fluid medium is written as the limit approached by the ratio between the displacement \( \delta s \) of an element along its path and the corresponding increment of time \( \delta t \) as the latter approaches zero: so

\[ \mathbf{u} = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} . \]

The pressure results from a normal compressive force acting on an area. If we were to measure this force per unit area acting on a submerged element, we would observe that it can either act inward, or place the element in compression. The quantity measured is therefore the *pressure* which must be the negative of the normal stress. When the shearing stresses are present, the normal stress components at a point may not be equal; however the pressure is still equal to the negative of the average normal stress. The *absolute pressure* is a scale measuring pressures, where zero is reached when an ideal vacuum is achieved. In many relationships, absolute scales must be used for pressure.

If the shear stress of a fluid is directly proportional to the velocity gradient, the fluid is said to be a *Newtonian fluid* and the coefficient of proportionality is evaluated as the *viscosity*, \( \mu \). This relation between shear stress and the velocity gradient also applies for an *incompressible fluid flow*, that is a flow for which the density is constant across the fluid. Fortunately, many common fluids, such as air, water and oil, are Newtonian and the viscosity by definition depends only on temperature and pressure, as well as the
The viscosity of a fluid is a measure of its resistance to deformation rate. Another important effect of viscosity is to cause the fluid to adhere to the surface: This is known as the no-slip condition. The viscosity in general is dependent on temperature in liquids in which cohesive forces play a dominant role. Note that the viscosity of liquids decreases with increased temperature. In this dissertation, we use a viscosity which is constant (incompressibility of the fluid).

Since the viscosity is often divided by the density in the derivation of equations (1.2.6) below, it has become useful and customary to define kinematic viscosity to be

\[ \nu = \frac{\mu}{\rho}. \]  

(1.1.1)

Now with the above definitions, we are able to establish easily the equations of an incompressible fluid flow.

### 1.2 Differential Equations of Incompressible Newtonian Fluid Flow

The theory of mechanics of continuous media, also known as continuum mechanics, allows the description of the constitutive equations laws that describe the deformations of fluid medium. These laws, in combination with the general conservation principles (conservation of mass and of momentum), form the system of partial differential equations, which are equal in number to the number of unknowns of the system. Namely, for 3-D motion there are four dependents variables: \( u, v, w \) and \( p \) and four independent variables: \( x, y, z \) and \( t \).

The written constitutive equations of a Newtonian fluid are based on the following considerations, according to B. Mohammadi [17]:

- At rest the fluid obeys the laws of statics.
• The equation of the fluid is objective, that is tensors are used. It is independent of the Galilean reference frame in which it is expressed, and independent of the observer.

• Constitutive relations governing the fluid are isotropic, which means, independent of the orientation of the coordinate system axes.

With these assumptions, we exploit Cauchy’s Laws, to obtain (B. Mohammadi [17]) the Navier-Stokes equations.

### 1.2.1 The Continuity Equation

To study the motion of a fluid which occupies a domain $\Omega \subset \mathbb{R}^3$ over a time interval $[0,T]$, we shall denote by $O$ any regular subdomain of $\Omega$ and by $x = (x,y,z)$ any point of $\Omega$.

To conserve mass, the rate of change of mass in fluid in $O$, $\frac{\partial}{\partial t} \int_O \rho$, has to be equal to the mass flux, $-\int_{\partial O} \rho \mathbf{u} \cdot \mathbf{n}$, across the boundary $\partial O$ of $O$, ($\mathbf{n}$ denotes the exterior normal to $\partial O$). Then,

$$\frac{\partial}{\partial t} \int_O \rho = -\int_{\partial O} \rho \mathbf{u} \cdot \mathbf{n}.$$  

By using the Stokes' formula

$$\int_O \nabla \cdot (\rho \mathbf{u}) = \int_{\partial O} \rho \mathbf{u} \cdot \mathbf{n}, \quad (1.2.1)$$

the mass conservation equation becomes

$$\int_O \left( \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u}) \right) = 0.$$  

The fact that $O$ is arbitrary, yields the equation of conservation of mass, expressed in differential form, and found to be

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (1.2.2)$$

It is also called the continuity equation.

The assumption to restrict our attention to incompressible flow with constant density, $\rho$, yields $\frac{\partial}{\partial t} \rho = 0$ and $\nabla \cdot (\rho \mathbf{u}) = \rho (\nabla \cdot \mathbf{u})$. Therefore, the continuity equation (1.2.2) takes
the final sought form

$$\nabla \cdot \mathbf{u} = 0.$$  \hspace{1cm} (1.2.3)

Which means that the velocity field, \( \mathbf{u} \), of an incompressible flow must be divergence free.

In cartesian coordinates \( \nabla \) is written:

$$\nabla = \mathbf{i} \partial_x + \mathbf{j} \partial_y + \mathbf{k} \partial_z$$

and recalling \( \mathbf{u} = (u, v, w) \), equation (1.2.3) reads:

$$u_x + v_y + w_z = 0.$$  \hspace{1cm} (1.2.4)

### 1.2.2 The Navier-stokes Equations

The Navier-Stokes equations are considered as the foundation of fluid mechanics, and were introduced by C. Navier in 1823 and developed by G. Stokes. However, these equations were first introduced by L. Euler. The main contribution by C. Navier was to add a friction forcing term due to interactions between fluids layers which move with different speeds. These equations are nothing but the momentum equations based on Newton’s second law, which relates the acceleration of a particle to the resulting volume and body forces acting on it. They are, accordingly, the differential form of Newton’s second law of motion.

Let us now write Newton’s second law for the arbitrary volume element \( O \) of fluid. By definition of the velocity \( \mathbf{u} \), a particle of the fluid at position \( \mathbf{x} = (x, y, z) \) at time \( t \) will be approximately at \( \mathbf{x} + \mathbf{u}(\mathbf{x}, t)\delta t \) at time \( t + \delta t \). Its acceleration is therefore

$$\lim_{\delta t \to 0} \frac{1}{\delta t} \left[ \mathbf{u}(\mathbf{x} + \mathbf{u}(\mathbf{x}, t)\delta t, t + \delta t) - \mathbf{u}(\mathbf{x}, t) \right] = \mathbf{u}_t + \sum_{j=1}^{3} u^j \mathbf{u}_{x_j} \equiv \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u},$$

where \( u^j \) is the \( j^{th} \) component of the vector \( \mathbf{u} \) and \( \mathbf{u}_{x_j} \) the partial derivative of \( \mathbf{u} \) with respect to the \( j^{th} \) coordinate of the point \( \mathbf{x} \).
If we disregard external forces like those due to gravity, electromagnetism, Coriolis, etc., the only remaining forces are the pressure force and the viscous force due to the motion of the fluid, and equal to \( \int_{\partial O} (\sigma - pI) n \), where \( \sigma \) is the stress tensor, \( I \) is the unit tensor and \( n \) denotes the unit outer normal to \( \partial O \). In this condition, Newton’s second law of motion for \( O \) is given by

\[
\int_O \rho (u_t + u \cdot \nabla u) = -\int_{\partial O} (pn - \sigma n) = \int_O (-\nabla p + \nabla \cdot \sigma)
\]

where we have used the Stokes’ formula (1.2.1) to establish the second equality.

The fact that \( O \) is arbitrary yields

\[
\rho (u_t + u \cdot \nabla u) = -\nabla p + \nabla \cdot \sigma. \tag{1.2.5}
\]

Now we need to relate the stress tensor \( \sigma \) to the velocity of the fluid: The hypothesis of Newtonian flow is a linear law relating \( \sigma \) to \( \nabla u \):

\[
\sigma = \mu (\nabla u + \nabla u^T) + (\nu - \frac{2\mu}{3}) I \nabla \cdot u
\]

where \( \nu \) is the second viscosity of the fluid. For air and water the second viscosity \( \nu \) is very small. For Newtonian fluids, we assume that \( \nu = 0 \). The stress tensor becomes

\[
\sigma = \mu (\nabla u + \nabla u^T) - \frac{2\mu}{3} I \nabla \cdot u.
\]

With this definition for \( \sigma \), equation (1.2.5) becomes

\[
\rho (u_t + u \cdot \nabla u) = -\nabla p + \nabla \cdot [\mu (\nabla u + \nabla u^T) - \frac{2\mu}{3} I \nabla \cdot u]
\]

or

\[
\rho (u_t + u \cdot \nabla u) + \nabla p - \mu [\nabla \cdot \nabla u + \nabla \cdot \nabla u^T] + \frac{2\mu}{3} \nabla \cdot (I \nabla \cdot u) = 0.
\]

Since \( \nabla \cdot \nabla u = \nabla^2 u \), \( \nabla \cdot \nabla u^T = \nabla (\nabla \cdot u) \) and \( \nabla \cdot (I \nabla \cdot u) = \nabla (\nabla \cdot u) \), the latter equation finally yields the equation of conservation of momentum written as

\[
\rho (u_t + u \cdot \nabla u) + \nabla p - \mu \nabla^2 u - \frac{\mu}{3} \nabla (\nabla \cdot u) = 0.
\]

Taking into account the continuity equation (1.2.3), the equation of conservation of momentum becomes the incompressible Navier-Stokes equations

\[
u_t + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 u
\]
where $\nu = \mu / \rho$ is the kinematic viscosity of the fluid defined in equation (1.1.1) and $p \to p / \rho$ is the reduced pressure. Note that the Navier-Stokes equations are non-linear because of the term $\mathbf{u} \cdot \nabla \mathbf{u}$ which is seen as the source of instability.

Now we write each term of (1.2.6) in cartesian coordinates:

$$
\mathbf{u}_t = (u_t, v_t, w_t)
$$

$$
\mathbf{u} \cdot \nabla \mathbf{u} = (u, v, w) \cdot \begin{pmatrix}
    u_x & v_x & w_x \\
    u_y & v_y & w_y \\
    u_z & v_z & w_z
\end{pmatrix}
= \begin{pmatrix}
    uu_x + vu_y + wu_z \\
    wv_y + vv_y + wv_z \\
    uw_x + vw_y + ww_z
\end{pmatrix}
$$

$$
\nabla p = (p_x, p_y, p_z)
$$

$$
\nabla^2 \mathbf{u} = \begin{pmatrix}
    u_{xx} + u_{yy} + u_{zz} \\
    v_{xx} + v_{yy} + v_{zz} \\
    w_{xx} + w_{yy} + w_{zz}
\end{pmatrix}
= \begin{pmatrix}
    \nabla^2 u \\
    \nabla^2 v \\
    \nabla^2 w
\end{pmatrix}
$$

Thus the Navier-Stokes equations for constant density, $\rho$, and constant viscosity, $\mu$, are written as:

**x-component**

$$
u_t + uu_x + vu_y + wu_z = -p_x + \nu \nabla^2 u,
$$

(1.2.7)

**y-component**

$$
u_t + wu_x + vv_y + wv_z = -p_y + \nu \nabla^2 v,
$$

(1.2.8)

**z-component**

$$
u_t + uw_x + vw_y + ww_z = -p_z + \nu \nabla^2 w.
$$

(1.2.9)
1.3 The Reynolds Number

The challenge of laminar-transition-turbulence started in 1883, when Osborne Reynolds of Manchester University (United Kingdom) made a prominent discovery that has remained a puzzle ever since. By introducing a small amount of ink into a horizontal glass pipe filled with water, he was able to check whether the flow was laminar or turbulent. Reynolds found that the transition from laminar to turbulent flow occurs spontaneously if a dimensionless quantity (see [31]), \( Re \), is larger than some critical value, about 2300. This quantity, which is known as the Reynolds number, has ever since become a quantity which engineers and scientists use to estimate if a fluid flow is laminar or turbulent. It is defined as the ratio of the inertia and viscous forces on the fluid.

Let us rewrite the Navier-stokes equations (1.2.4), (1.2.7), (1.2.8), (1.2.9) in non-dimensional form.

Let \( U \) the characteristic velocity scale of the flow under study, \( L \) the characteristic length scale and \( T_1 \) a characteristic time (which is a priori equal to \( L/U \), we put

\[
\begin{align*}
    u' &= \frac{u}{U}; \quad v' = \frac{v}{U}; \quad w' = \frac{w}{U}; \\
    x' &= \frac{x}{L}; \quad y' = \frac{y}{L}; \quad z' = \frac{z}{L}; \\
    t' &= \frac{Ut}{L}; \quad p' = \frac{p}{U^2}; \quad \nu' = \frac{\nu}{LU}.
\end{align*}
\]

To simplify the notation, the primes are dropped, and the non-dimensional form of the Navier-Stokes equations (1.2.4), (1.2.7), (1.2.8), (1.2.9) are respectively

\[
\begin{align*}
    u_x + v_y + w_z &= 0 \quad (1.3.1) \\
    u_t + uu_x + vv_y + ww_z &= -p_x + Re^{-1} \nabla^2 u \quad (1.3.2) \\
    v_t + uv_x + vv_y + wv_z &= -p_y + Re^{-1} \nabla^2 v \quad (1.3.3) \\
    w_t + uw_x + vw_y + ww_z &= -p_z + Re^{-1} \nabla^2 w \quad (1.3.4)
\end{align*}
\]

where the Reynolds number \( Re \) is defined as

\[
Re = \frac{UL}{\nu}. \quad (1.3.5)
\]
It is clear that $Re$ compares the importance of inertia $UL$ to the effects of viscosity, characterized by $\nu$.

A limiting form of the Navier-Stokes equations is obtained when $Re \to \infty$. These equations are the *Euler equations*; they describe the dynamics of perfect (inviscid) incompressible fluids. These flows have a characteristic Reynolds number of the order of millions, and the Euler equations are a valid model when the effects of turbulence are neglected. The case $Re \to 0$ corresponds to slow flows (creeping flows) and the previous non-dimensionalization is no longer appropriate. It is necessary to relate the characteristic scale to the dominant physical phenomenon, i.e., the viscosity, so as to obtain the Reynolds number zero limit that yields non-dimensional equations called the Stokes equations.

### 1.4 Significance of the Non-Dimensionalization

An important consideration in all differential equations written thus far, has been the dimensional homogeneity. At times it has been necessary to use proper conversion factors for an answer to be correct numerically and have the proper units. The idea of dimensional consistency can be used in another way; by the procedure of dimensional analysis, to group the variables in a given situation into non-dimensional parameters that are less numerous than the original variables. By combining the variables into a smaller number of non-dimensional parameters, the work and time required to reduce and correlate experimental data, are decreased substantially.

The dimensional homogeneity of equations like the Navier-Stokes’, requires that each term in the equation has the same units. The ratio of one term in the equation to another must then, of necessity, be dimensionless. With the knowledge of the physical meaning of the various terms in the equation, we are then able to give some physical interpretation of the non-dimensional parameters thus formed. In the previous case of the non-dimensional form of the Navier-Stokes equations, the only non-dimensional
parameter formed was the Reynolds number $Re$ seen as the ratio of two forces

\[
\frac{\text{inertia force}}{\text{viscous force}} = \frac{UL}{\nu} = Re.
\]

The non-dimensionalization of governing equations has at least three advantages in comparison with dimensional relationships: Firstly, the relationship so derived is independent both numerically and dimensionally of the system of unit used in expressing the variables themselves. Secondly, the number of terms are usually reduced by the number of dimensional categories involved. Thirdly, the variables are so grouped as to facilitate the further study of their functional interrelationship. In any event, several facts should by now have become apparent. The non-dimensional parameters that are obtained are not limited to few commonly named numbers, but are of many varied forms. As these forms increase in number, so do their functional combinations.

To avoid any confusion during the non-dimensionalization of governing equations like the preceding Navier-Stokes equations, it is advisable to specify clearly the reference length, reference velocity, etc. (as in the previous section) when reporting a value to any dimensionless parameter.

In the present case (incompressible fluid), $Re$ is the perturbation parameter because it is the only parameter of the Navier-Stokes equations. Therefore, it is possible to derive $Re$, so as to explain, at least in a crude way, how the transition from laminar to turbulence flow usually occurs.

### 1.5 Basics of Laminar, Transition and Turbulent Flow

The existence of two types of viscous flow is a broadly accepted phenomenon:

The word *laminar* deriving from the Latin word *lámina*, which means stream or sheet, indicates the regularity. Therefore a laminar motion gives the idea of a regular streaming motion. In the opposite the word *turbulent* is used in every day experience to indicate something which is not regular. In Latin the word *turba* means something confusing
or something which does not follow an ordered plan. A turbulent boy, in all Italian schools, is a young fellow who rebels against ordered schemes. Following the same line, the behavior of a flow which rebels against the deterministic rules of classical dynamics, is called turbulent.

A flow can be both laminar in a given region, and turbulent in another region, which means that a region of transition exists between these two types of flow. A good example is the flow of a rising smoke from a cigarette: The smoke initially travels in smooth, straight lines (laminar flow), then starts to "wave" back and forth (transition flow), and finally seems to randomly mix (turbulent flow). There are other examples like the pipe flow, or the flow between two concentric cylinders, in which the three types of flow regions coexist.

Laminar flow is generally observed when adjacent fluid layers slide smoothly over one another with mixing between layers or lamina occurring only at molecular level; whereas turbulent flow regime happens when small packets of fluid particles are transferred between layers, giving it a fluctuating nature.

The existence of a transition regime from laminar to turbulent flow, although recognized earlier, was first described quantitatively by Reynolds in 1883, with his legendary and classic experiment evoked in section 1.3. At low rates of flow the pattern of the ink (dye) he injected, was regular and formed a single line of color. At high flow rates, however, the ink became dispersed throughout the pipe cross section because of the irregular fluid motion. The difference in appearance of the ink streak was, of course, due to the orderly nature of laminar flow in the first case, and to the fluctuating character of turbulent flow in the latter. The transition from laminar to turbulent flow in pipes is thus a function of the fluid velocity. Reynolds found that fluid velocity was the only variable determining the nature of pipe flow. The other variables are pipe diameter, fluid density, and viscosity. That is why these other variables were combined into the single dimensionless parameter, the Reynolds number.

Generally in pipe flow, as shown below, a fluid is laminar from $Re=0$ to some critical value at which transition flow begins. This critical value is about 2300, above which small
disturbances will cause a transition to turbulent flow, and below which disturbances are
damped out and laminar flow prevails. In the transition range, the flow becomes semi-
irregular and is on the verge of becoming turbulent. Finally, the flow becomes unstable
as $Re$ increases, leading us to the turbulent flow in which there is increased mixing that
results in viscous losses, which are generally much higher than those in laminar flow. At
this level the flow is characterized by the following:

- The flow is fully three dimensional.

- It is time dependent with a marked random character.

- It is dissipative; in other words, the viscous terms of the Navier-Stokes equation can
  not be neglected, even though the Reynolds number can be very high.

Our attention here being the transition flow only, it follows that the occurrence of tran-
sition is expressed in terms of the Reynolds number only; while a variety of factors other
than $Re$ actually influence transition when fluctuations have started. The Reynolds
number remains, however, the principal parameter for predicting transition.

1.6 A Concrete Example: The Transition to Turbu-
ience in the Wake of a Circular Cylinder

A concrete example is illustrated in the publication [24], where Hélène Persillon and
Marianna Braza studied and represented the transition to turbulence of the flow around
a circular cylinder, namely the transition to turbulence in the wake of a circular cylinder.
This study together with the one by G. E. Karniadakis and G. S. Triantafyllou [12],
about the wake formation behind bluff bodies, has received a great deal of attention
over more than a century from both an experimental and a numerical point of view.
Hélène Persillon and Marianna Braza have computed the three dimensional flow around a
circular cylinder in the Reynolds number range of 100-300. The time-dependent evolution
of the $u$- and $v$-velocity components are presented in both two and three-dimensional cases, for Reynolds number 200 and 300. This evolution is done at a spatial point of investigation $x/D = 0.97$, $y/D = 0$ and $z = 0$, where $D$ is the diameter of the cylinder. The drawings show the quasi-periodic character of the studied flow, and one can see that the amplitudes of the oscillations increase with the Reynolds number. Many other factors, like the frequency of the oscillations of $u$- and $v$-velocity components are taken into account and are the same for each Reynolds number. The establishment of the quasi-periodic character is more rapid for the $v$-component than for the $u$-component, because the periodic character of this component is masked by the overall convection effect. The amplitudes of the oscillations decrease as the sampling point moves downstream. It is clearly illustrated that the stability of the flow depends on the Reynolds number, and the flow is more perturbed as the Reynolds number increases. So, a general characteristic is that the amplitude of the oscillations increases as Reynolds number increases. This is what we are going to explain in the following chapters.

Does there exist any mathematical explanation for the fact that the flow which was previously laminar, starts to fluctuate between laminar and turbulent state to finally become unstable? The Reynolds number is certainly linked to the answer. Recall that it is the only parameter appearing in the non-dimensional form of the Navier-Stokes equations. Thus, it is convenient to say that one way to understand why the fluid flow suddenly becomes irregular and breaks up into the transition from laminar to turbulent state as the Reynolds number becomes larger, is to investigate the equations governing the flow. This investigation can be done thanks to two important theories: Lie Groups Theory and the finite element method.
Chapter 2

Lie Groups Theory and Finite Element Method

In this chapter we review two effective methods applicable to solve differential equations in general and Navier-Stokes equations in particular. Firstly, Lie Group theoretical method is used for obtaining closed forms of solutions of differential equations analytically. Secondly, the finite element method is numerical and determines an approximation of the sought solution, with hypothesis of suitable boundary and initial conditions in an appropriate space. We will present in a concise manner, the basic theories that lie at the core of both modern group and finite element analysis. We will also compare the results of the methods, with the hope that there are agreements.

2.1 Lie Groups Theory of Differential Equations

This theory is a method for solving linear and non-linear differential equations analytically. We discuss the general view and define the basic concepts of the theory, according to Ibragimov [9] and Olver [20].
2.1.1 General Principles of Lie Groups Theory

The applications of Lie groups to solve differential equations date back to the original work of the Norwegian mathematician Sophus Lie (1842-1899) in the 1870s. One of Lie’s striking achievements (see [14] and [15]) was the discovery that the majority of ad hoc methods of integration of differential equations, could be explained and deduced simply by means of group theory. Moreover, he classified differential equations in terms of their symmetry groups and showed that the second-order equations integrable by his method, can be reduced to merely four distinct canonical forms by the change of variables. Before going further let us outline a general view of what we mean by a symmetry group.

**Definition 2.1.1.** A symmetry group of a differential equation is a Lie-group action on some space of independent and dependent variables, transforming solutions of a given differential equation into other solutions.

More detailed explanations will be given in the next section.

Lie’s theory of differential equations unifies the many ad hoc methods developed in the 18th century, and known for solving differential equations. It also provides powerful new ways to finding solutions. A familiarity with the elements of the theory of differential equations, is therefore a prerequisite to Lie group theory. There are many well-known techniques for obtaining exact solutions, but what is often not recognized is that these techniques are usually special cases of a few powerful symmetry methods. Lie’s theory has applications to both ordinary and partial differential equations, and was introduced in order to study symmetry properties of these differential equations. Symmetry groups method is one of the known means for finding concrete solutions to complicated equations like those encountered in fluids mechanics.

Using symmetry method is an easy means to solving differential equations. This allows Lie group theory to use symmetry analysis (properties) to simplify a system of partial differential equations, thereby making it a valuable asset for solving non-linear models. Since the method of determining symmetries is applicable to almost any system of differential equations, a general method for solving differential equations may be formulated.
from this analysis.

The method has been successful in solving some equations (Olver [20]). Research is ongoing on others.

Lie’s theory, as described above, has become what today’s applied mathematicians, physicists or engineers call the *classical Lie method*, since it has shown its limitations for more complicated models describing fluid flow, like the Navier-Stokes equations (1.3.1)-(1.3.4). In the last few years, a variety of methods have been developed in order to find special classes of solutions of partial differential equations which cannot be easily determined by the classical Lie method. It has been shown that the common theme of all these methods has been the appearance of some form of group invariance. For the model (1.3.1)-(1.3.4) under investigation, considerable progress in resolving it can be achieved by means of a symmetry approach, which can be seen as a *non classical method*. However, the classical Lie method remains the core of finding symmetry reduction of the Navier-Stokes equations by symmetry approach, which makes it necessary to explore the basic definitions and concepts characterizing the theory.

### 2.2 Basic Concepts

The distinction between a group and a manifold is important to make, because Lie group is connected to both ideas. Manifolds are the generalizations of the familiar concepts of smooth curves and smooth surfaces in a 3-D space. In general, they are spaces which locally look Euclidian, but may be quite different globally. Mathematical objects like differential equations, functions, or symmetry groups, are defined on open subsets of an Euclidian space, and despite any particular coordinate system used to describe these subsets, their underlying geometrical features are defined to be coordinate independent. So a *manifold* is a set $M$ that contains a countable number of subsets $U_\alpha$ (called coordinate charts) and one-to-one functions $\Upsilon_\alpha$ (called coordinate maps), which map the $U_\alpha$ onto connected open subsets of an Euclidian space. The change of coordinates should be smooth.
A simple example of a manifold is the Euclidian space $\mathbb{R}^m$. There is a single coordinate chart equal to $\mathbb{R}^m$, therefore any open subset $U_\alpha$ is a $m$-manifold whose local coordinate is the identity.

The concept of a group is closely related to that of invariance or symmetry of mathematical objects (Differential equations, functions, symmetry groups etc...). So given any object $\mathcal{M}$, the set $G$ of all invertible transformations $T$ leaving the object $\mathcal{M}$ unaltered:

$$T: \mathcal{M} \rightarrow \mathcal{M},$$

contains the identity transformation $I$, the inverse $T^{-1}$ of any transformation $T \in G$ and the multiplication (or composition) $T_1 T_2$ of any two transformations $T_1, T_2 \in G$.

$G$ is then called a symmetry group of the object $\mathcal{M}$. A symmetry group of an object $\mathcal{M}$ is also termed a group admitted by this object.

The set of integers is an example of a group with operation, the addition whereby the identity element is zero, and the inverse of any given element is its negative.

An $r$-parameter Lie group is further defined to be a group $G$ which also carries the structure of an $r$-dimensional smooth manifold so that its group elements can be continuously varied. We assume that Lie groups are connected, that is, they cannot be written as the disjoint unions of two open sets. The set of real numbers is a Lie group.

It is worthwhile mentioning the definition of Lie sub-groups. The proper definition of a Lie sub-group is modelled on that of a sub-manifold. From this remark, follows the theorem that if $H$ is a closed sub-group of a Lie group $G$, then $H$ is also a regular sub-manifold of $G$ and hence a Lie group in its own right.

Lie groups often arise as transformations on some manifold, and the transformation does not need to be defined for all elements of the group, or all points on the manifold, since it can act locally.

The present section introduces a comprehensible method for solving differential equations via the use of symmetry groups. A symmetry group of a system of differential equations
transforms solutions of the system to other solutions, and is the largest local group of the transformations acting on the independent and dependent variables of the system. Before attempting to determine the symmetry groups of systems of differential equations, it is helpful to introduce the concept for simple equations or a function.

Given a system of equations defined for one or several variables in a manifold, we define a symmetry group of the system to be a local group of transformations that acts on the manifold by transforming solutions of the system to other solutions. We can also examine the invariance of a function under a group of transformations. Given a group of transformations acting on a manifold, and a function which maps from that manifold to another manifold, if for all group elements and all points on the manifold the function yields the same mapping for both a point on the manifold and for that same point acted on by the group operation, the function is said to be invariant for that group. In Lie group theory, one can replace the above criteria for invariant functions and subsets by an equivalent linear condition of infinitesimal invariance under the corresponding infinitesimal generators of the group action.

More concretely, we consider the general system of \( r \) homogeneous linear partial differential equations for one unknown function \( u = u(x) \):

\[
\sum_{i=1}^{n} \zeta^i_{\alpha}(x)p_i = 0, \quad \alpha = 1, \ldots, r, \tag{2.2.1}
\]

where \( x = (x^1, \ldots, x^n) \) are the independent variables and \( p_i = \partial u/\partial x^i \) are the partial derivatives. After introducing \( r \) partial differential operators of the form

\[
\chi_{\alpha} = \zeta^1_{\alpha}(x)\frac{\partial}{\partial x^1} + \ldots + \zeta^n_{\alpha}(x)\frac{\partial}{\partial x^n}, \quad \alpha = 1, \ldots, r, \tag{2.2.2}
\]

the system (2.2.1) is written in the compact form

\[
\chi_1(u) = 0, \quad \ldots, \quad \chi_r(u) = 0 \tag{2.2.3}
\]

The commutator of any two operators (2.2.2), \( \chi_{\alpha} \) and \( \chi_{\beta} \) is also a differential operator defined by

\[
[\chi_{\alpha}, \chi_{\beta}] = \chi_{\alpha}\chi_{\beta} - \chi_{\beta}\chi_{\alpha},
\]
or in the following explicit form

\[
[\chi_\alpha, \chi_\beta] = \sum_{i=1}^{n} \left( \chi_\alpha (\xi^i_\beta) - \chi_\beta (\xi^i_\alpha) \right) \frac{\partial}{\partial x^i}
\]  

(2.2.4)

In the case of \( n = 2 \), \( G \) is considered as the group of transformations in the plane \((x, y)\) given by

\[
\bar{x} = f(x, y, a) \approx x + a\xi(x, y), \quad \bar{y} = \varphi(x, y, a) \approx y + a\eta(x, y)
\]  

(2.2.5)

depending on a parameter \( a \), and where we have taken a linear part (in the parameter \( a \)) in the Taylor expansion of the initial transformations (called finite transformations). Lie’s theory therefore reduces the construction of the largest symmetry group \( G \) to the determination of its \textit{infinitesimal transformations} (2.2.5). Its representation by linear differential operators (2.2.2) becomes

\[
\chi = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}
\]  

(2.2.6)

called by Lie the \textit{symbol} of the infinitesimal transformation, but in modern literature is referred to as the \textit{infinitesimal operator}, or simply the \textit{generator} of the group \( G \).

Lastly, we shall consider the concept of Lie algebras of operators i.e. vector spaces of linear differential operators of the form (2.2.2) endowed with the commutators of the form defined by (2.2.4).

**Definition 2.2.1.** A \textit{Lie algebra} is a vector space \( L \) of operators \( \chi = \xi^i(x)\frac{\partial}{\partial x^i} \) with the property (2.2.4) verified for any two of its elements

\[
\chi_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}, \quad \chi_2 = \xi_2^i(x)\frac{\partial}{\partial x^i}
\]

such that their commutator \([\chi_1, \chi_2]\) is also an element of \( L \).

### 2.2.1 Symmetry Groups of Differential Equations

Let \( G \) be a one parameter group of the following transformations involving many independent variables \( x = (x^1, ..., x^n) \) and differential variable \( u = (u^1, ..., u^m) \):

\[
\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i
\]  

(2.2.7)
\[ \overline{u}^\alpha = \varphi^\alpha(x, u, a), \quad \varphi^\alpha|_{a=0} = u^\alpha, \] (2.8.2)

the generator of \( G \) is written in the form

\[ \chi = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \] (2.9.2)

where

\[ \xi^i(x, u) = \frac{\partial f^i(x, u, a)}{\partial a} \bigg|_{a=0}, \quad \eta^\alpha(x, u) = \frac{\partial \varphi^\alpha(x, u, a)}{\partial a} \bigg|_{a=0}. \] (2.10.2)

There are special formulae, called prolongation formulae, which are used to extend the generator (2.9.2) to the order of the highest derivative of the equation. Before continuing, let us define the frame of a differential equation which is also very important in our study.

**Definition 2.2.2.** Given a differentiable function \( F \) of order \( k \), (that is, a locally analytic function of the variables (sequences) \( Z = (x, u, u^{(1)}, ..., u^{(k)}) \) with elements \( Z^\nu, \nu \geq 1 \), where, for example \( Z^i = x^i (1 \leq i \leq n) \), \( Z^{n+\alpha} = u^\alpha (1 \leq \alpha \leq m) \)), equation

\[ F(x, u, u^{(1)}, ..., u^{(k)}) = 0 \] (2.11.2)

defines a manifold in the space of variables \( x, u, u^{(1)}, ..., u^{(k)} \). This manifold is called the frame of the \( k^{th} \)-order partial differential equation

\[ F \left( x, u, \frac{\partial u}{\partial x}, ..., \frac{\partial^k u}{\partial x^k} \right) = 0 \] (2.12.2)

Now we consider a system of \( k^{th} \)-order differential equations. The class of solutions being fixed, this system can be identified by its frame

\[ F_\sigma(x, u, u^{(1)}, ..., u^{(k)}) = 0, \quad \sigma = 1, ..., s, \] (2.13.2)

where \( F_\sigma \) are differential functions, \( u^{(i)} \) the set of all \( i^{th} \) derivative of \( u \) and the order \( k \) refers to the highest derivative appearing in (2.13.2).

**Definition 2.2.3.** This system of \( k^{th} \)-order differential equations is said to be invariant under a group \( G \) if its frame (2.13.2) is an invariant manifold for the extension of the group \( G \) to the \( k^{th} \)-order derivatives.
The definition is of great importance in Lie's theory since it helps us establish what is called the determining equations, which are important for the establishment of the admissible Lie algebra. Thus, there is a theorem (Olver [20]) which states that the system of differential equations (2.2.13) is invariant under the group with a generator $\chi$ if and only if

$$\chi F_\sigma \big|_{(2.2.13)} = 0, \quad \sigma = 1, \ldots, s,$$

where $\chi$ is extended to all derivatives involved in $F_\sigma(x, u, u^{(1)}, \ldots, u^{(k)})$ and the symbol $\big|_{(2.2.13)}$ means evaluated on the frame (2.2.13).

**Definition 2.2.4.** Equations (2.2.14) determine all infinitesimal symmetries of a system (2.2.13) and therefore they are known as *determining equations*.

It is worthwhile adding that the solutions of any determining equations, form a Lie algebra.

### 2.2.2 Lie Reduction of the Navier-Stokes Equations

After applying the appropriate prolonged operator (2.2.9) to each equation of the system (1.3.1)-(1.3.4), we find, via the determining equations, all the operators admitted by the Navier-Stokes equations. In other words, we seek the admitted Lie algebra spanned by operators in the form

$$\chi = \sum_i \xi^i \frac{\partial}{\partial q^i} + \sum_j \eta^j \frac{\partial}{\partial \phi^j}.$$  \hspace{1cm} (2.2.15)

We calculate the components of prolonged operator

$$\chi_p = \chi + \sum_{i,j} \zeta_{ij} \frac{\partial}{\partial \phi^j} + \sum_{i,j,k} \zeta_{ijk} \frac{\partial}{\partial \phi^{jk}}$$  \hspace{1cm} (2.2.16)

with the formulae:

$$\zeta^i_j = D_j \eta^i - \sum_k \phi^i_k D_j \xi^k; \quad \zeta^i_{jk} = D_k \zeta^i_j - \sum_l \phi^i_{jl} D_k \xi^l;$$
\[ D_j = \frac{\partial}{\partial q^j} + \sum_k \phi^j_k \frac{\partial}{\partial \phi^k} + \sum_{i,k} \phi^k_{ij} \frac{\partial}{\partial \phi^j_i} \]  
(2.2.17)

(j = 1, 2, 3, 4.)

where

\[ q = (t, x, y, z), \quad \phi = (u, v, w, p), \quad \phi^k_i = \frac{\partial \phi^k}{\partial q^i}, \quad \phi^k_{ij} = \frac{\partial \phi^k_i}{\partial q^j} \]

(i, j, k = 1, 2, 3, 4.).

We are now able to reduce the Navier-Stokes equations to a system of ordinary differential equations in order to describe the solution.

The problem of finding solutions of the non-linear system (1.3.1)-(1.3.4) is an important, but rather complicated one. There are some ways to solve it (see Boisvert et al. [3] or Ovsiannikov [22]). This can be achieved by means of a symmetry approach since our system has non-trivial symmetry properties. It was found long ago (Ovsiannikov [22]), by means of Lie method, that the maximal Lie invariance algebra of the Navier-Stokes equations (1.3.1)-(1.3.4) is an infinite-dimensional algebra. So the full Lie group which leaves the Navier-Stokes equations invariant, can be established, and some of its different subgroups can be utilized to construct a number of exact solutions of the Navier-Stokes equations. This is done in the next chapter.

\section*{2.3 The Finite Element Method}

\subsection*{2.3.1 Basic Principles of the Finite Element Method}

The advent of modern and sophisticated digital computers has enabled applied scientists, mathematicians, physicists and engineers, to make significant progress in the solutions of previously intractable problems. Indeed, it is now possible to access the validity of previously unproven concepts related to complex problems. This trend is particularly
valid in fluid mechanics (transition to turbulent fluid flows), where there is an increasing need to test previously advocated fundamental concepts, and to develop new computer-based numerical techniques. In fact, it is now apparent that new concepts can be tested via numerical methods in general, and the finite element method in particular. One can therefore begin with an introductory definition (in the sense of David V. Hutton [7]) of the finite element method.

**Definition 2.3.1.** The finite element method, sometimes referred to as finite element analysis, is a computational technique used to obtain approximate solutions of boundary value problems in applied sciences (applied mathematics, engineering, etc.).

Gene Oliver [19] considers it as a computer-aided, mathematical technique for obtaining approximate numerical solutions of the complex equations of calculus, that predict the response of physical systems subjected to external influences.

Such problems arise in many areas of applied sciences in general, and in the area of fluid mechanics in particular, with abstract models like the one under investigation: The transition to turbulence.

Before seeing how we can apply the finite element method to the model of our study, let us outline the basic principle of the method.

The finite element method involves discretization of the domain of the solution into a finite number of sub-domains or cells: the finite elements. Adjacent elements touch without overlapping, and there are no gaps between the elements. The shapes of the elements are intentionally made as simple as possible, such as triangles and quadrilaterals in two-dimensional domains, and tetrahedra, pentahedra and hexahedra in three-dimensions. Now we use variational concepts (weak formulations of the problem), together with boundary conditions to construct an approximation of the solution over the collection of finite elements. In each element the sought solution is approximated by a characteristic form which often is a linear solution (function). The construction of an approximated solution depends on the domain, and is traditionally based on the Galerkin and Crank Nicolson methods. A Sobolev space is needed for the mathematical treatment of the
variational formulation of the model.

2.3.2 The Finite Element Method and Navier-Stokes Equations

The method of finite element is one of the main tools for the numerical treatment of complex partial differential equations. Because it is based on the variational formulation of the differential equation, it is much more flexible than other numerical methods (finite difference method, finite volume method, boundary element method etc...), and can therefore be applied to more complicated models like the transition to turbulent flow with its Navier-Stokes equations. Since no classical solution exists for the latter equations, we often have to work with a so-called weak solution. This has consequences for both the theory, and the numerical treatment. While it is true that classical solutions do exist under appropriate regularity hypotheses, for numerical calculations we usually cannot set up our analysis in a framework in which the existence of classical solutions is guaranteed.

One manner to obtain a suitable framework for treating our Navier-Stokes problem is to pose it as a variational one. The numerical treatment of the system (1.3.1)-(1.3.4), by the finite element method consists of computing the primitive variables $u$ (velocity), and $p$ (pressure), using a special Galerkin method based on a variational formulation. The spatial and time discretizations of the Navier-Stokes problem are constructed in an appropriate function space, and ”discrete” approximations will be determined in certain finite dimensional subspaces, consisting of piecewise polynomial functions.

Progress on the development in numerical solutions of the Navier-Stokes equations by the finite element method, has been successfully performed (Gunzburger [8]). So the global material for the numerical treatment of the system (1.3.1-1.3.4) can be presented as follows:

- **Presentation of the model** (which has already been done in the first chapter by the equations (1.3.1)-(1.3.4)) with appropriate initial and boundary conditions. It is worth mentioning that in some hypothesis of transition to turbulent flows (for example, flows
with Reynolds number in the range $2300 \leq Re \leq 10^5$), the numerical solution of this system involves typical difficulties like complicated flow structure (which implies fine meshes), dominant non-linear effects (which implies the stability), the constraint $(1.3.1)$, \( \nabla \cdot u = 0 \) (which implies implicit solution), etc...

- **Spatial discretization**, by finite elements Method, of the Navier-Stokes equations $(1.3.1)$, $(1.3.2)$, $(1.3.3)$ and $(1.3.4)$ which is a three-dimensional non-stationary flow problem.

- **Time discretization** since the model is non stationary.

There are many time discretization algorithms that fall into one of four classes of methods, see Max D. Gunzburger [8], namely single-step and multistep methods of both fully implicit and semi-implicit type. Multistep methods are more cumbersome to implement than single-step methods, but the former yield higher time accuracy. Fully implicit methods are likewise more cumbersome to implement than semi-implicit methods, but the former have better stability properties. Fully implicit methods are generally unconditionally stable. We will focus on the Crank-Nicolson scheme that lies in the single-step fully implicit method.

- **Solution of the algebraic system matrices** resulting from the above discretizations, and are exploited by the iterative solution method.

- **Estimation of the discretization error** in quantities of physical interest. The method is used to provide an approximate solution, with a margin of error. The use of a finite element Galerkin discretization provides the appropriate framework for a mathematically rigorous error analysis.

A concrete application of all these steps to the problem $(1.3.1)$- $(1.3.4)$ is presented in the fourth chapter.
Chapter 3

Lie Group Treatment

This chapter is on the treatment of the Navier-Stokes equations using the methods of Lie. We follow the approach by Robert Eugene Boisvert [2] and R.E. Boisvert et al. [3], in reducing the Navier-Stokes equations (1.3.1), (1.3.2), (1.3.3) and (1.3.4), to the steady state. We try to establish an equivalence group of transformations of the Navier-Stokes equations, in order to find a solution and express it explicitly.

3.1 Equivalence Group of Transformations

We consider the model (1.3.1)-(1.3.4) for viscous Newtonian incompressible flow established in the first chapter:

\[
\begin{align*}
\nu \nabla^2 u - p_x - (u_t + uu_x + vu_y + wu_z) &= 0 \\
\nu \nabla^2 v - p_y - (v_t + vv_x + vv_y + wv_z) &= 0 \\
\nu \nabla^2 w - p_z - (w_t + uw_x + vw_y + ww_z) &= 0 \\
\end{align*}
\]

subject to the incompressibility condition

\[u_x + v_y + w_z = 0\]
where we assume that all variables have been non-dimensionalized, so that the kinematic viscosity $\nu$ can be taken as the inverse of the Reynolds number $Re$: $(\nu = Re^{-1})$.

In order to find the Lie algebra, $L$, admitted by these equations, we apply the second extension, $\chi^2$, of the generator operator, $\chi$, of the form (2.2.15) to each equation; we therefore obtain the invariance conditions written as

$$\chi^2 [\nu \nabla^2 u - p_x - (u_t + uu_x + vu_y + wu_z)] = 0 \quad (3.1.5)$$

$$\chi^2 [\nu \nabla^2 v - p_y - (v_t + uv_x + vv_y + wv_z)] = 0 \quad (3.1.6)$$

$$\chi^2 [\nu \nabla^2 w - p_z - (w_t + uw_x + vw_y + ww_z)] = 0 \quad (3.1.7)$$

subject to the incompressibility condition invariance

$$\chi^2 [u_x + v_y + w_z] = 0 \quad (3.1.8)$$

whenever (3.1.1)-(3.1.4) are verified. In fact we look for operators (2.2.15) that take the form

$$\chi = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \eta^3 \frac{\partial}{\partial w} + \eta^4 \frac{\partial}{\partial p} \quad (3.1.9)$$

where we have considered the variables $t$, $x$, $y$, and $z$ as independent variables and $u$, $v$, $w$ and $p$ as differential variables on the space $(t, x, y, z)$. The coordinates $\xi^1$, $\xi^2$, $\xi^3$, $\xi^4$, $\eta^1$, $\eta^2$, $\eta^3$ and $\eta^4$ of the operator (3.1.9) are sought as functions of $t$, $x$, $y$, $z$, $u$, $v$, $w$ and $p$. Thus equation (2.2.16) implies that the first extension, $\chi^1$, and the second extension, $\chi^2$, of $\chi$ take the form:

$$\chi^1 = \chi + \xi^1 \frac{\partial}{\partial u_t} + \xi^2 \frac{\partial}{\partial u_x} + \xi^3 \frac{\partial}{\partial u_y} + \xi^4 \frac{\partial}{\partial u_z} + \eta^1 \frac{\partial}{\partial v_t} + \eta^2 \frac{\partial}{\partial v_x} + \eta^3 \frac{\partial}{\partial v_y} + \eta^4 \frac{\partial}{\partial v_z}$$

$$+ \xi^3 \frac{\partial}{\partial w_t} + \xi^4 \frac{\partial}{\partial w_x} + \xi^3 \frac{\partial}{\partial w_y} + \xi^4 \frac{\partial}{\partial w_z} + \xi^4 \frac{\partial}{\partial p_t} + \xi^4 \frac{\partial}{\partial p_x} + \xi^4 \frac{\partial}{\partial p_y} + \xi^4 \frac{\partial}{\partial p_z}$$

and

$$\chi^2 = \chi + \xi^1 \frac{\partial}{\partial u_t} + \xi^2 \frac{\partial}{\partial u_x} + \xi^3 \frac{\partial}{\partial u_y} + \xi^4 \frac{\partial}{\partial u_z} + \eta^1 \frac{\partial}{\partial v_t} + \eta^2 \frac{\partial}{\partial v_x} + \eta^3 \frac{\partial}{\partial v_y} + \eta^4 \frac{\partial}{\partial v_z}$$
with the formulae (2.2.17) explicitly given by:

\[
\begin{align*}
\zeta_1^3 & = D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2) - u_y D_t(\xi^3) - u_z D_t(\xi^4), \\
\zeta_2^3 & = D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3) - u_z D_x(\xi^4), \\
\zeta_3^3 & = D_y(\eta^1) - u_t D_y(\xi^1) - u_x D_y(\xi^2) - u_y D_y(\xi^3) - u_z D_y(\xi^4), \\
\zeta_4^3 & = D_z(\eta^1) - u_t D_z(\xi^1) - u_x D_z(\xi^2) - u_y D_z(\xi^3) - u_z D_z(\xi^4), \\
\zeta_1^2 & = D_t(\eta^2) - v_t D_t(\xi^1) - v_x D_t(\xi^2) - v_y D_t(\xi^3) - v_z D_t(\xi^4), \\
\zeta_2^2 & = D_x(\eta^2) - v_t D_x(\xi^1) - v_x D_x(\xi^2) - v_y D_x(\xi^3) - v_z D_x(\xi^4), \\
\zeta_3^2 & = D_y(\eta^2) - v_t D_y(\xi^1) - v_x D_y(\xi^2) - v_y D_y(\xi^3) - v_z D_y(\xi^4), \\
\zeta_4^2 & = D_z(\eta^2) - v_t D_z(\xi^1) - v_x D_z(\xi^2) - v_y D_z(\xi^3) - v_z D_z(\xi^4), \\
\zeta_1^1 & = D_t(\eta^3) - w_t D_t(\xi^1) - w_x D_t(\xi^2) - w_y D_t(\xi^3) - w_z D_t(\xi^4), \\
\zeta_2^1 & = D_x(\eta^3) - w_t D_x(\xi^1) - w_x D_x(\xi^2) - w_y D_x(\xi^3) - w_z D_x(\xi^4), \\
\zeta_3^1 & = D_y(\eta^3) - w_t D_y(\xi^1) - w_x D_y(\xi^2) - w_y D_y(\xi^3) - w_z D_y(\xi^4), \\
\zeta_4^1 & = D_z(\eta^3) - w_t D_z(\xi^1) - w_x D_z(\xi^2) - w_y D_z(\xi^3) - w_z D_z(\xi^4), \\
\end{align*}
\]
\(\frac{\partial}{\partial z}\left(\nu^3\right) - w_tD_z(\xi_1) - w_xD_z(\xi_2) - w_yD_z(\xi_3) - w_zD_z(\xi_4),\)
\(\frac{\partial}{\partial t}\left(\nu^4\right) - p_tD_t(\xi_1) - p_xD_t(\xi_2) - p_yD_t(\xi_3) - p_zD_t(\xi_4),\)
\(\frac{\partial}{\partial x}\left(\nu^4\right) - p_tD_x(\xi_1) - p_xD_x(\xi_2) - p_yD_x(\xi_3) - p_zD_x(\xi_4),\)
\(\frac{\partial}{\partial y}\left(\nu^4\right) - p_tD_y(\xi_1) - p_xD_y(\xi_2) - p_yD_y(\xi_3) - p_zD_y(\xi_4),\)
\(\frac{\partial}{\partial z}\left(\nu^4\right) - p_tD_z(\xi_1) - p_xD_z(\xi_2) - p_yD_z(\xi_3) - p_zD_z(\xi_4).\)

(3.1.11)
\[\zeta_3^2 = D_y(\zeta_4^3) - v_{zt}D_y(\xi^1) - v_{xx}D_y(\xi^2) - v_{zy}D_y(\xi^3) - v_{zz}D_y(\xi^4),\]
\[\zeta_4^2 = D_z(\zeta_4^3) - v_{zt}D_z(\xi^1) - v_{xx}D_z(\xi^2) - v_{zy}D_z(\xi^3) - v_{zz}D_z(\xi^4),\]
\[\zeta_3^1 = D_t(\zeta_1^3) - w_{tt}D_t(\xi^1) - w_{tx}D_t(\xi^2) - w_{ty}D_t(\xi^3) - w_{tz}D_t(\xi^4),\]
\[\zeta_4^1 = D_t(\zeta_1^3) - w_{zt}D_t(\xi^1) - w_{xx}D_t(\xi^2) - w_{xy}D_t(\xi^3) - w_{xz}D_t(\xi^4),\]
\[\zeta_{12}^3 = D_t(\zeta_2^3) - w_{xt}D_t(\xi^1) - w_{xx}D_t(\xi^2) - w_{xy}D_t(\xi^3) - w_{xz}D_t(\xi^4),\]
\[\zeta_{13}^3 = D_t(\zeta_3^3) - w_{yt}D_t(\xi^1) - w_{yx}D_t(\xi^2) - w_{yy}D_t(\xi^3) - w_{yz}D_t(\xi^4),\]
\[\zeta_{14}^3 = D_t(\zeta_4^3) - w_{zt}D_t(\xi^1) - w_{xx}D_t(\xi^2) - w_{xy}D_t(\xi^3) - w_{xz}D_t(\xi^4),\]
\[\zeta_{22}^3 = D_x(\zeta_2^3) - w_{xt}D_x(\xi^1) - w_{xx}D_x(\xi^2) - w_{xy}D_x(\xi^3) - w_{xz}D_x(\xi^4),\]
\[\zeta_{23}^3 = D_x(\zeta_3^3) - w_{yt}D_x(\xi^1) - w_{yx}D_x(\xi^2) - w_{yy}D_x(\xi^3) - w_{yz}D_x(\xi^4),\]
\[\zeta_{24}^3 = D_x(\zeta_4^3) - w_{zt}D_x(\xi^1) - w_{xx}D_x(\xi^2) - w_{xy}D_x(\xi^3) - w_{xz}D_x(\xi^4),\]
\[\zeta_{33}^3 = D_y(\zeta_3^3) - w_{yt}D_y(\xi^1) - w_{yx}D_y(\xi^2) - w_{yy}D_y(\xi^3) - w_{yz}D_y(\xi^4),\]
\[\zeta_{34}^3 = D_y(\zeta_4^3) - w_{zt}D_y(\xi^1) - w_{xx}D_y(\xi^2) - w_{xy}D_y(\xi^3) - w_{xz}D_y(\xi^4),\]
\[\zeta_{44}^3 = D_z(\zeta_4^3) - w_{zt}D_z(\xi^1) - w_{xx}D_z(\xi^2) - w_{xy}D_z(\xi^3) - w_{xz}D_z(\xi^4),\]
\[\zeta_{11}^4 = D_t(\zeta_1^4) - p_{tt}D_t(\xi^1) - p_{tx}D_t(\xi^2) - p_{ty}D_t(\xi^3) - p_{tz}D_t(\xi^4),\]
\[\zeta_{12}^4 = D_t(\zeta_2^4) - p_{xt}D_t(\xi^1) - p_{xx}D_t(\xi^2) - p_{xy}D_t(\xi^3) - p_{xz}D_t(\xi^4),\]
\[\zeta_{13}^4 = D_t(\zeta_3^4) - p_{yt}D_t(\xi^1) - p_{yx}D_t(\xi^2) - p_{yy}D_t(\xi^3) - p_{yz}D_t(\xi^4),\]
\[\zeta_{14}^4 = D_t(\zeta_4^4) - p_{zt}D_t(\xi^1) - p_{xx}D_t(\xi^2) - p_{xy}D_t(\xi^3) - p_{xz}D_t(\xi^4),\]
\[\zeta_{22}^4 = D_x(\zeta_2^4) - p_{xt}D_x(\xi^1) - p_{xx}D_x(\xi^2) - p_{xy}D_x(\xi^3) - p_{xz}D_x(\xi^4),\]
\[\zeta_{23}^4 = D_x(\zeta_3^4) - p_{yt}D_x(\xi^1) - p_{yx}D_x(\xi^2) - p_{yy}D_x(\xi^3) - p_{yz}D_x(\xi^4),\]
\[\zeta_{24}^4 = D_x(\zeta_4^4) - p_{zt}D_x(\xi^1) - p_{xx}D_x(\xi^2) - p_{xy}D_x(\xi^3) - p_{xz}D_x(\xi^4),\]
\[\zeta_{33}^4 = D_y(\zeta_3^4) - p_{yt}D_y(\xi^1) - p_{yx}D_y(\xi^2) - p_{yy}D_y(\xi^3) - p_{yz}D_y(\xi^4),\]
\[\zeta_{34}^4 = D_y(\zeta_4^4) - p_{zt}D_y(\xi^1) - p_{xx}D_y(\xi^2) - p_{xy}D_y(\xi^3) - p_{xz}D_y(\xi^4),\]
\[\zeta_{44}^4 = D_z(\zeta_4^4) - p_{zt}D_z(\xi^1) - p_{xx}D_z(\xi^2) - p_{xy}D_z(\xi^3) - p_{xz}D_z(\xi^4),\]

with the total derivatives given by following sums:

\[D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + u_{tz} \frac{\partial}{\partial u_z}\]
After extending the determining equations (3.1.5)-(3.1.8), we find all the generators admitted by the Navier-Stokes equations (3.1.1)-(3.1.4), see Birkhoff [1] or Wilzynski [33]. As shown in the transformation (2.2.5) together with the generator (3.1.9): \( \chi = \xi^1 \frac{\partial}{\partial t} + \)
\[ \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \eta^3 \frac{\partial}{\partial w} + \eta^4 \frac{\partial}{\partial p}, \]

we look for the group of transformations of the forms

\[
\begin{align*}
\bar{t} &= t + \varepsilon \xi^1(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{x} &= x + \varepsilon \xi^2(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{y} &= y + \varepsilon \xi^3(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{z} &= z + \varepsilon \xi^4(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{u} &= u + \varepsilon \eta^1(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{v} &= v + \varepsilon \eta^2(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{w} &= w + \varepsilon \eta^3(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{p} &= p + \varepsilon \eta^4(t, x, y, z, u, v, w, p) + O(\varepsilon^2),
\end{align*}
\]

(3.1.14)

which leave the Navier-Stokes equations (3.1.1)-(3.1.4) invariant. Boisvert [2] proved that this group (called the full group) is obtained by the transformations (3.1.14) with

\[
\begin{align*}
\xi^1 &= \alpha + 2\beta t \tag{3.1.15} \\
\xi^2 &= \beta x - \gamma y - \lambda z + f(t) \tag{3.1.16} \\
\xi^3 &= \beta y + \gamma x - \sigma z + g(t) \tag{3.1.17} \\
\xi^4 &= \beta z + \lambda x + \sigma y + h(t) \tag{3.1.18} \\
\eta^1 &= -\beta u - \gamma v - \lambda w + f'(t) \tag{3.1.19} \\
\eta^2 &= -\beta v + \gamma u - \sigma w + g'(t) \tag{3.1.20} \\
\eta^3 &= -\beta w + \lambda u + \sigma v + h'(t) \tag{3.1.21} \\
\eta^4 &= -2\beta p + j(t) - xf''(t) - yg''(t) - zh''(t) \tag{3.1.22}
\end{align*}
\]

where \(\alpha, \beta, \gamma, \lambda, \text{ and } \sigma\) are five arbitrary parameters and \(f(t), g(t), h(t), \text{ and } j(t)\) are arbitrary, sufficiently smooth, functions of \(t\). Each of the arbitrary parameters corresponds to the well known transformation. The parameter \(\alpha\) corresponds to a translation with respect to time, \(t\); \(\beta\) represents a stretching (dilatation) in all coordinates; \(\gamma, \lambda, \sigma\) represent a space rotation. With \(f(t), g(t)\) and \(h(t)\) as constants, it is clear that translations in the various coordinate directions are also included. Moving coordinate transformations
are also included as long as these changes are reflected in $\eta^1$, $\eta^2$, $\eta^3$, $\eta^4$, as shown in (3.1.19),(3.1.20),(3.1.21),(3.1.22).

From the generator (3.1.9): $\chi = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \eta^3 \frac{\partial}{\partial w} + \eta^4 \frac{\partial}{\partial p}$, we find the infinitesimal operator associated with each parameter by setting the studied parameter equal to one, while all other parameters and arbitrary functions are equal to zero. Then we obtain the following generators:

- translation with respect to time, $t$ (associated with $\alpha$)
  \[ \chi_1 = \frac{\partial}{\partial t}, \]  
  (3.1.23)

- scale (dilatation) transformation (associated with $\beta$)
  \[ \chi_2 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} - w \frac{\partial}{\partial w} - 2p \frac{\partial}{\partial p}, \]  
  (3.1.24)

- space rotations (associated with $\gamma$, $\lambda$, $\sigma$)
  \[ \chi_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}. \]  
  (3.1.25)

  \[ \chi_4 = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} + u \frac{\partial}{\partial w} - w \frac{\partial}{\partial u}, \]  
  (3.1.26)

  \[ \chi_5 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} + w \frac{\partial}{\partial v} - v \frac{\partial}{\partial w}, \]  
  (3.1.27)

- moving coordinates (associated with the arbitrary functions) and obtained

  infinitely in the forms:

  \[ \chi_6 = f(t) \frac{\partial}{\partial x} + f'(t) \frac{\partial}{\partial u} - z f''(t) \frac{\partial}{\partial p}, \]  
  (3.1.28)

  \[ \chi_7 = g(t) \frac{\partial}{\partial y} + g'(t) \frac{\partial}{\partial v} - y g''(t) \frac{\partial}{\partial p}, \]  
  (3.1.29)

  \[ \chi_8 = h(t) \frac{\partial}{\partial z} + h'(t) \frac{\partial}{\partial w} - z g''(t) \frac{\partial}{\partial p}, \]  
  (3.1.30)
pressure changes
\[ \chi_9 = j(t) \frac{\partial}{\partial p}. \] (3.1.31)

The operators (3.1.23)-(3.1.27) generate a finite-dimensional Lie algebra, called \( L_5 \), which is five-dimensional subalgebra of the infinite-dimensional algebra \( L_\infty \) generated by the operators (3.1.23)-(3.1.31).

### 3.2 Solutions of the Navier-Stokes equations

We are now able to find a solution of the three-dimensional Navier-Stokes equations (3.1.1)-(3.1.4) by utilizing a different subgroup of the full group (3.1.15)-(3.1.22) with, for simplicity, \( \beta = \gamma = \lambda = \sigma = 0 \) and \( \alpha = 1 \). This subgroup becomes

\[ \begin{align*}
\xi^1 &= 1; \quad \xi^2 = f(t); \quad \xi^3 = g(t); \quad \xi^4 = h(t) \\
\eta^1 &= f'(t); \quad \eta^2 = g'(t); \quad \eta^3 = h'(t); \quad \eta^4 = j(t) - xf''(t) - yg''(t) - zh''(t)
\end{align*} \]

and has the associated operator (3.1.9):

\[ \chi = \frac{\partial}{\partial t} + f(t) \frac{\partial}{\partial x} + g(t) \frac{\partial}{\partial y} + h(t) \frac{\partial}{\partial z} + f'(t) \frac{\partial}{\partial u} + g'(t) \frac{\partial}{\partial v} + h'(t) \frac{\partial}{\partial w} + \\
+ [j(t) - xf''(t) - yg''(t) - zh''(t)] \frac{\partial}{\partial p}. \]

Now we can utilize the useful result mentioned in Boisvert et al. [3] which states that: Any steady-state solution to the three-dimensional equations can be transformed by means of

\[ \begin{align*}
\tilde{x} &= x - F(t), \quad \tilde{y} = y - G(t), \quad \tilde{z} = z - H(t) \\
\end{align*} \] (3.2.1)

with

\[ \begin{align*}
u = \tilde{v}(\tilde{x}, \tilde{y}, \tilde{z}) + f(t), \quad w = \tilde{w}(\tilde{x}, \tilde{y}, \tilde{z}) + g(t), \quad w = \tilde{w}(\tilde{x}, \tilde{y}, \tilde{z}) + h(t) \\
p = \tilde{p}(\tilde{x}, \tilde{y}, \tilde{z}) - xf'(t) - yg'(t) - zh'(t) + k(t),
\end{align*} \] (3.2.2)
where \( F' = f, \ G' = g, \ H' = h, \ k = \frac{1}{2}[f^2 + g^2 + h^2] + \int j \, dt, \) into a time-dependent solution involving four arbitrary functions of time variable. Then, the transformations (3.2.1)-(3.2.2) yield:

\[
\begin{align*}
u_t &= \frac{\partial \tilde{u}}{\partial x} \frac{\partial \tilde{x}}{\partial t} + \frac{\partial \tilde{u}}{\partial y} \frac{\partial \tilde{y}}{\partial t} + \frac{\partial \tilde{u}}{\partial z} \frac{\partial \tilde{z}}{\partial t} + f'(t) \\
&= -\tilde{u}_x F' - \tilde{u}_y G' - \tilde{u}_z H' + f'(t) \\
&= -(\tilde{u}_x f + \tilde{u}_y g + \tilde{u}_z h) + f'(t),
\end{align*}
\]

\[
\begin{align*}
p_x &= \frac{\partial \tilde{p}}{\partial x} \frac{\partial \tilde{x}}{\partial x} - f'(t) \\
&= \tilde{p}_x - f'(t), \quad \left( \frac{\partial \tilde{x}}{\partial x} = 1 \right)
\end{align*}
\]

\[
\begin{align*}\nu_x &= \frac{\partial(\tilde{u} + f)}{\partial x} \\
&= \frac{\partial \tilde{u}}{\partial x} \frac{\partial \tilde{x}}{\partial x} \\
&= \tilde{u}_x, \quad \left( \frac{\partial \tilde{y} \text{ or } \tilde{z}}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial x} = 0 \right).
\end{align*}
\]

In the same manner, \( u_y = \tilde{u}_y, \ u_z = \tilde{u}_z, \) and

\[
\nabla^2 \nu = (\nu_{xx} + \nu_{xx} + \nu_{xx}) \\
= (\tilde{u}_{xx} + \tilde{u}_{yy} + \tilde{u}_{zz}).
\]

We do the same for the \( v- \) and \( w- \)components.

After substituting into the time-dependent Navier-Stokes equations (3.1.1)-(3.1.4), we find that the functions \( \tilde{u}, \ \tilde{v} \) and \( \tilde{w} \) satisfy the steady Navier-Stokes equations:

\[
\begin{align*}
\tilde{u} \tilde{u}_x + \tilde{v} \tilde{u}_y + \tilde{w} \tilde{u}_z &= -\tilde{p}_x + \nu[\tilde{u}_{xx} + \tilde{u}_{yy} + \tilde{u}_{zz}], \\
\tilde{u} \tilde{v}_x + \tilde{v} \tilde{v}_y + \tilde{w} \tilde{v}_z &= -\tilde{p}_y + \nu[\tilde{v}_{xx} + \tilde{v}_{yy} + \tilde{v}_{zz}], \\
\tilde{u} \tilde{w}_x + \tilde{v} \tilde{w}_y + \tilde{w} \tilde{w}_z &= -\tilde{p}_z + \nu[\tilde{w}_{xx} + \tilde{w}_{yy} + \tilde{w}_{zz}], \quad (3.2.3) \\
\tilde{u}_x + \tilde{v}_y + \tilde{w}_z &= 0.
\end{align*}
\]

Another interesting result of the transformation (mentioned in the same Boisvert et al. [3] ), is that different subgroups of the reduced (time-independent), full group may now
be used to study (3.2.3), and transform it into a system of ordinary differential equations. Consequently, from the full group (3.1.15)-(3.1.22), it follows that the dilatation subgroup generated by \( \beta \) (obtained with \( \beta = 1 \) and all other parameters and functions vanishing) will leave (3.2.3) invariant. This subgroup becomes

\[
\xi^1 = 2t; \quad \xi^2 = x; \quad \xi^3 = y; \quad \xi^4 = z
\]

\[
\eta^1 = -u; \quad \eta^2 = -v; \quad \eta^3 = -w; \quad \eta^4 = -2p
\]

and has the associated operator (3.1.9):

\[
\chi = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w} - 2p \frac{\partial}{\partial p}.
\]

We obtain the invariants, \( I \), of this subgroup, by integrating the associated \( \chi I = 0 \).

Then the characteristic equations (Boisvert et al. [3]) are given by

\[
\frac{d \tilde x}{\tilde x} = \frac{d \tilde y}{\tilde y} = \frac{d \tilde z}{\tilde z} = \frac{d \tilde u}{\tilde u} = \frac{d \tilde v}{\tilde v} = \frac{d \tilde w}{\tilde w} = \frac{d \tilde p}{\tilde p}
\]

leading to the invariants

\[
\eta_1 = \frac{\tilde y}{\tilde x}, \quad \eta_2 = \frac{\tilde z}{\tilde x}
\]

and

\[
\tilde u = \tilde x^{-1} \Gamma(\eta_1, \eta_2), \quad \tilde v = \tilde x^{-1} \Lambda(\eta_1, \eta_2)
\]

\[
\tilde w = \tilde x^{-1} \Phi(\eta_1, \eta_2), \quad \tilde p = \tilde x^{-2} \Omega(\eta_1, \eta_2)
\]

where \( \Gamma, \Lambda, \Phi, \Omega \) satisfy the partial differential equations:

\[
\begin{align*}
-\Gamma^2 &- \eta_1 \Gamma \eta_1 - \eta_2 \Gamma \eta_2 + \Lambda \eta_1 + \Phi \eta_2 - 2\Omega - \eta_1 \Omega \eta_1 - \eta_2 \Omega \eta_2 \\
-\nu(2\Gamma &+ 4\eta_1 \Gamma \eta_1 + 4\eta_2 \Gamma \eta_2 + \Gamma \eta_1 + \Gamma \eta_2 + \eta_1^2 \Gamma \eta_1 + 2\eta_1 \eta_2 \Gamma \eta_1 \eta_2 + \eta_2^2 \Gamma \eta_2) = 0,
\end{align*}
\]

\[
-\Gamma \Lambda - \eta_1 \Gamma \Lambda \eta_1 - \eta_2 \Gamma \Lambda \eta_2 + \Lambda \eta_1 + \Phi \Lambda \eta_2 + \Omega \eta_1 - \nu(2\Lambda &+ 4\eta_1 \Lambda \eta_1 \\
+ 4\eta_2 \Lambda \eta_2 + \Lambda \eta_1 \eta_1 + \Lambda \eta_2 \eta_2 + \eta_1^2 \Lambda \eta_1 \eta_1 + 2\eta_1 \eta_2 \Lambda \eta_1 \eta_2 + \eta_2^2 \Lambda \eta_2 \eta_2) = 0,
\]

(3.2.4)

\[
\begin{align*}
-\Gamma \Phi &- \eta_1 \Gamma \Phi \eta_1 - \eta_2 \Gamma \Phi \eta_2 + \Lambda \Phi \eta_1 + \Phi \Phi \eta_2 + \Omega \eta_2 - \nu(2\Phi &+ 4\eta_1 \Phi \eta_1 \\
+ 4\eta_2 \Phi \eta_2 + \Phi \eta_1 \eta_1 + \Phi \eta_2 \eta_2 + \eta_1^2 \Phi \eta_1 \eta_1 + 2\eta_1 \eta_2 \Phi \eta_1 \eta_2 + \eta_2^2 \Phi \eta_2 \eta_2) = 0,
\end{align*}
\]

\[
-\Gamma - \eta_1 \Gamma \eta_1 - \eta_2 \Gamma \eta_2 + \Lambda \eta_1 + \Phi \eta_2 = 0.
\]
However, no further group reduction is possible (Boisvert et al. [3]). But by setting $\eta = \eta_1 - \eta_2$, the system (3.2.4) is reduced to the system of ordinary differential equations

\[-\Gamma^2 - \eta \Gamma \Gamma_{\eta} + \Lambda \Gamma_{\eta} - \Phi \Gamma_{\eta} - 2\Omega - \eta \Gamma_{\eta} - \nu(2\Gamma + 4\eta \Gamma_{\eta} + 2\Gamma_{\eta\eta} + \eta^2 \Lambda_{\eta\eta}) = 0, \quad (3.2.5)\]

\[-\Gamma \Lambda - \eta \Gamma \Lambda_{\eta} + \Lambda \Lambda_{\eta} - \Phi \Lambda_{\eta} + \Omega_{\eta} - \nu(2\Lambda + 4\eta \Lambda_{\eta} + 2\Lambda_{\eta\eta} + \eta^2 \Lambda_{\eta\eta}) = 0, \quad (3.2.6)\]

\[-\Gamma \Phi - \eta \Gamma \Phi_{\eta} + \Lambda \Phi_{\eta} - \Phi \Phi_{\eta} - \Omega_{\eta} - \nu(2\Phi + 4\eta \Phi_{\eta} + 2\Phi_{\eta\eta} + \eta^2 \Phi_{\eta\eta}) = 0, \quad (3.2.7)\]

\[-\Gamma - \eta \Gamma_{\eta} + \Lambda_{\eta} - \Phi_{\eta} = 0. \quad (3.2.8)\]

The last of these is satisfied when

\[\Lambda - \Phi = \eta \Gamma - c_1 = 0, \quad (3.2.9)\]

where $c_1$ is an arbitrary constant. The substitution of (3.2.9) into (3.2.5) yields

\[-\Gamma^2 - c_1 \Gamma_{\eta} - 2\Omega - \eta \Omega_{\eta} - \nu(2\Gamma + 4\eta \Gamma_{\eta} + 2\Gamma_{\eta\eta} + \eta^2 \Lambda_{\eta\eta}) = 0 \quad (3.2.10)\]

The substitution of (3.2.9) into (3.2.6) and (3.2.7) and then subtracting yields

\[-\eta \Gamma^2 + 2\Omega_{\eta} - \nu(6\eta \Gamma + 4\Gamma_{\eta} + 6\eta^2 \Gamma_{\eta} + 2\eta \Gamma_{\eta\eta} + \eta^3 \Gamma_{\eta\eta}) = 0.\]

Solving the latter equation for $\Omega_{\eta}$ yields

\[\Omega_{\eta} = \frac{1}{2}[\eta \Gamma^2 + \nu(6\eta \Gamma + 4\Gamma_{\eta} + 6\eta^2 \Gamma_{\eta} + 2\eta \Gamma_{\eta\eta} + \eta^3 \Gamma_{\eta\eta})] \quad (3.2.11)\]

and replacing it into (3.2.10) gives

\[\Omega = \frac{1}{2}[-\Gamma^2 - \frac{1}{2} \eta^2 \Gamma^2 - c_1 \Gamma_{\eta} - \nu(2\Gamma + 3\eta^2 \Gamma + 6\eta \Gamma_{\eta} + 3\eta^3 \Gamma_{\eta\eta} + \frac{1}{2} \eta^4 \Gamma_{\eta\eta})]. \quad (3.2.12)\]

The differentiation of equation (3.2.12) with respect to $\eta$ and setting it equal to (3.2.11) implies that

\[2\eta \Gamma^2 + 2\Gamma \Gamma_{\eta} + \eta^2 \Gamma \Gamma_{\eta} + c_1 \Gamma_{\eta\eta} + \nu(12\eta \Gamma + 12\Gamma_{\eta} + 18\eta^2 \Gamma_{\eta}) + 12\eta \Gamma_{\eta\eta} + 6\eta^3 \Gamma_{\eta\eta} + 2\Gamma_{\eta\eta\eta} + 2\eta^2 \Gamma_{\eta\eta\eta} + \frac{1}{2} \eta^4 \Gamma_{\eta\eta\eta} = 0 \quad (3.2.13)\]

One solution of (3.2.13) is

\[\Gamma = -6\nu. \quad (3.2.14)\]
The corresponding values for $\Lambda$ and $(\Lambda - \Phi)$ from (3.2.12) and (3.2.9) are

$$\Lambda = -12\nu. \quad (3.2.15)$$

$$\Lambda - \Phi = -6\nu\eta - c_1. \quad (3.2.16)$$

Substitution of (3.2.14), (3.2.15), (3.2.16) into (3.2.7) results in

$$4\Phi - \left[\frac{c_1}{\nu} + 4\eta\right] \Phi_\eta - (\eta^2 + 2)\Phi_{\eta\eta} = 0,$$

whose general solution, for the case $c_1 = 0$, is

$$\Phi = c_2\nu\eta - c_3\nu \left[\frac{1}{4} + \frac{1}{8}\eta^2(\eta^2 + 2)^{-1} - \frac{3\eta}{8\sqrt{2}} \arctan\left(\frac{\eta}{\sqrt{2}}\right) \right]. \quad (3.2.17)$$

and the substitution into (3.2.16) yields

$$\Lambda = -6\nu\eta + c_2\nu\eta - c_3\nu \left[\frac{1}{4} + \frac{1}{8}\eta^2(\eta^2 + 2)^{-1} - \frac{3\eta}{8\sqrt{2}} \arctan\left(\frac{\eta}{\sqrt{2}}\right) \right]. \quad (3.2.18)$$

Using these last expressions together with the relations (3.2.1)-(3.2.2), we rewrite the sought solutions $u, v, w$ and $p$ in the original variables, which leads to the solution of the unsteady three-dimensional Navier-Stokes equations (3.1.1)-(3.1.4). We will also use $\eta$ as follows:

$$\eta = \eta_1 - \eta_2 = \frac{\bar{y}}{\bar{x}} - \frac{\bar{z}}{\bar{x}} = \bar{x}^{-1}(\bar{y} - \bar{z})$$

$$= (x - F(t))^{-1}(y - F(t) - z + F(t)) = (x - F(t))^{-1}R$$

with

$$R = y - F(t) - z + F(t).$$

Then,

$$u = \tilde{u} + f(t)$$

$$= \bar{x}^{-1}\Gamma + f(t)$$

$$= (x - F(t))^{-1}\Gamma + f(t)$$

$$u = -6\nu(x - F(t))^{-1} + f(t) \quad (3.2.19)$$

$$v = \tilde{v} + g(t)$$

$$= \bar{x}^{-1}\Lambda + g(t)$$

$$= (x - F(t))^{-1}\Lambda + g(t)$$
\[ u = -\frac{1}{Re} \frac{1}{(x - F(t))^{-1} + f(t)} \]
\[ v = \frac{1}{Re} \left\{ (c_2 - 6)(x - F(t))^{-2}R - c_3 \left[ \frac{1}{4}(x - F(t))^{-1} \right] \right. \\
+ \frac{1}{8}(x - F(t))^{-3}R^2 \left( (x - F(t))^{-2}R^2 + 2 \right)^{-1} - \frac{3}{8\sqrt{2}}(x - F(t))^{-1}R \\
\times \arctan \left[ \frac{(x - F(t))^{-1}R}{\sqrt{2}} \right] \left\} + g(t) \] (3.2.24)

\[ w = \frac{1}{Re} \left\{ c_2(x - F(t))^{-2}R - c_3 \left[ \frac{1}{4}(x - F(t))^{-1} \right] \right. \\
+ \frac{1}{8}(x - F(t))^{-3}R^2 \left( (x - F(t))^{-2}R^2 + 2 \right)^{-1} - \frac{3}{8\sqrt{2}}(x - F(t))^{-1}R \\
\times \arctan \left[ \frac{(x - F(t))^{-1}R}{\sqrt{2}} \right] \left\} + h(t) \] (3.2.25)

\[ p = -12 \left( \frac{1}{Re} \right)^2 (x - F(t))^{-2} - xf'(t) - yg'(t) - zh'(t) + k(t). \] (3.2.26)

All these solutions of the equations governing the flow are expressed in their explicit forms, and one can see that the Reynolds number, \( Re \), clearly appears. This proves the fact that the Reynolds number influences the three types of fluid flow’s regimes observed experimentally. As example, the time-dependent evolutions of the \( u \)- and \( v \)-velocity components are presented by Persillon and Braza [24], in both two and three-dimensional case, for Reynolds number 200 and 300. Their drawings (in the sixth part of the same article [24]) show the quasi-periodic character of the studied flow, and one can see that the amplitudes of the oscillations increase with the Reynolds number.
Chapter 4

Finite Element Treatment

In this chapter we discuss the treatment of the Navier-Stokes equations using the method of finite elements. We now consider the domain $\Omega$ defined in the first chapter and the established model for viscous Newtonian flow (1.2.6), subject to body forces in this case, and which is given by the Navier-Stokes equations

$$u_t + u \cdot \nabla u + \nabla p - \rho \nu \nabla^2 u = f \quad \text{in} \quad \Omega \times (0, T], \quad (4.0.1)$$

subject to the incompressibility condition,

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0, T], \quad (4.0.2)$$

the homogeneous no-slip boundary condition,

$$u = 0 \quad \text{on} \quad \Gamma_{\text{rigid}} \times (0, T], \quad (4.0.3)$$

the inflow condition

$$u^{\text{in}} = u \quad \text{on} \quad \Gamma_{\text{in}}, \quad (4.0.4)$$

and the initial condition,

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega, \quad (4.0.5)$$

where $f$ is the body force per unit mass (note that we will assume that $f = 0$). $\Gamma_{\text{rigid}}$ and $\Gamma_{\text{in}}$ are the rigid part and the inflow part of the boundary $\Gamma$, respectively. We assume that $\Omega$ does not change in time.
4.1 Function Spaces, Norms, and Forms

The finite element discretization of the Navier-Stokes problem is based on the variational formulation, and the use of Sobolev spaces is needed for the mathematical treatment of the variational formulation of the model. We use sub-spaces of the usual Hilbert space $L^2(\Omega) = \{ f : \int_{\Omega} |f|^2 \, dx < \infty \}$ of square-integrable functions on $\Omega$, where integration is in the sense of Lebesgue.

$\begin{align*}
L^2_0(\Omega) = \{ f : f \in L^2(\Omega), \ (f, 1) = 0 \},
\end{align*}$

and the corresponding inner products and norms

$$(f, g) = \int_{\Omega} fg \, dx, \quad \| f \|_0 = (f, f)^{1/2}.$$ 

Next, for any non-negative integer $k$, we define the Sobolev space

$H^k(\Omega) = \{ f : f \in L^2(\Omega), \ D^s f \in L^2(\Omega), \ for \ s = 1, \ldots, k \}$

of square integrable functions, all of whose derivatives of order up to $k$, are also square integrable, where $D^s$ denotes any and all derivatives of order $s$. $H^k(\Omega)$ comes with the norm

$$\| f \|_k = \left( \| f \|_0^2 + \sum_{s \leq k, \ s \neq 0} \| D^s f \|_0^2 \right)^{1/2}.$$

The following definitions can now be stated:

$H^0(\Omega) = L^2(\Omega)$

$H^1(\Omega) = \{ f : f \in L^2(\Omega), \ \partial_i f \in L^2(\Omega), \ 1 \leq i \leq 3 \}$

$$\| \nabla f \|_0 = (\nabla f, \nabla f)^{1/2}$$

$$\| f \|_1 = \left( \| f \|_0^2 + \| \nabla f \|_0^2 \right)^{1/2} = \left( \| f \|_0^2 + \sum_{i=1}^{3} \left\| \frac{\partial f}{\partial x_i} \|_0^2 \right) \right)^{1/2}. \quad (4.1.1)$$

Of particular interest is the subspace of $H^1_0(\Omega)$ of $H^1(\Omega)$ defined by

$H^1_0(\Omega) = \{ f : f \in H^1(\Omega), \ f = 0 \ on \ \Gamma \}$.
whose elements vanish on the boundary $\Gamma$.

For functions belonging to $H^1(\Omega)$, the semi-norm

$$|f|_1 = \left( \sum_{i=1}^{3} \left\| \frac{\partial f}{\partial x_i} \right\|_0^2 \right)^{1/2}$$

(4.1.2)
defines a norm equivalent to (4.1.1). The proof of this statement is not our aim in this dissertation, but it can be found in Dietrich Braess [4]. Thus for such functions, (4.1.2) may be used instead of (4.1.1).

We denote by $H^{-1}(\Omega)$ the dual space consisting of bounded linear functionals on $H^1_0(\Omega)$, i.e., $f \in H^{-1}(\Omega)$ implies that $(f, w) \in \mathbb{R}$ for all $w \in H^1_0(\Omega)$. A norm for $H^{-1}(\Omega)$ is given by

$$\|f\|_{-1} = \sup_{0 \neq w \in H^1_0(\Omega)} \frac{(f, w)}{|w|_1}$$

Since the velocity field $u = u(u, v, w) = (u_i)_{i=1,2,3}$ is a vector valued function, we use the spaces

$$H^k(\Omega) = H^k(\Omega)^3 = \left\{ u : u_i \in H^k(\Omega) \text{ for } i = 1, 2, 3 \right\},$$

$$H^1_0(\Omega) = H^1_0(\Omega)^3 = \left\{ u : u_i \in H^1_0(\Omega) \text{ for } i = 1, 2, 3 \right\},$$

and

$$H^{-1}(\Omega) = H^{-1}(\Omega)^3 = \left\{ u : u_i \in H^{-1}(\Omega) \text{ for } i = 1, 2, 3 \right\},$$

For $k \geq 0$, $H^k(\Omega)$ is equipped with the norm

$$\|u\|_k = \left( \sum_{i=1}^{3} \|u_i\|_k^2 \right)^{1/2}.$$ 

Alternatively, for functions belonging to $H^1_0(\Omega)$, we may use

$$|u|_1 = \left( \sum_{i=1}^{3} |u_i|_1^2 \right)^{1/2}.$$ 

The inner product for functions belonging to $L^2(\Omega) = H^0(\Omega) = L^2(\Omega)^3$ is also given by

$$(u, w) = \int_{\Omega} u \cdot w \, dx.$$
Before stating the weak variational formulation for our model, let us discuss some preliminaries concerning the existence and uniqueness of a solution of the Navier-Stokes equations.

### 4.2 Existence and Uniqueness for a Solution of Navier-Stokes Equations

From the mathematical point of view, two questions concerning the Navier-Stokes equations are of main interest. Given a set of data which are sufficiently smooth:

1. Does a solution of (4.0.1)-(4.0.5) exist?

2. If a solution exists, is it unique?

First, we have to clarify the notion of a solution of (4.0.1)-(4.0.5). There exists several concepts of the notion of a solution of the above system, the most important of which are the classical solution and the weak solution.

**Definition 4.2.1. (classical solution)**

A pair \((u, p)\) is called a classical solution of the Navier-Stokes (4.0.1)-(4.0.5) if:

1. \((u, p)\) satisfies the Navier-Stokes problem (4.0.1)-(4.0.5).

2. \(u\) and \(p\) are infinitely many times differentiable with respect to space and time variables.

Then, according to J. Volker and S. Kaya [29], the existence of a classical solution of (4.0.1) – (4.0.5) cannot yet be proven, but if a classical solution exists, it is unique.

To define a weak solution, we first need to transform (4.0.1) into a weak form by

- multiplying (4.0.1) with a suitable vector valued function \(\varphi\) (test function),
- integrating over \(\Omega \times (0, T]\),
applying integration by parts (Green's theorem).
The last step is possible only if there are some restrictions on the domain. For the test
function $\varphi$, one requires
\begin{itemize}
  \item $\varphi \in C_{0, \text{div}}^\infty(\Omega)$ for each time $t$, where $C_{0, \text{div}}^\infty(\Omega) = \{ f : f \in C_0^\infty(\Omega), \nabla \cdot f = 0 \}$,
  \item $\varphi$ is infinitely differentiable with respect to time,
  \item $\varphi(., T) = 0$.
\end{itemize}
This gives the weak formulation of the Navier-Stokes equations
\begin{equation}
\int_0^T \left[ - (u, \varphi_t) + (u \cdot \nabla u, \varphi) + \text{Re}^{-1}(\nabla u, \nabla \varphi) \right] dt = \int_0^T (f, \varphi) dt + (u_0, \varphi(., 0)). \tag{4.2.1}
\end{equation}
which has the following features:
\begin{itemize}
  \item There is no time derivative of $u$
  \item There is no second order spatial derivative with respect to $u$
  \item The pressure vanishes, since the Green's formula yields
    \begin{equation}
    (\nabla p, \varphi) = \int_{\partial \Omega} p \varphi \cdot n \, ds - (p, \nabla \cdot \varphi) = 0
    \end{equation}
    because $\varphi \cdot n = 0$ on $\partial \Omega$ and $\nabla \cdot \varphi = 0$.
\end{itemize}

**Definition 4.2.2. (weak solution)**
A function $u$ is called weak solution of the Navier-Stokes equations if:
\begin{itemize}
  \item $u$ satisfies (4.2.1) for all test functions $\varphi$ with the properties on $\varphi$ given above,
  \item $u$ has the following regularity
    \begin{equation}
    u \in L^2(0, T; H^1_{0, \text{div}}(\Omega)) \cap L^\infty(0, T; L^2_{\text{div}}(\Omega)),
    \end{equation}
\end{itemize}
where the subscript $\text{div}$ means space of divergence-free functions; for instance
\begin{equation}
C_{0, \text{div}}^\infty(\Omega) = \{ f : f \in C_0^\infty(\Omega), \nabla \cdot f = 0 \}
\end{equation}
and
\begin{equation}
L^2(0, T; H^1_0(\Omega)) = \left\{ f(x, t) : \int_0^T \| f \|_{H^1_0}^2 dt < \infty \right\}.
\end{equation}
More generally
\begin{equation}
L^q(t_0, t_1; X) = \left\{ f(x, t) : \int_{t_0}^{t_1} \| f \|_X^q dt < \infty \right\} \text{ for any } q \in [1, \infty),
\end{equation}
is the space of strongly measurable maps $f : [t_0, t_1] \to X$, such that

$$\|f\|_{L^q(t_0, t_1; X)} = \left( \int_{t_0}^{t_1} \|f\|^q_X \, dt \right)^{1/q} < \infty \quad \text{for } q \in [1, \infty)$$

and $X$ is a Banach space. Furthermore

$$L^\infty(t_0, t_1; X) = \left\{ f(x, t) : \text{ess sup}_{t_0 \leq t \leq t_1} \|f\|_X < \infty \right\}$$

with

$$\|f\|_{L^\infty(t_0, t_1; X)} = \text{ess sup}_{t_0 \leq t \leq t_1} \|f\|_X < \infty \quad \text{for } q = \infty.$$ 

It is obvious that all these spaces are needed for the weak formulation given in the next section.

The existence of a weak solution of (4.0.1)-(4.0.5) was proved in 1934 by Jean Leray [13]. The weak solution is unique if every other weak solution satisfies an additional regularity assumption, Serrin’s condition, see J. Serrin [28], or J. Volker and S. Kaya [29] . But it is not known in 3-D if every weak solution possesses such additional condition.

According to the same article [29], the existence of a weak solution of the Navier-Stokes equations can be proven in arbitrary domains, but the uniqueness cannot yet be proven.

The answer to the question of uniqueness of the weak solution in 3-D, or existence of a classical solution in 3-D is one of the major mathematical challenges of this century (J. Volker and S. Kaya [29]). There is a prize of one million US-Dollars for people who can answer these questions.

### 4.3 A Galerkin-Type Weak Formulation

We introduce the bilinear forms

$$a(u, w) = Re^{-1}(\nabla u, \nabla w) = Re^{-1} \int_\Omega \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \quad \text{for all } u, w \in H^1(\Omega) \quad (4.3.1)$$

$$b(p, u) = -(p, \nabla \cdot u) \quad \text{for all } u \in H^1(\Omega) \text{ and } p \in L^2(\Omega) \quad (4.3.2)$$
and the trilinear form

\[ c(u, v, w) = (u \cdot \nabla v, w) = \int_{\Omega} \sum_{i,j=1}^{3} u_j \frac{\partial v_i}{\partial x_j} w_i, \quad \text{for all } u, v, w \in H^1(\Omega). \] (4.3.3)

In addition to the above spaces, we will need to use the space

\[ H = \{ u \in L^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega; \ u = 0 \text{ on } \Gamma \}, \]

which consists of (weakly) divergence free functions, i.e. functions whose divergence vanishes almost everywhere.

Recall that \( \partial \Omega = \Gamma = \Gamma_{\text{rigid}} \cup \Gamma_{\text{in}}, \) then, following the same procedure mentioned earlier of defining the weak solution, the weak (variational) formulation of the Navier-Stokes equations (4.0.1)-(4.0.5), reads as follows:

Given

\[ f \in L^2(0,T; H^{-1}(\Omega)) \text{ and } u_0 \in H, \]

find functions \( u \in L^2(0,T; H^1_0(\Omega)) \cup L^\infty(0,T; H) \) and \( p \in L^2[0,T; L^2_0(\Omega)] \) such that

\[
\begin{align*}
\left( \frac{\partial u}{\partial t}, v \right) + a(u, v) + c(u, u, v) + b(p, v) &= (f, v) \quad \text{for all } v \in H^1_0(\Omega) \\
b(q, u) &= 0 \quad \text{for all } q \in L^2_0(\Omega) \\
u(0, x) &= u_0(x) \quad \text{for } x \in \Omega
\end{align*}
\] (4.3.4)

where the first two equations of (4.3.4) hold on \((0, T),\) in the sense of distributions.

### 4.4 Spatial Discretizations

To discretize the above problem with the spatial variables, we introduce the triangulation, named \( T_h, \) of \( \overline{\Omega}, \) with width \( h \) into (closed) cells \( K \) (tetrahedra) such that the following regularity conditions are satisfied:

- \( \overline{\Omega} = \bigcup \{ K \in T_h \}. \)
- Any two cells \( K, K' \) only intersect in common faces, edges or vertices.
• The decomposition $\mathbb{T}_h$ matches the decomposition $\partial \Omega = \Gamma = \Gamma_{\text{rigid}} \cup \Gamma_{\text{in}}$.

On the finite element mesh $\mathbb{T}_h$, one defines spaces of "discrete" trial and test functions with the following constructions:

For each $h$, let $W^h$ and $Q^h$ be two finite-dimensional spaces such that

$$W^h \subset H^1(\Omega), \quad Q^h \subset L^2(\Omega)$$

and throughout this chapter we assume that $Q^h$ contains the constant functions.

We set

$$V^h_0 = W^h \cap H^1_0(\Omega) = \{ v^h \in W^h : v^h = 0 \text{ on } \Gamma \}$$

$$S^h_0 = Q^h \cap L^2_0(\Omega) = \{ q^h \in Q^h : \int_{\Omega} q^h dx = 0 \}$$

(4.4.1)

There are many pairs of these finite element spaces. Some of them are stable, others are not. Naturally, one would like to know which are best. It is generally thought that elements which at least yield elementwise mass conservation, are superior. This judgement is largely based on the examination of graphical representations of solutions, e.g. streamline plots. For details on the choice of pairs of finite element spaces, consult Gunzburger [8].

With these spaces, the finite element approximation of the problem (4.3.4) is given by:

Find a pair $(u^h, p^h) \in V^h_0 \times S^h_0$ such that

$$\left( \frac{\partial u^h}{\partial t}, v^h \right) + a(u^h, v^h) + c(u^h, u^h, v^h) + b(p^h, v^h) = (f, v^h) \quad \text{for all } v^h \in V^h_0 \text{ and } t \in (0, T)$$

(4.4.2)

$$b(q^h, u^h) = 0 \quad \text{for all } q^h \in S^h_0 \text{ and } t \in (0, T)$$

(4.4.3)

$$u^h(0, x) = u^h_0 \in V^h_0 \quad \text{for } x \in \Omega,$$

(4.4.4)

where $u^h_0$ is an approximation to the initial function $u^h(0, x)$.

In order that (4.4.2)-(4.4.4) is a stable approximation of (4.3.4) as $h \to 0$, it is crucial that we relate the continuous and discrete spaces by the following hypotheses (for a complete and rigorous analysis of these approximations, refer to Girault and Raviart [6]):

**Hypothesis H1** (Approximation property of $V^h_0$)

There exists an operator $r^h \in L([H^2(\Omega) \cap H^1_0(\Omega)]^2; V^h_0)$ and an integer $l$ such that

$$\| \varphi - r^h \varphi \|_1 \leq Ch^m \| \varphi \|_{m+1} \quad \text{for all } \varphi \in H^{m+1}(\Omega), \ 1 \leq m \leq l.$$  

(4.4.5)
**Hypothesis H2** (Approximation property of \( Q^h \))

There exists an operator \( s^h \in L(L^2(\Omega); Q^h) \) such that

\[
\| q - s^h q \|_0 \leq C h^m \| q \|_m \quad \text{for all} \quad q \in H^m(\Omega), \quad 0 \leq m \leq l.
\]  

(4.4.6)

**Hypothesis H3** (Uniform inf-sup condition)

For each \( q^h \in S^h_0 \), there exists a \( v^h \in V^h_0 \) such that

\[
\begin{align*}
    b(q^h, v^h) &= \| q^h \|^2_0 \\
    |v^h|_1 &\leq C \| q^h \|_0
\end{align*}
\]

(4.4.7)

where the constant \( C > 0 \) is independent of \( h, q^h \) and \( v^h \); \( L(Y, W) \) is the space of linear operators from \( Y \) to \( W \); \( \| \cdot \|_0 \) and \( \| \cdot \|_m \) are the standard norms in \( L^2(\Omega) \) and \( H^m(\Omega) \) respectively; \( | \cdot |_1 \) is the standard semi-norm in \( H^1(\Omega) \).

Now we may choose specific bases for \( V^h_0 \) and \( S^h_0 \) which are both finite-dimensional in such a way that the system (4.4.2)-(4.4.4) becomes equivalent to a system of non-linear ordinary differential equations with linear algebraic constraints. Indeed if \( \{ q_j(\mathbf{x}) \}_{j=1}^J \) and \( \{ v_k(\mathbf{x}) \}_{k=1}^K \) denote bases for \( S^h_0 \) and \( V^h_0 \), respectively, we can then write

\[
p^h(t, \mathbf{x}) = \sum_{j=1}^J \alpha_j(t) q_j(\mathbf{x}) \quad \text{and} \quad u^h(t, \mathbf{x}) = \sum_{k=1}^K \beta_k(t) v_k(\mathbf{x}).
\]

The system (4.4.2)-(4.4.4) is therefore equivalent to the system of ordinary differential equations

\[
\begin{align*}
    \sum_{k=1}^K (v_k, v_l) \frac{d\beta_k}{dt} + \sum_{k=1}^K a(v_k, v_l) \beta_k(t) + \sum_{k,m=1}^K c(v_m, v_k, v_l) \beta_k(t) \beta_m(t) \\
    + \sum_{j=1}^J b(v_l, q_j) \alpha_j(t) &= (f, v_l) \quad \text{for} \quad l = 1, \ldots, K,
\end{align*}
\]

(4.4.8)

with initial data \( \beta_k(0), \quad k = 1, \ldots, K \) satisfying

\[
\sum_{k=1}^K v_k \beta_k(0) = u^h_0
\]

(4.4.9)

and are subject to the linear algebraic constraints

\[
\sum_{k=1}^K b(v_k, q_i) \beta_k(t) = 0 \quad \text{for} \quad i = 1, \ldots, J.
\]

(4.4.10)
The system of ordinary differential equations (4.4.8), or equivalently (4.4.2)-(4.4.4), may now be discretized with respect to time. In this regard, it is convenient to rewrite the semi-discrete system (4.4.8) as

$$\left( \frac{\partial u^h}{\partial t}, v^h \right) = F(f, u^h, p^h; v^h) \quad \text{for all } v^h \in V_0^h$$

(4.4.11)

where the linear functional $F(., ., .; v^h)$ is defined, for any $u^h \in V_0^h$ and $p^h \in S_0^h$ and any $f$, by

$$F(f, u^h, p^h; v^h) = (f, v^h) - a(u^h, v^h) - c(u^h, u^h, v^h) - b(p^h, v^h) \quad \text{for all } v^h \in V_0^h.$$  

(4.4.12)

### 4.5 Time Discretizations

In chapter 2 we saw that there are many time discretization algorithms that fall into one of four classes of methods, (see Max D. Gunzburger [8]), namely single-step and multistep methods of both fully implicit and semi-implicit type. We will focus on the Crank-Nicolson scheme which is a single-step fully implicit method. Explicit methods are not in common use for time discretizations of the Navier-Stokes equations, because of their severe stability restriction. We apply the Crank-Nicolson extrapolation scheme to the time discretization of the system (4.4.8).

We subdivide the time interval $[0, T]$ into $M$ intervals of uniform length $\delta$ so that $\delta = \frac{T}{M}$. Throughout, $u^m$ and $p^m$, $m = 0, ..., M$, will respectively denote approximations to $u^h(m\delta, \mathbf{x})$ and $p^h(m\delta, \mathbf{x})$ where $u^h$ and $p^h$ denote the solution of (4.4.2)-(4.4.4). Likewise, for $m = 0, ..., M$, $k = 1, ..., K$, and $j = 1, ..., J$, $\alpha^m_j$ and $\beta^m_k$ denote approximations to $\alpha_j(m\delta)$ and $\beta_k(m\delta)$, respectively, where $\alpha_j$, $j = 1, ..., J$, and $\beta_k$, $k = 1, ..., K$, denote the solution of (4.4.8) and (4.4.10). Also, throughout, $f^m = f(m\delta, \mathbf{x})$.

Given $u^0$ (which may be chosen to be $u_0$), $\{u^m, p^m\}$ for $m = 0, ..., M$, are determined from

$$\frac{1}{\delta} (u^m - u^{m-1}, v^h) = F(f^m_\delta, u^m_\delta, p^m_\delta; v^h) \quad \text{for all } v^h \in V_0^h$$

(4.5.1)
and

\[ b(q^h, u^m) = 0 \quad \text{for all } q^h \in S^h_0 \]  

(4.5.2)

where

\[ u^m_\theta = \frac{u^m - u^{m-1}}{2} \quad \text{and} \quad p^m_\theta = \frac{p^m - p^{m-1}}{2} \]  

(4.5.3)

and likewise for \( f^m_\theta \).

The scheme (4.5.1)-(4.5.2) requires an initial condition for the pressure since \( p^1 \) is obtained from \( p^0_\theta \) and \( p^0 \) through (4.5.3). Due to the fact that we are interested in the velocity field only, there is no need to compute \( p^1 \), since the latter will not be used in the velocity computation at the next time level, i.e., \( t = 2\delta \). However, one may use Taylor’s theorem \( p^1 = p^0_\theta + O(\delta/2) + O(\delta^2) \) and therefore take \( p^1_\theta \) as the pressure approximation at \( t = \delta \), and again never need \( p^0 \). This procedure results in a loss of accuracy in the pressure.

Clearly, (4.5.1)-(4.5.2) is a system of non-linear algebraic equations. In order to minimize the cost of computing each pair \( (u^m, p^m) \), one should solve the equivalent problem

\[ \frac{2}{\delta}(u^m_\theta, v^h) - F(f^m_\theta, u^m_\theta, p^m_\theta; v^h) = \frac{2}{\delta}(u^{m-1}, v^h) \quad \text{for all } v^h \in V^h_0 \]  

(4.5.4)

and

\[ b(q^h, u^m_\theta) = \begin{cases} 
(1/2) b(q^h, u^0) & \text{if } m = 1 \\
0 & \text{if } m > 1
\end{cases} \quad \text{for all } q^h \in S^h_0 \]  

(4.5.5)

and then set

\[ u^m = 2u^m_\theta - u^{m-1} \quad \text{and} \quad p^m = 2p^m_\theta - p^{m-1}. \]  

(4.5.6)

Substituting (4.4.12) into (4.5.4) yields

\[ a(u^m_\theta, v^h) + c(u^m_\theta, u^m_\theta, v^h) + b(p^m_\theta, v^h) + \frac{2}{\delta}(u^m_\theta, v^h) = \]  

\[ (f^m_\theta, v^h) + \frac{2}{\delta}(u^{m-1}, v^h) \quad \text{for all } v^h \in V^h_0. \]  

(4.5.7)

When \( \delta \) is small, a good starting guess for any iterative method for solving (4.5.5) and (4.5.7) is the solution \( u^{m-1} \) at the previous time step. On the other hand, due to the
fully implicit character of the scheme (4.5.1)-(4.5.3), a different non-linear system has to be solved for each \( m \).

As has been noted, the above scheme is second-order accurate with respect to \( \delta \), i.e., \( O(\delta^2) \). This observation is with respect to both the \( L^2(\Omega) \) and \( H^1_0(\Omega) \) - norms of the differences \( u(m\delta,.) - u^m \), \( m = 1, ..., M \).

Keeping in mind the above scheme of the finite element method applied to the non stationary Navier-stokes equations, we may investigate the stability of the model being studied, in order to understand how a laminar flow may develop into a turbulent flow. Most of the traditional theory for fluid flow is of qualitative nature, based on eigenvalue criteria through a hydrodynamic stability argument. We investigate the case associated with energy stability analysis.

### 4.6 Energy Stability Analysis

In the global theory, energy methods have an important place. These methods lead to a variational problem for the first critical Reynolds number (or viscosity) of the energy theory, and to a definite criterion which is sufficient for the global stability of the (basic) flow.

Bear in mind that the procedure which follows, has already been mentioned in the introduction of this dissertation. We consider the flow \( u \) in the domain \( \Omega \), which is mathematically represented by the non stationary Navier-stokes equations (4.0.1)-(4.0.5). For simplicity, we denote \( u(x,0) = u_0(x) \) by \( U(x) \) and thus at \( t = 0 \), the flow \( u \) has the velocity field \( u(x,0) = U(x) \). Suppose that at this instant, we perturb the flow with a perturbation \( w(x,0) \). The subsequent departure of the perturbed flow from the given flow is denoted by \( w(x,t) \), so that the perturbed flow is henceforth given by \( u(x,t) + w(x,t) \), where \( u \) denotes the subsequent unperturbed flow. Nothing is assumed concerning the size of the initial disturbance \( w(x,0) \) relative to the size of the given flow \( U(x) \). We assume that both unperturbed flow \( u \) and the disturbed flow \( u + w \).
satisfy the unsteady Navier-Stokes equations (4.0.1)-(4.0.5), and have the same, possibly homogeneous, values at the boundary $\Gamma$. Thus we have

$$\nabla \cdot w = 0, \quad \text{in } \Omega \times (0, T) \quad \text{and} \quad w = 0 \quad \text{on } \Gamma_{\text{rigid}} \times (0, T)$$

However, due to the non-linear convection term, $w$ does not satisfy the unsteady Navier-Stokes equation (4.0.1).

To assign a definite meaning to the word "(un)stable", the average energy of the disturbance

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} w \cdot w \, dx$$

is introduced, where we assume again that all variables have been non-dimensionalized, so that the kinematic viscosity $\nu$ can be taken as the inverse of the Reynolds number $Re$.

**Definition 4.6.1.** We say that the given flow $u$ is stable in the energy sense, (see Joseph [10] or Gunzburger [8]), if

$$\mathcal{E}(t) \to 0 \quad \text{as} \quad t \to \infty.$$ 

On the basis that both unperturbed flow $u$ and the disturbed flow $u+w$ satisfy the unsteady Navier-Stokes equations (4.0.1)-(4.0.5) and $\nabla \cdot w = 0$, in $\Omega \times (0, T)$ and $w = 0$, on $\Gamma_{\text{rigid}} \times (0, T)$, we are led to

$$\frac{d\mathcal{E}(t)}{dt} = \frac{1}{2} \frac{d}{dt} \int_{\Omega} w \cdot w \, dx$$

$$= \int_{\Omega} w \cdot \frac{d}{dt} w \, dx$$

$$= \int_{\Omega} w \cdot \left( \frac{\partial w}{\partial t} + U \cdot \nabla w \right) \, dx, \quad \text{since} \quad \nabla w = 0$$

$$= \int_{\Omega} w \cdot \left( -w \cdot \nabla U - w \cdot \nabla w + Re^{-1} \nabla^2 w \right) \, dx, \quad \text{since} \quad u \quad \text{and} \quad u + w \quad \text{satisfy} \quad (4.0.1)$$

$$= \int_{\Omega} w \cdot \left( -\nabla U \cdot w + Re^{-1} \nabla^2 w \right) \, dx, \quad \text{since} \quad \nabla w = 0$$

$$= - \int_{\Omega} w \cdot \nabla U \cdot w \, dx + Re^{-1} \int_{\Omega} w \cdot \nabla^2 w \, dx$$

$$= - \int_{\Omega} w \cdot \nabla U \cdot w \, dx - Re^{-1} \int_{\Omega} \nabla w \cdot \nabla w \, dx,$$
where we have applied the divergence theorem to the second integral and the fact that \( \mathbf{w} = 0 \) on \( \Gamma_{\text{rigid}} \times (0, T) \). Then,

\[
\frac{dE(t)}{dt} = -\int_{\Omega} \left[ \mathbf{w} \cdot D(\mathbf{U}) \cdot \mathbf{w} + Re^{-1} \nabla \mathbf{w} \cdot \nabla \mathbf{w} \right] d\mathbf{x}
\]  

(4.6.1)

with \( \nabla \mathbf{U} = (\partial_i u_j)_{ij} = (D_{ij}[\mathbf{U}])_{ij} = D(\mathbf{U}) \), where

\[
D(\mathbf{U}) = \frac{1}{2} (\nabla \mathbf{U} + (\nabla \mathbf{U})^T)
\]

is the rate of strain or the rate of deformation tensor of the given flow \( \mathbf{U} \), and \( (\nabla \mathbf{U})^T \), the transpose of \( \nabla \mathbf{U} \).

In equation (4.6.1), the term \( \int_{\Omega} \nabla \mathbf{w} \cdot \nabla \mathbf{w} d\mathbf{x} \) truly represents the average dissipation, and the term \( \int_{\Omega} \mathbf{w} \cdot D(\mathbf{U}) \cdot \mathbf{w} d\mathbf{x} \) represents the production integral which couples the given flow \( \mathbf{U} \) (with stretching tensor \( D(\mathbf{U}) \)) to the disturbance \( \mathbf{w} \).

If the right-hand side of (4.6.1) is negative (i.e., the derivative of \( E \) is less than zero), then \( E \) will decrease as \( t \) increases, characterizing the stability of the flow according to the previous definition. Now let

\[
\frac{1}{Re} = \bar{\nu} = \max_{\mathbf{v}} \left( -\frac{\int_{\Omega} \mathbf{v} \cdot D(\mathbf{U}) \cdot \mathbf{v} d\mathbf{x}}{\int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{v} d\mathbf{x}} \right),
\]  

(4.6.2)

where the maximum is sought over all the vector fields \( \mathbf{v} \) satisfying \( \nabla \cdot \mathbf{v} = 0 \) in \( \Omega \), and \( \mathbf{v} = 0 \) on \( \Gamma_{\text{rigid}} \). The allowed perturbation \( \mathbf{w} \) satisfies these two constraints so that (4.6.2) implies that

\[
-\frac{\int_{\Omega} \mathbf{w} \cdot D(\mathbf{U}) \cdot \mathbf{w} d\mathbf{x}}{\int_{\Omega} \nabla \mathbf{w} \cdot \nabla \mathbf{w} d\mathbf{x}} \leq \bar{\nu}
\]

or

\[
-\int_{\Omega} \mathbf{w} \cdot D(\mathbf{U}) \cdot \mathbf{w} d\mathbf{x} \leq \bar{\nu} \int_{\Omega} \nabla \mathbf{w} \cdot \nabla \mathbf{w} d\mathbf{x}
\]

combining with (4.6.1) yields

\[
\frac{dE(t)}{dt} \leq -(\nu - \bar{\nu}) \int_{\Omega} \nabla \mathbf{w} \cdot \nabla \mathbf{w} d\mathbf{x}
\]  

(4.6.3)

or

\[
\frac{dE(t)}{dt} \leq - \left( \frac{1}{Re} - \frac{1}{\bar{Re}} \right) \int_{\Omega} \nabla \mathbf{w} \cdot \nabla \mathbf{w} d\mathbf{x}
\]  

(4.6.4)
so that if the solution $\frac{1}{Re} = \tilde{\nu}$ of the maximization problem (4.6.2) satisfies $\tilde{\nu} < \nu$ (or equivalent to $Re < \tilde{Re}$), then $\frac{dE(t)}{dt} < 0$ and the flow is stable.

From equation (4.6.4), we can say that there is a critical value of the Reynolds number for which the transition from a stable state to an unstable state occurs. This is in concordance with the energy stability theorems developed in the book by D. D. Joseph [10]. We are led to the problem of energy stability limit. This limit is defined by

$$\frac{1}{\tilde{Re}} = \max_\nu \left( -\frac{\int_\Omega \mathbf{v} \cdot D(U) \cdot \mathbf{v} \, d\mathbf{x}}{\int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{v} \, d\mathbf{x}} \right), \quad (4.6.5)$$

where $\tilde{Re}$ is seen as the critical value.

One may actually understand a mathematical explanation of the fact that laminar flows break down, their subsequent development, and trigger their eventual transition to turbulence as the Reynolds number becomes large. This theoretical result is in good agreement with experiments (O. Reynolds [26], W. Orr [21]) concerning the critical Reynolds number at which the first bifurcation occurs. This bifurcation triggers the beginning of the instability of the flow.

4.6.1 The Energy Eigenvalue Problem

In this, we want to convert (4.6.5) into an eigenvalue problem, to characterize the set of eigenvalues with respect to completeness, and to show that $\tilde{Re}$ defined by (4.6.5) can also be found as the principal eigenvalue of a differential equation. For simplicity we use the notation

$$\langle \cdot \rangle = \int_\Omega (\cdot) d\mathbf{x}.$$

The following fundamental lemma of the calculus of variations, for vector fields (see D. D. Joseph [10]) proves to be usefull:

**Lemma 4.6.1.** If a fixed function $F(\mathbf{x}) \in C^1(\mathbf{x})$ and if $\langle F \cdot \phi \rangle = 0$ for all vectors fields $\phi \in C^3(\Omega)$ such that $\phi \cdot \mathbf{n} = 0$ on $\partial \Omega$, then there exists a single-valued potential $s = s(\mathbf{x})$
such that

\[ F = -\nabla s. \]

Let us consider a slightly more general problem than (4.6.5), i.e.

\[ \frac{1}{\rho} = \max_{v} \left( \frac{\mathcal{F}}{\mathcal{D}} \right) \tag{4.6.6} \]

where

\[ \mathcal{F} = -\langle v \cdot D(U) \cdot v \rangle, \quad \mathcal{D} = \langle 2D(v) : D(v) \rangle \]

Suppose that the maximum of (4.6.6) is attained when \( v = \nabla \). Consider the values of \( \mathcal{F}/\mathcal{D} \) when \( v_i = \overline{v}_i + \varepsilon \eta_i \) where \( \eta_i = \left. \frac{\partial v_i}{\partial \varepsilon} \right|_{\varepsilon=0} \) is an arbitrary vector (satisfying (4.6.5)). For each fixed \( \eta_i \) we have

\[ \frac{1}{\rho(\varepsilon)} = \frac{\mathcal{F}(\varepsilon)}{\mathcal{D}(\varepsilon)}. \tag{4.6.7} \]

Clearly \( 1/\rho(\varepsilon) \) is a maximum when \( \varepsilon = 0 \). Then

\[ \rho(\varepsilon)\mathcal{F}(\varepsilon) - \mathcal{D}(\varepsilon) = 0 \]

and

\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} [\rho(\varepsilon)\mathcal{F} - \mathcal{D}] = \rho(0) \frac{d\mathcal{F}}{d\varepsilon} - \frac{d\mathcal{D}}{d\varepsilon} = 0. \tag{4.6.8} \]

Using equation (4.6.6), we may write (4.6.8) as

\[ \rho \left( \nabla \cdot D(U) \cdot \frac{\partial v}{\partial \varepsilon} \right) + 2 \left( D(\nabla) : \frac{\partial D(v)}{\partial \varepsilon} \right) = 0. \tag{4.6.9} \]

Here all quantities are evaluated at \( \varepsilon = 0 \) (then \( v \simeq \nabla \)) and, we have used the symmetry of \( D \) to write

\[ \left( \frac{\partial v}{\partial \varepsilon} \cdot D(U) \cdot \nabla \right) = \left( \nabla \cdot D(U) \cdot \frac{\partial v}{\partial \varepsilon} \right). \]

Equation (4.6.9) may be regarded as Euler’s functional equation. It holds for every vector field \( \partial v/\partial \varepsilon \) such that \( \nabla \cdot (\partial v/\partial \varepsilon) = 0 \) in \( \Omega \) and \( \partial v/\partial \varepsilon = 0 \) on \( \partial \Omega \). To convert this equation into an eigenvalue problem for a system of differential equations, we note that

\[ 2 \left( D(\nabla) : \frac{\partial D(v)}{\partial \varepsilon} \right) = 2 \left( \nabla \cdot \left( D(\nabla) \cdot \frac{\partial v}{\partial \varepsilon} \right) \right) - \left( \frac{\partial v}{\partial \varepsilon} \cdot \nabla^2 \nabla \right). \]
Therefore, the equality (4.6.9) becomes
\[
\varrho\left(\nabla \cdot \mathbf{D}(\mathbf{U}) \cdot \frac{\partial \mathbf{v}}{\partial \varepsilon}\right) = -2 \left\langle \nabla \cdot \left(\mathbf{D}(\mathbf{v}) \cdot \frac{\partial \mathbf{v}}{\partial \varepsilon}\right) \right\rangle + \left\langle \frac{\partial \mathbf{v}}{\partial \varepsilon} \cdot \nabla^2 \mathbf{v} \right\rangle.
\] (4.6.10)

But the divergence theorem (\( \int_{\Omega} \nabla \cdot \mathbf{F} \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, ds \)) yields
\[
-2 \left\langle \nabla \cdot \left(\mathbf{D}(\mathbf{v}) \cdot \frac{\partial \mathbf{v}}{\partial \varepsilon}\right) \right\rangle = -2 \int_{\Omega} \nabla \cdot \left(\mathbf{D}(\mathbf{v}) \cdot \frac{\partial \mathbf{v}}{\partial \varepsilon}\right) \, d\mathbf{x}
\]
\[
= -2 \int_{\partial\Omega} \mathbf{D}(\mathbf{v}) \cdot \frac{\partial \mathbf{v}}{\partial \varepsilon} \cdot \mathbf{n} \, ds
\]
\[
= 0, \quad \text{since } \frac{\partial \mathbf{v}}{\partial \varepsilon} = 0 \text{ on } \partial\Omega.
\]

Then, (4.6.10) becomes
\[
\left\langle \left(\varrho \mathbf{v} \cdot \mathbf{D}(\mathbf{U}) - \nabla^2 \mathbf{v}\right) \cdot \frac{\partial \mathbf{v}}{\partial \varepsilon} \right\rangle = 0.
\]

If we set \( \mathbf{F} = \varrho \mathbf{v} \cdot \mathbf{D}(\mathbf{U}) - \nabla^2 \mathbf{v}, \) then \( \mathbf{F} \) satisfies the conditions of the fundamental lemma 4.6.1. Applying the lemma to \( \mathbf{F} = \varrho \mathbf{v} \cdot \mathbf{D}(\mathbf{U}) - \nabla^2 \mathbf{v}, \) we obtain
\[
\varrho \mathbf{v} \cdot \mathbf{D}(\mathbf{U}) - \nabla^2 \mathbf{v} = -\nabla s,
\]
which is the same as
\[
\mathbf{v} \cdot \mathbf{D}(\mathbf{U}) - \frac{1}{\varrho} \nabla^2 \mathbf{v} = -\nabla s.
\] (4.6.11)

Finally, we obtain the Euler equations corresponding to the maximization problem (4.6.5) and given by
\[
\lambda \nabla^2 \mathbf{w} - \nabla s = \mathbf{w} \cdot \mathbf{D}(\mathbf{U}) \quad \text{in } \Omega,
\] (4.6.12)
\[
\nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega,
\] (4.6.13)
and
\[
\mathbf{w} = 0 \quad \text{on } \partial\Omega,
\] (4.6.14)
where we have set \( \frac{1}{\varrho} = \frac{1}{\text{Re}} = \lambda \) and where \( s(x) \) is seen as Lagrange multiplier associated with the constraint \( \nabla \cdot \mathbf{w} = 0. \) Given the velocity field \( \mathbf{U}(x), \) the system (4.6.12)-(4.6.14) is a \textit{self-adjoint linear eigenvalue problem} for the triple \( \mathbf{w}(x) \neq 0 \) (the perturbation), \( s(x) \neq 0, \) and \( \lambda \in \mathbb{R}. \)

The solution \( \tilde{\nu} = \frac{1}{\text{Re}} \) of the maximization problem (4.6.5), is then given by the largest eigenvalue (M. D. Gunzburger [8]) of the system (4.6.12)-(4.6.14). The existence of a
non-negative $\tilde{\nu}$ follows from the fact that the trace $[D(U)] = \nabla \cdot U = 0$. Then if $\nu > \tilde{\nu}$ (or $Re < \tilde{Re}$), the given flow $U$ is stable.

Even for simple flows $U$ in simple domains $\Omega$, it is not possible to determine the eigenvalues of the system (4.6.12)-(4.6.14), except through numerical procedures. Thus in the following section, we investigate the finite element approximations of the eigenvalues of the system (4.6.12)-(4.6.14).

### 4.6.2 Finite Element Approximations of the Eigenvalues

In order to define such approximations, with the help of the general principles stated in section 4.3, one first recasts the system (4.6.12)-(4.6.14) into the following weak form:

Given $U \in H^r(\Omega)$ for some positive integer $r$, find $w \in H^1_0(\Omega), w \neq 0, s \in L^2_0(\Omega), s \neq 0$, and $\lambda \in \mathbb{R}$ such that

$$\lambda \tilde{a}(w, v) + b(s, v) = d(U; w, v) \quad \text{for all } v \in H^1_0(\Omega) \quad (4.6.15)$$

and

$$b(q, w) = 0 \quad \text{for all } q \in L^2_0(\Omega) \quad (4.6.16)$$

where $\tilde{a}(\cdot, \cdot) = Re a(\cdot, \cdot)$ and the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined in (4.3.1) and (4.3.2), respectively, and where

$$d(U; w, v) = -\int_{\Omega} w \cdot D(U) \cdot v \, dx.$$ 

Therefore we are interested in finding an approximation for $\tilde{\nu} = \frac{1}{\tilde{Re}}$, which now denotes the largest eigenvalue of the system (4.6.15)-(4.6.16). We denote by $m$ the algebraic multiplicity of the eigenvalue $\tilde{\nu} = \frac{1}{\tilde{Re}}$ and by $\mathcal{R}_\tilde{\nu}$ the space spanned by the eigenvectors $(s, w)$ of (4.6.15)-(4.6.16) corresponding to the eigenvalue $\tilde{\nu}$. Due to the fact that (4.6.15)-(4.6.16) is self-adjoint, $m$ is also the geometric multiplicity of the eigenvalue $\tilde{\nu}$ and hence the dimension of the eigenspace $\mathcal{R}_\tilde{\nu}$.

Now, following the same principles as in Section 4.4, the finite element formulation is given by the following problem:
Given $U \in \mathbf{H}^r(\Omega)$ for some positive integer $r$, find $w^h \in \mathbf{V}_0^h \subset \mathbf{H}_0^1(\Omega)$, $w^h \neq 0$, $s^h \in S_0^h \subset L_0^2(\Omega)$, $s^h \neq 0$ and $\lambda^h \in \mathbb{R}$ such that
\begin{equation}
\lambda^h \tilde{a}(w^h, v^h) + b(s^h, v^h) = d(U; w^h, v^h) \quad \text{for all } v^h \in \mathbf{V}_0^h
\end{equation} (4.6.17)
and
\begin{equation}
b(q^h, w^h) = 0 \quad \text{for all } q^h \in S_0^h.
\end{equation} (4.6.18)

We assume that the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ and the approximating subspaces $\mathbf{V}_0^h$ and $S_0^h$ satisfy all the hypotheses $\mathbf{H1}, \mathbf{H2}, \mathbf{H3}$ (Section 4.4), required for suitable approximations. We also assume that for the given velocity field $U$, the bilinear form $d(U; w, v)$ is continuous for all $w, v \in \mathbf{H}_0^1(\Omega)$; this assumption is valid whenever, e.g., $U \in \mathbf{H}_0^1(\Omega)$. Of interest here, are the following results (for more details, see the PhD thesis by Peterson J. [25] or see Max D. Gunzburger [8]): Firstly there are exactly $m$ eigenvalues of (4.6.17)-(4.6.18), counted according to the multiplicity, which as the discretization parameter $h \to 0$, converge to the eigenvalue $\tilde{\nu} = \frac{1}{\tilde{\nu}}$ of the system (4.6.15)-(4.6.16). Thus if we denote these $m$ eigenvalues by $\{\tilde{\nu}^h_j\}, j = 1, \ldots, m$ then we have
\begin{equation}
\tilde{\nu}^h_j \to \tilde{\nu} \quad \text{as, } h \to 0.
\end{equation} (4.6.19)

In addition, we also have the error estimate: For $h$ sufficiently small, there exists a constant $C$ such that
\begin{equation}
|\tilde{\nu} - \tilde{\nu}^h_j| \leq C(\xi^h)^2 \quad \text{for, } j = 1, \ldots, m
\end{equation} (4.6.20)
where
\begin{equation}
\xi^h = \sup_{(s, w) \in \mathbb{R}^p} \inf_{\|s\|_0 + |w| = 1} \left( |w - v^h|_1 + |s - q^h|_0 \right).
\end{equation} (4.6.21)

From the latter equations, we see that the usual situation concerning eigenvalue approximations by the finite element methods, is obtained in the present case; namely that the error in the eigenvalue is the square of the error for the eigenfunction, the latter being measured in the ”natural” norm sense. In this way we calculate an approximation of the critical Reynolds number,
\begin{equation}
\widetilde{Re} = \frac{1}{\tilde{\nu}}
\end{equation}
Figure 1. Stability limits for the basic flow. $R_E$, $R_G$, and $R_L$ are the critical values of the Reynolds number, depending on the type of instability.

for which the transition from laminar to turbulent flow occurs. Figure 1 shows the stability limits for the basic flow, and the different types of (in)stability. It clearly shows different zones of (in)stability which change with values of the Reynolds number.

Next, let us investigate the stationary case of the Navier-Stokes equations to show that the Reynolds number is again at the core of all factors that may trigger flow instabilities.

### 4.6.3 The Stationary Case of Navier-Stokes Equations

We now consider our domain $\Omega$ defined in the first chapter and assume that all the variables in the system (4.0.1)-(4.0.5) are independent of the time. We therefore obtain the following stationary Navier-Stokes problem:

$$u \cdot \nabla u + \nabla p - Re^{-1} \nabla^2 u = f, \quad \nabla \cdot u = 0, \quad \text{in} \ \Omega \quad (4.6.22)$$
with the following boundary conditions
\[ u|_{r_{\text{rigid}}} = 0, \quad u|_{r_{\text{in}}} = u^{\text{in}} \] (4.6.23)
having the following weak formulation (see Section 4.3):

Find functions \( u \in u^{\text{in}} + H^1_0(\Omega) \) and \( p \in L^2_0(\Omega) \) such that
\[
\begin{aligned}
a(u, v) + c(u, u, v) + b(p, v) &= (f, v) \quad \text{for all } v \in H^1_0(\Omega) \\
b(q, u) &= 0 \quad \text{for all } q \in L^2_0(\Omega)
\end{aligned}
\] (4.6.24)
discretized as follows (see Section 4.4):

Find a pair \((u^h, p^h) \in u^{\text{in}} + V^h_0 \times S^h_0\) such that
\[
\begin{aligned}
a(u^h, v^h) + c(u^h, u^h, v^h) + b(p^h, v^h) &= (f, v^h) \quad \text{for all } v^h \in V^h_0 \\
b(q^h, u^h) &= 0 \quad \text{for all } q^h \in S^h_0.
\end{aligned}
\] (4.6.25)

We assume that the the bilinear forms \(a(., .)\) and \(b(., .)\) and the approximating subspaces \(V^h_0 \text{ and } S^h_0\) satisfy all the hypotheses \textbf{H1}, \textbf{H2}, \textbf{H3} in Section 4.4 required for suitable approximations. For any \( f \in H^{-1}(\Omega) \), the system (4.6.25)-(4.6.26) has a solution \((u^h, p^h)\).

It is well known (see Girault and Raviart [6] or Gunzburger [8]) that the solution is unique for ”sufficiently small” data \( f \) or ”sufficiently small” Reynolds number, \( Re \). In fact if we set
\[
Z = \{ v \in H^1_0(\Omega) : b(q, v) = 0 \quad \text{for all } q \in L^2_0(\Omega) \}
\] (4.6.27)
the space of divergence free functions, and
\[
\mathcal{N} = \sup_{u, v, w \in Z} \frac{c(w, u, v)}{|u||v||w|} \quad (4.6.28)
\]
then given \( f \in H^{-1}(\Omega) \), and if \( (Re)^2 \mathcal{N} \|f\|_{-1} < 1 \), then the problem (4.6.24) has a unique solution \((u, p) \in Z \times L^2_0(\Omega)\).

From the previous assertion, we can state one of the fundamental properties (see D. D. Joseph [10]) of the solutions of the Navier-Stokes equations (4.6.22)-(4.6.23) which reads as follows: \textit{When the viscosity is large (or the Reynolds number is small), all solutions of the Navier-Stokes equations tend to a single basic flow.} So what is the final destiny of
all these uniquely determined solutions of the Navier-Stokes equations? For large values of the Reynolds number, the final set of flows which evolve from a given set of initial fields is generally "turbulent".

As just proven, the Navier-Stokes equations have in general more than one solution, unless the data satisfies very stringent requirements. However, it can also be shown that in many practical examples these solutions are mostly isolated, i.e. there exists a neighborhood in which each solution is unique. Furthermore, it can be established (Girault and Raviart [6]) that the solutions depend continuously on the Reynolds number, $Re$ (which is inversely proportional to the kinematic viscosity). Thus as the Reynolds number varies along an interval, each solution of the Navier-Stokes equations describe an isolated branch. In particular, this means that the bifurcation phenomenon can be rare or occurs infrequently (Girault and Raviart [6]). This situation, frequently encountered in practice, is expressed mathematically by the notion of branches of non-singular solutions. So the solutions of the problem (4.6.22)-(4.6.23) are "in general" non-singular.

But when the Reynolds number, $Re$ is large (small viscosity), compared to the other parameters of the fluid, there arises a boundary layer in the neighborhood of $\partial \Omega$ where the viscosity predominates while it is negligible in the interior of $\Omega$. At the same time, the transition to turbulence occurs. Thus the solutions of the Navier-Stokes are seriously discontinuous at certain values of the Reynolds number, $Re$.

In the coming sections, we show that under the hypotheses $H1, H2, H3$, the problem (4.6.25)-(4.6.26) possess a branch of non-singular solutions that, as $h \to 0$, converges to a given branch of non-singular solutions of (4.6.24). This convergence depends on the Reynolds number $Re$. We also use a finite element method, the Newton method, to show that the stability of such a convergence is guaranteed for small variations of $Re$. 
4.6.4 Non-singular Solutions and a Finite Element Approximation

Let $Z$ and $\mathcal{Z}$ be two Banach spaces and $\Lambda$ a compact interval of the real line $\mathbb{R}$. We are given a $C^p$-mapping ($p \geq 1$)

$$F : (Re, U) \in \Lambda \times Z \to F(Re, U) \in \mathcal{Z}$$

and we want to solve the equation

$$F(Re, U) = 0$$  \hspace{1cm} (4.6.29)

i.e. we want to find pairs $(Re, U) \in \Lambda \times Z$ which are solutions of (4.6.29).

Let $\{(Re, U(Re)); Re \in \Lambda\}$ be a branch of solutions of equation of (4.6.29). This means that

$$Re \to U(Re) \text{ is a continuous function from } \Lambda \text{ into } Z$$  \hspace{1cm} (4.6.30)

and

$$F(Re, U(Re)) = 0.$$  \hspace{1cm} (4.6.31)

Moreover, we suppose that these solutions are non-singular in the sense that:

$$D_UF(Re, U(Re)) \text{ is an isomorphism from } Z \text{ onto } \mathcal{Z} \text{ for all } Re \in \Lambda.$$  \hspace{1cm} (4.6.32)

As an immediate consequence of (4.6.32), it follows from the implicit function theorem (see [6]) that $Re \to U(Re)$ is a $C^p$-function from $\Lambda$ into $Z$.

Let us show that our problem for the Navier-Stokes equations (4.6.24) fits into the above framework. We first set:

$$Z = \mathcal{Z} = H^1(\Omega) \times L^2_0(\Omega),$$  \hspace{1cm} (4.6.33)

and we introduce the intermediate space

$$Y = H^{-1}(\Omega) \times \left\{ g \in H^{1/2}(\Gamma); \int_{\Gamma} g \cdot \mathbf{n} \, dx = 0 \right\}.$$  \hspace{1cm} (4.6.34)
Next we define a linear operator $T$ as follows: given $(f_*, g_*) ∈ Y$, we denote by $(u_*, p_*) = T(f_*, g_*) ∈ Z$ the solution of the Dirichlet problem for the Stokes equations:

$$
\begin{aligned}
-\nabla^2 u_* + \nabla p_* &= f_* \text{ in } \Omega \\
\nabla \cdot u_* &= 0 \text{ in } \Omega \\
u_*|_{\Gamma} &= g_* 
\end{aligned}
$$

(4.6.35)

Finally, with the data $(f, g) ∈ Y$, we associate a $C^\infty$-mapping $G$ from $\mathbb{R}_+ × Z$ into $Y$, defined for a $U = (w, q) ∈ Z$, by

$$
G : (Re, U) \to G(Re, U) = (Re(w \cdot \nabla w - f), -g)
$$

(4.6.36)

and we set

$$
F(Re, U) = U + TG(Re, U).
$$

(4.6.37)

It is clear that $(f, g) = (f, 0) ∈ Y$ and (4.6.36) becomes

$$
G(Re, U) = (Re(w \cdot \nabla w - f), 0)
$$

(4.6.38)

or simply

$$
G(Re, U) = Re(w \cdot \nabla w - f)
$$

(4.6.39)

Now we may state the lemma:

**Lemma 4.6.2.** The pair $(u, p) ∈ H^1(Ω) × L^2_0(Ω)$ is a solution of Problem (4.6.22) – (4.6.23) if and only if $(Re, U)$, with $U = (u, Re p)$ is a solution of (4.6.29), where the spaces $Z$ and $Z$ are defined by (4.6.33) and the compound mapping $F$ is defined by (4.6.37) and (4.6.38).

**Proof:** If $(u, p)$ is a solution of the problem (4.6.22)-(4.6.23) then

$$
\begin{aligned}
-\nabla^2 u + \nabla(Re p) &= Re(f - u \cdot \nabla u) \text{ in } \Omega \\
\nabla \cdot u &= 0 \text{ in } \Omega \\
u &= 0 \text{ on } \Gamma_{rigid}.
\end{aligned}
$$
From both (4.6.35) and \((u_*, p_*) = T(f_*, g_*)\), applied to \(f_* = Re(f - u \cdot \nabla u)\) and \(p_* = Rep\), we can write
\[
(u, Rep) = T(Re(f - u \cdot \nabla u), 0)
\]
or
\[
(u, Rep) - T(Re(f - u \cdot \nabla u), 0) = 0 \quad \text{(since the operator } T \text{ is linear)}.
\]
Then, equation (4.6.39) yields
\[
U + TG(Re, U) = 0
\]
and (4.6.37) yields
\[
F(Re, U) = 0
\]
which means
\[
(Re, U) \text{ is a solution of (4.6.29)}.
\]

\[\Box\]

From lemma 4.6.2, it is clear that if \((u, p)\) is a solution of the problem (4.6.22)-(4.6.23), then \((Re, U), \text{ where } U = (u, Re p)\) is a non-singular solution of (4.6.29).

We may now state the following proposition:

**Proposition 4.6.3.** Assume that the hypotheses \(H1, H2\) and \(H3\) hold. Let
\[
\{(Re, (u(Re), Rep(Re))); Re \in \Lambda\}
\]
be a branch of non-singular solutions of the Navier-Stokes problem (4.6.24). Then there exists a neighborhood \(O\) of the origin in \(H_0^1(\Omega) \times L_0^2(\Omega)\) and for \(h \leq h_0\) sufficiently small a unique \(C^\infty\) branch \(\{(Re, (u^h(Re), Rep^h(Re))); Re \in \Lambda\}\) of non-singular solutions of problem (4.6.25) – (4.6.26) such that:
\[
(u^h(Re), Rep^h(Re)) \in (u(Re), Rep(Re)) + O \quad \text{for all } Re \in \Lambda.
\]

Moreover, we have the convergence property:
\[
\limsup_{h \to 0} \{||u^h(Re) - u(Re)||_1 + ||p^h(Re) - p(Re)||_0\} = 0. \quad (4.6.40)
\]
In addition, if the mapping \( \text{Re} \to (\mathbf{u}(\text{Re}), p(\text{Re})) \) is continuous from \( \Lambda \) into \( H^{m+1}(\Omega)^3 \times H^m(\Omega) \) for some integer \( m \) with \( 1 \leq m \leq l \), we have for all \( \text{Re} \in \Lambda \):

\[
|\mathbf{u}^h(\text{Re}) - \mathbf{u}(\text{Re})|_1 + \|p^h(\text{Re}) - p(\text{Re})\|_0 \leq Kh^m. \tag{4.6.41}
\]

Our goal is not to prove this proposition, but to use the results. A similar proof of this proposition can be found in Girault and Raviart [6].

It is also possible to derive an \( L^2 \)-estimate for the velocity. But the following regularity must be satisfied:

\[
\text{The mapping } (\phi, \mu) \mapsto \nabla \mu - \text{Re}^{-1} \nabla^2 \phi \text{ is an isomorphism from } [H^2(\Omega) \cap S] \times [H^1(\Omega) \cap L^2_0(\Omega)] \text{ onto } L^2(\Omega). \tag{4.6.42}
\]

where \( S \) is a closed subspace of \( Z \).

**Proposition 4.6.4.** We retain the hypotheses of Proposition 4.6.3 and we assume that (4.6.42) holds. If the mapping \( \text{Re} \to (\mathbf{u}(\text{Re}), p(\text{Re})) \) is continuous from \( \Lambda \) into \( H^{m+1}(\Omega)^3 \times H^m(\Omega) \) for some integer \( m \in [1, l] \), then we have the following \( L^2 \)-estimate for all \( \text{Re} \in \Lambda \):

\[
\|\mathbf{u}^h(\text{Re}) - \mathbf{u}(\text{Re})\|_0 \leq Kh^{m+1}. \tag{4.6.43}
\]

These propositions show that the convergence of finite element approximations is guaranteed, and the stability constants depend on \( \text{Re} \).

### 4.6.5 Stability of Newton’s Method

The method discussed here is one of the finite element methods that is intended to solve the Navier-stokes equations. We saw in Lemma 4.6.2 that it suffices to investigate equations of the type (4.6.29):

\[
F(\text{Re}, \mathcal{U}) = 0 \tag{4.6.44}
\]
where \( F \) is defined by (4.6.37) and is a \( C^p \)-mapping \((P \geq 1)\) defined on \( \Lambda \times Z \) (with \( \Lambda \) a compact interval of the real line \( \mathbb{R} \) and \( Z = H^1(\Omega) \times L^2_0(\Omega) \)). For simplicity, let us set and fix
\[
\lambda = \text{Re} \in \Lambda
\]
and assume that \( U = U(\lambda) \) is a non-singular solution of (4.6.44). Then
\[
F(\lambda, U) = 0, \quad D_U F(\lambda, U) \text{ is an isomorphism from } Z \text{ onto } Z
\]
where \( Z \) is a Banach space.

Since \( U \) is an isolated solution of (4.6.44) and since \( F \) is at least differentiable, an efficient way to approximate \( U \) is by the Newton method. The Newton algorithm reads as:

Starting from an initial guess \( U^0 \), construct the sequence \( \{U^n\}_n \) in \( Z \) by:
\[
U^{n+1} = U^n - [D_U F(\lambda, U^n)]^{-1} \cdot F(\lambda, U^n) \quad n \geq 0
\]
(4.6.45)

or equivalently
\[
D_U F(\lambda, U^n) \cdot (U^{n+1} - U^n) = -F(\lambda, U^n).
\]

As \( D_U F(\lambda, U) \) is a linear operator, each step of Newton’s method requires the solution of a different problem relative to \( D_U F(\lambda, U^n) \). If this is too costly, the simplest alternative is to replace (4.6.45) by:
\[
U^{n+1} = U^n - [D_U F(\lambda, U^0)]^{-1} \cdot F(\lambda, U^n) \quad n \geq 0,
\]
(4.6.46)

or equivalently
\[
D_U F(\lambda, U^0) \cdot (U^{n+1} - U^n) = -F(\lambda, U^n).
\]

The drawback of Newton’s method is that the stability of its (quadratic) convergence (see [6]) can only be ensured when the first guess \( U^0 \) is sufficiently near the solution \( U \). If this solution is part of a branch of non-singular solutions, and if we know the solution at a neighboring point, say \( U(\lambda - \nabla \lambda) \) for an adequate increment \( \nabla \lambda \), then we can derive from this value the first guess to start Newton’s algorithm.
Since $\lambda \mapsto U(\lambda)$ is a branch of non-singular solution of (4.6.44), then $F$ is a $C^P$-mapping ($P \geq 2$), so is the mapping $U(\lambda)$ and we can differentiate both sides of (4.6.44):

$$D_{u}F(\lambda, U(\lambda)) \cdot \left( \frac{dU(\lambda)}{d\lambda} \right) + D_{\lambda}F(\lambda, U(\lambda)) = 0 \quad \text{for all } \lambda \in \Lambda$$

(4.6.47)

i.e. we find a first order differential equation of the form

$$\frac{dU(\lambda)}{d\lambda} = -\phi(\lambda)$$

(4.6.48)

where

$$\phi(\lambda) = [D_{u}F(\lambda, U(\lambda))]^{-1} D_{\lambda}F(\lambda, U(\lambda)).$$

The simplest way to solve (4.6.48) is to use the one-step, explicit Euler’s method; this brings us to choose

$$U^0(\lambda) = U(\lambda - \Delta \lambda) - \phi(\lambda - \Delta \lambda) \Delta \lambda.$$ 

(4.6.49)

In other words $U^0(\lambda)$ is defined by

$$D_{u}F(\lambda - \Delta \lambda, U(\lambda - \Delta \lambda)) \cdot (U^0(\lambda) - U(\lambda - \Delta \lambda)) = -D_{\lambda}F(\lambda - \Delta \lambda, U(\lambda - \Delta \lambda)) \cdot \Delta \lambda$$

Let us estimate the difference $U(\lambda) - U^0(\lambda)$. From (4.6.48), we infer that

$$U(\lambda) = U(\lambda - \Delta \lambda) - \int_{\lambda - \Delta \lambda}^{\lambda} \phi(\xi) d\xi.$$

Subtracting (4.6.49) from this equality yields

$$U(\lambda) - U^0(\lambda) = - \left[ \int_{\lambda - \Delta \lambda}^{\lambda} \phi(\xi) d\xi - \phi(\lambda - \Delta \lambda) \cdot \Delta \lambda \right]$$

$$= - \int_{\lambda - \Delta \lambda}^{\lambda} \phi'(\theta \xi) \cdot (\xi - \lambda + \Delta \lambda) d\xi.$$

Hence

$$\|U(\lambda) - U^0(\lambda)\|_Z \leq \left[ (\Delta \lambda)^2 / 2 \right] \max_{\theta \in (\lambda - \Delta \lambda, \lambda)} \|\phi'(\theta)\|_Z.$$

Thus $\|U(\lambda) - U^0(\lambda)\|_Z$ is $O((\Delta \lambda)^2)$ and if $\Delta \lambda$ is small enough, the solutions $U$ and $U^0$ stay close, characterizing the stability and $U^0$ defined by (4.6.49) is an adequate starting value for Newton’s algorithm.
Finally, we have used finite element theory to show that the Reynolds number is once again at the core of the stability of a fluid motion. We have shown a way to compute the critical Reynolds number at which the first bifurcation (appearance of the alternating vortex pattern), occurs. This critical Reynolds number varies according to the type of the flow and physical condition imposed on it.
Conclusion

The present study contributes, among other numerical and analytical studies to the knowledge of physical phenomena related to the transition from laminar to turbulence in Newtonian fluid flow. The strategy adopted in this work consists of investigating the governing equations, for the case of incompressible viscous Newtonian fluid flow. These equations are the continuity and the Navier-Stokes equations.

Experimental studies carried out by many authors proved the existence of three different regimes of the flow, which are laminar, transition and turbulent regimes. Experimental studies also showed that this situation depends on the Reynolds number. Our main objective in this work has been to explain this dependence mathematically.

Therefore, we have used two effective methods: Lie group theory, and the finite element method, to explain why the Reynolds number influences the different regimes of Newtonian fluid flows:

By Lie group theory, we have solved analytically the Navier-Stokes equations, using the symmetry approach. Finally, we have succeeded to express explicitly a solution; the $u$-, $v$-, and $w$-velocity components as well as the pressure $p$. We found them to be functions of the Reynolds number, even though there are other analytic functions appearing in their expressions. This explains, for example, the figures drawn in the sixth part of the article [24], which show that the time evolutions of velocity components become more perturbed as the Reynolds number increases.

Secondly, we have used the finite element method to show a way to compute the critical
Reynolds number at which the first bifurcation occurs. The stability theory has helped us to prove that the stability of a flow motion is proportional to the Reynolds number. One can now understand how a laminar flow may develop into a turbulent flow, through the critical Reynolds number, for which the transition regime occurs. Thereafter, we investigated the stationary flow. We have seen that solutions of the Navier-Stokes equations governing it, are in general non-singular. Therefore, we have chosen a finite element method; the Newton method, to show that the stability of its convergence is guaranteed for small variations of the Reynolds number.

The results obtained from both methods are almost the same: Laminar flow occurs at low Reynolds numbers, where viscous forces are dominant, and is characterized by smooth, constant fluid motion, while turbulent flow, on the other hand, occurs at high Reynolds numbers and is dominated by inertial forces, producing random eddies, vortices and other flow fluctuations. The transition between laminar and turbulent flow is indicated by a critical Reynolds number, which depends on the exact flow configuration.

Reynolds numbers are of extreme importance in the study of Newtonian fluid flows. As our focus here is the transition flow only, it follows that the occurrence of transition is expressed in terms of the Reynolds number mainly, though there are other factors, not taken into consideration in the present work. Accordingly, it seems easier to comprehend the experiments carried out by many scientists: Firstly, the one of pipe flow done by Osborne Reynolds [26], where intermittent flashes of turbulence could be seen as the Reynolds number increased beyond a critical value. Secondly, the experimental studies of the wake formation behind bluff bodies, pointed out by Roshko [27], who first observed the existence of a transition regime in the wake of the cylinder and found distinct irregularities in the wake velocity fluctuation. He showed that there exist three different regimes of the flow at low moderate Reynolds numbers, namely laminar, transition and irregular turbulent regimes. In the transition regime, he reported that the low-frequency irregularities obtained experimentally are related to the pressure of three-dimensionalities in the flow, which lead to the development of turbulent motion further downstream. Thirdly, in the same type of flow, Williamson [32] observed the existence of two modes of formation of streamwise vorticity in the near wake, each occurring at
a different range of Reynolds numbers, and both being related to the three-dimensional transition between Reynolds numbers from 180 to 260. The first mode occurs beyond Reynolds number 180 and is characterized by a continuous change in the wake formation, as the primary vortices become unstable and generate large-scale vortex loops. The second, beyond Reynolds number 260, corresponds to the appearance of small-scale streamwise vortex structures.

From a numerical point of view, a large number of numerical studies have been devoted to the analysis of unsteady flow around a circular cylinder in the low and moderate Reynolds number regime. But these studies are only two-dimensional simulations. Reliable three-dimensional numerical simulations of this category of flow have only very recently appeared, due to the increased capacities and evolution of supercomputing technology. Karniadakis and Triantafyllou [12] have computed the three dimensional flow around a circular cylinder in the Reynolds number range of 200-500, by using the spectral-element method by Patera [23]. In the same way, Hélène Persillon and Marianna Braza [24] succeeded to compute the three dimensional flow around a circular cylinder in the Reynolds number range of 100-300.

Thanks to the direct observations of fluids, like those studied by Reynolds, researchers know that the profile of a fluid during laminar flow is parabolic. This can also seen by solving the Navier-Stokes equations. The non-dimensional form of the Navier-Stokes equations: (1.3.1), (1.3.2), (1.3.3) and (1.3.4) clearly shows that the Reynolds number is the only parameter of the fluid flow (chapter 1). The Navier-Stokes equations generally have more than one solution, unless the data satisfies very stringent requirements, as we saw in Chapter 4. However, it can also be shown in many practical examples that these solutions are mostly isolated, i.e. there exists a neighborhood in which each solution is unique. Furthermore, it can be established [6] that these solutions depend continuously on the Reynolds number, $Re$ (which is inversely proportional to the kinematic viscosity). Thus as the Reynolds number varies along an interval, each solution of the Navier-Stokes equations describes an isolated branch. In particular, this means that the bifurcation phenomenon can be rare. This situation, very frequently encountered in practice, is expressed mathematically by the notion of branches of non-singular solutions (chapter 4).
Essentially, saying that a solution belongs to a non-singular branch of solutions, means that we are not at bifurcation points or turning points. But when the Reynolds number, $Re$ is large (small viscosity) compared to the other parameters of the fluid, there arises a boundary layer in the neighborhood of $\partial \Omega$ where the viscosity predominates while it is negligible in the interior of $\Omega$. At the same time the transition to turbulence occurs. Thus the solutions of the Navier-Stokes equations are seriously discontinuous at certain values of the Reynolds number, $Re$. The former case, where the solutions depend continuously on the Reynolds number, can be seen as a characterization of laminar flow (since the bifurcations are rare), while the latter case where the solutions are seriously discontinuous at high Reynolds numbers, can be seen as a characterization of turbulent flow.

Even though we have chosen simplified hypotheses throughout the present work, others may be inspired to give relevant meanings to all these experimental observations. However, transition to turbulence remains complex, and its study is far from being fully achieved. In this work, we tried to provide only basic explanations. In the study of transition to turbulence, there are many other elements that still have to be taken into consideration: For example, the prediction of the frequency modulation, and the formation of a discontinuity region delimited by two frequency steps within a given Reynolds number range. Another example is the birth of streamwise vorticity and the kinetic energy distribution in the studied region, where the similarity laws do not always hold.

It is encouraging to know that great works in the field, are still in progress all over the world.
Bibliography


SOME MATHEMATICAL EXPLANATIONS OF THE TRANSITION FROM LAMINAR TO TURBULENT FLOW IN NEWTONIAN FLUIDS, USING THE LIE GROUP AND FINITE ELEMENT METHODS

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Abstract

In this work, we show more simple ways to obtain the equations governing the flow in incompressible viscous Newtonian fluids, namely the continuity and the Navier-Stokes equations. These equations are investigated to explain how the transition from laminar to turbulent flow depends on the Reynolds number. Two effective methods are used: Lie theoretical and the finite element methods. By Lie’s theory, we explicitly express a solution of the Navier-Stokes equations, in terms of the Reynolds number. By the finite element method, together with the stability theory, we show a way to compute the critical Reynolds number, for which the transition to turbulence occurs.

1 Introduction

More than a century after Reynolds’ paper, the understanding how turbulent regions grow (in a pipe flow, for example) and to bring laminar flow to fully developed turbulence, is not completely achieved. It has since been known (O. Reynolds [26]) that the transition to turbulence occurs in an intermittent fashion. As the Reynolds number increased beyond a critical value of about 2300 (although the precise value depends on the pipe used and on the experimental conditions at the inlet), intermittent flashes of turbulence can be seen in the pipe. Reynolds proved that the transition from laminar to turbulent flow in pipes, is a function of the fluid velocity.

Furthermore, the reason for this intermittency is well known, at least in a crude way: Laminar flow at a given flow has a lower drag than turbulent flow, and as the pressure drop driving the flow is increased, there arises a critical interval of flow rate within which laminar flow offers too low a resistance to the pressure drop, and turbulent flow provides too high a resistance. In this intermediate case, the flow cycles between the two types of flow. This is manifested in the pipe through the regular occurrence of what Reynolds called turbulent ”flashes”, nowadays known as ”slugs” or ”puffs” depending on their provenance. The resultant flow oscillates, producing an oscillatory outlet flow.

However, because understanding of the transition requires an understanding of laminar and turbulence flow, both are explored in this article. With the assistance of existing
experimental information, it is possible to develop a mathematical model of the transition between the two types of flow.

Another example is the wake formation behind bluff bodies where Karniadakis and Triantafyllou [12] observed the existence of a transitional regime, depending on the Reynolds number. Over more than a century, it has received a great deal of attention from an experimental and a numerical point of view. Other researchers like Williamson [32] observed the existence of two modes of formation of streamwise vorticity in the near wake, each occurring at a different range of Reynolds numbers, and both being related to the three-dimensional transition between Reynolds numbers belonging to a specific real interval.

It always happens, as shown in the second part of Girault and Raviart [6], that the occurrence of transition is expressed in terms of the Reynolds number mainly, though there are other factors that are not of our interest in this work. Thus, when the Reynolds number, $Re$ is large (small viscosity) compared to the other parameters of the fluid, there arises a boundary layer in the neighborhood of the controlled domain, $\partial \Omega$, where the viscosity predominates while it is negligible in the interior of $\Omega$. At the same time the transition to turbulence occurs.

So why does the flow suddenly become unstable and break up into turbulent swirls at large enough values of the Reynolds number? There are certain standard ways to answer this question: One can simply try to solve analytically or numerically the Navier-Stokes equations governing the flow, so as to express explicitly the solutions in terms of the Reynolds number. Another way is to study the response of a fluid when it is subject to infinitesimally small disturbances, using a mathematical tool called hydrodynamic stability theory.

Hydrodynamic Stability Theory
Stability theory in general, according to Daniel D. Joseph [10], is the body of mathematical physics which enables one to deduce from first principles, the critical values which separate the different regimes of flow, as well as the forms of the fluid motions in these different regimes.

In the case of this report, we seek the critical Reynolds number at which the transition from laminar flow to turbulent flow occurs.

Drazin & Reid [5] gave a more explicit definition of hydrodynamic instability, suitable for the scope of this article, and which is defined as that branch of hydrodynamic concerned with “when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence”. From this definition, we can propose the following general procedure for studying hydrodynamic stability mathematically:
1. Start with a laminar or non-perturbed solution of the Navier-Stokes equations,

2. Perturb this solution with small disturbances,

3. Substitute the disturbed solution into the Navier-Stokes equations to derive disturbance equations. This usually yields an eigenvalue problem.

4. Solve the eigenvalue problem to study the (in)stability from the obtained equations.

We will try to explain mathematically the transition to turbulence by investigating the equations governing the flows of incompressible Newtonian fluids. The complete modelling must include not only these equations, but also the physical boundary and initial conditions imposed on the fluid. At this level we will treat the model (problem) analytically and numerically, thanks to two chosen methods: The Lie Group analysis and finite elements method.

Lie Group Analysis is a method for solving linear or non-linear differential equations analytically. It augments intuition in understanding and using symmetry for formulation of mathematical models, and often discloses possible approaches to solving complex problems. For the Navier-Stokes problem, this method uses general symmetry groups to explicitly determine solutions, which are themselves invariant under some subgroups of the full symmetry group of the system. These group-invariant solutions are found by solving a reduced system of ordinary differential equations, involving fewer independent variables than the original system (which presumably makes it easier to solve).

The finite element method requires discretization of the domain into sub regions or cells. In each cell the sought function is approximated by a characteristic form which is often a linear function. The method is traditionally based on the Galerkin weighted residual and Crank-Nicolson methods. One manner to obtain a suitable framework for treating our Navier-Stokes problem is to pose it as a variational one. The numerical treatment of the system of the Navier-Stokes equations by the finite element method, consists of computing the primitive variables $u$ (velocity), and $p$ (pressure), using a special Galerkin method based on a variational formulation. The spatial and time discretizations of the Navier-Stokes problem are constructed in appropriate function spaces, and "discrete" approximations will be determined in certain finite dimensional subspaces, consisting of piecewise polynomial functions.

In the coming sections, we use both methods to treat the Navier-Stokes equations governing the fluid flows. We end by discussing a few results experimentally observed by great researchers.
2 Basic Considerations

This section is to provide some basic definitions and a brief presentation of the equations governing an incompressible Newtonian fluid flow.

Notations
The following notations will be considered throughout this report:

For simplicity, we keep vectors represented in bold character.
Ω, open and bounded domain of \( \mathbb{R}^3 \)
x = (x, y, z), point in Ω
t, time over the time interval \([0;T]\)
∂Ω or Γ, boundary of Ω
\( n \), outward normal to Γ
\( s, t \), tangents to Γ
\( \mathbf{u} = (u, v, w) = (u(t, x, y, z), v(t, x, y, z), w(t, x, y, z)) \), fluid velocity vector field with components \( u \), \( v \) and \( w \) at the point \((x,y,z)\) and time \( t \).
Note: \( x = x(t), y = y(t), z = z(t) \), \( u = \frac{dx}{dt}, v = \frac{dy}{dt}, w = \frac{dz}{dt} \)
\( p = p(t,x,y,z) \), pressure
\( \rho \), constant density (assumed)
\( \mu \), constant viscosity (assumed)
\( \nu \), kinematic viscosity
\( u^i \), the \( i \)th component (or coordinate) of \( \mathbf{u} \)
\( u_{x^i}, \frac{\partial u}{\partial x^i} \) or \( \partial_i \mathbf{u} \), partial derivative of \( u \) with respect to the \( i \)th coordinate
\( \mathbf{u}_t, \frac{\partial \mathbf{u}}{\partial t} \) or \( \partial_t \mathbf{u} \), partial derivative in \( t \) of \( \mathbf{u} \)
\( u_{tt} \), second time derivative
\( u_{x^i x^j} \), second derivative of \( u \) with respect to the \( i \)th coordinate
\( \nabla p = (p_x, p_y, p_z) \), vector gradient of \( p \)
\( \nabla \cdot \mathbf{u} = u_x + v_y + w_z \), divergence of \( \mathbf{u} \)
\( \nabla \mathbf{u} = \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \), second order tensor (velocity gradient)
\( \mathbf{u}^T \), transpose of \( \mathbf{u} \).

New notations will be defined as we go along.

The following concepts are defined according to the book by Yuan [30].

The fluids considered in this article are those liquids or gasses that move under the action of a shear stress, irrespective of how small that shear stress may be: This means that even a very small shear stress results in motion in the fluid. In our study, it is convenient to assume that fluids are continuously distributed throughout a region of interest, that is, the fluid is treated as a continuum.
The primary property used to determine if the continuum assumption is appropriate, is the density of the fluid defined as the mass per unit volume. The density may vary significantly throughout the fluid. The concept of density at a mathematical point is defined as

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V},$$

where $\Delta m$ is the incremental mass contained in the incremental volume $\Delta V$.

The velocity, $u$, at any point of a fluid medium is written as the limit approached by the ratio between the displacement $\delta s$ of an element along its path and the corresponding increment of time $\delta t$ as the latter approaches zero: Therefore

$$u = \lim_{\delta t \to 0} \frac{\delta s}{\delta t}.$$

The pressure results from a normal compressive force acting on an area. If we were to measure this force per unit area acting on a submerged element, we would observe that it can either act inward or place the element in compression. The quantity measured is therefore the pressure which must be the negative of the normal stress. When the shearing stresses are present, the normal stress components at a point may not be equal;

If the shear stress of a fluid is directly proportional to the velocity gradient, the fluid is said to be a newtonian fluid and the coefficient of proportionality is evaluated as the viscosity, $\mu$. This relation between shear stress and the velocity gradient also applies for an incompressible fluid flow, that is a flow for which the density is constant across the fluid.

The viscosity of a fluid is a measure of its resistance to deformation rate. Another important effect of viscosity is to cause the fluid to adhere to the surface: This is known as the no-slip condition. The viscosity in general is dependent on temperature in liquids in which cohesive forces play a dominant role. Note that the viscosity of liquids decreases with increased temperature. In this report, we use a viscosity which is constant (incompressibility of the fluid).

Since the viscosity is often divided by the density in the derivation of equations (3.6) below, it has become useful, and customary, to define kinematic viscosity to be

$$\nu = \frac{\mu}{\rho},$$

(2.1)

The existence of two types of viscous flow is a broadly accepted phenomenon.

The word laminar deriving from the Latin word lámina, which means stream or sheet, indicates the regularity. Therefore a laminar motion gives the idea of a regular streaming
motion. In the opposite the word *turbulent* is used in every day experience to indicate something which is not regular. In Latin the word *turba* means something confusing or something which does not follow an ordered plan. A turbulent boy, in all Italian schools, is a young fellow who rebels against ordered schemes. Following the same line, the behavior of a flow which rebels against the deterministic rules of classical dynamics, is called turbulent.

A concrete example, the transition to turbulence in the wake of a circular cylinder, is illustrated in the publication [24], where Héléne Persillon and Marianna Braza studied and represented the transition to turbulence of the flow around a circular cylinder, namely, the transition to turbulence in the wake of a circular cylinder. This study together with the one by G. E. Karniadakis and G. S. Triantafyllou [12], about the wake formation behind bluff bodies, has received a great deal of attention over more than a century from both an experimental and a numerical point of view. Héléne Persillon and Marianna Braza have computed the three dimensional flow around a circular cylinder in the Reynolds number range of 100-300. The time-dependent evolution of the $u$- and $v$-velocity components are presented in both two and three-dimensional cases, for Reynolds number 200 and 300. This evolution is done at a spatial point of investigation $x/D = 0.97$, $y/D = 0$ and $z = 0$, where $D$ is the diameter of the cylinder. The drawings show the quasi-periodic character of the studied flow, and one can see that the amplitudes of the oscillations increase with the Reynolds number. This is what we are trying to explain.

3 Differential Equations of Incompressible Newtonian Fluid Flow

The theory of mechanics of continuous media, also known as continuum mechanics, allows the description of the constitutive equations laws that describe the deformations of fluid medium. These laws, in combination with the general conservation principles (conservation of mass and of momentum), form the system of partial differential equations, which are equal in number to the number of unknowns of the system. Namely, for 3-D motion there are four dependents variables: $u$, $v$, $w$ and $p$ and four independent variables: $x$, $y$, $z$ and $t$.

The written constitutive equations of a Newtonian fluid are based on the following considerations, according to B. Mohammadi [17]:

- At rest the fluid obeys the laws of statics.
- The equation of the fluid is objective, that is tensors are used. It is independent of the
Galilean reference frame in which it is expressed, and independent of the observer.

- Constitutive relations governing the fluid are isotropic, which means, independent of the orientation of the coordinate system axes.

With these assumptions, we exploit Cauchy’s Laws to obtain (B. Mohammadi [17]) the Navier-Stokes equations.

3.1 The Continuity Equation

To study the motion of a fluid which occupies a domain $\Omega \in \mathbb{R}^3$ over a time interval $[0,T]$, we shall denote by $O$ any regular subdomain of $\Omega$ and by $x = (x,y,z)$ any point of $\Omega$.

To conserve mass, the rate of change of mass in fluid in $O$, $\frac{\partial}{\partial t} \int_\Omega \rho$, has to be equal to the mass flux, $-\int_{\partial O} \rho \cdot \mathbf{n}$, across the boundary $\partial O$ of $O$, ($\mathbf{n}$ denotes the exterior normal to $\partial O$). Then,

$$\frac{\partial}{\partial t} \int_\Omega \rho = -\int_{\partial O} \rho \mathbf{u} \cdot \mathbf{n}.$$

By using the Stokes’ formula

$$\int_\Omega \nabla \cdot (\rho \mathbf{u}) = \int_{\partial O} \rho \mathbf{u} \cdot \mathbf{n},$$

the mass conservation equation becomes

$$\int_\Omega (\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u})) = 0.$$

The fact that $O$ is arbitrary, yields the equation of conservation of mass, expressed in differential form, and found to be

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u}) = 0.$$

(3.2)

It is also called the continuity equation.

The assumption to restrict our attention to incompressible flow with constant density, $\rho$, yields $\frac{\partial}{\partial t} \rho = 0$ and $\nabla \cdot (\rho \mathbf{u}) = \rho (\nabla \cdot \mathbf{u})$. Therefore, the continuity equation (3.2) takes the final sought form

$$\nabla \cdot \mathbf{u} = 0.$$

(3.3)

Which means that the velocity field, $\mathbf{u}$, of an incompressible flow must be divergence free.

In cartesian coordinates $\nabla$ is written :

$$\nabla = i \partial_x + j \partial_y + k \partial_z$$
and recalling $\mathbf{u} = (u, v, w)$, equation (3.3) reads:

$$u_x + v_y + w_z = 0. \quad (3.4)$$

### 3.2 The Navier-stokes Equations

The Navier-Stokes equations are considered as the foundation of fluid mechanics, and were introduced by C. Navier in 1823 and developed by G. Stokes. However, these equations were first introduced by L. Euler. The main contribution by C. Navier was to add a friction forcing term due to interactions between fluids layers which move with different speeds. These equations are nothing but the momentum equations based on Newton’s second law, which relates the acceleration of a particle to the resulting volume and body forces acting on it. They are, accordingly, the differential form of Newton’s second law of motion.

Let us now write Newton’s second law for the arbitrary volume element $O$ of fluid. By definition of the velocity $\mathbf{u}$, a particle of the fluid at position $x = (x,y,z)$ at time $t$ will be approximately at $x + \mathbf{u}(x,t)\delta t$ at time $t + \delta t$. Its acceleration is therefore

$$\lim_{\delta t \to 0} \frac{1}{\delta t} \left[ \mathbf{u}(x + \mathbf{u}(x,t)\delta t, t + \delta t) - \mathbf{u}(x, t) \right] = \mathbf{u}_t + \sum_{j=1}^{3} \mathbf{u}' u_{x_j} \equiv \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u},$$

where $\mathbf{u}'$ is the $j^{th}$ component of the vector $\mathbf{u}$ and $u_{x_j}$ the partial derivative of $\mathbf{u}$ with respect to the $j^{th}$ coordinate of the point $x$.

If we disregard external forces like those due to gravity, electromagnetism, Coriolis, etc., the only remaining forces are the pressure force and the viscous force due to the motion of the fluid, and equal to $\int_{\partial O} (\sigma - p\mathbf{I}) \mathbf{n}$, where $\sigma$ is the stress tensor, $\mathbf{I}$ is the unit tensor and $\mathbf{n}$ denotes the unit outer normal to $\partial O$. In this condition, Newton’s second law of motion for $O$ is given by

$$\int_{O} \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = - \int_{\partial O} (p\mathbf{n} - \sigma) \mathbf{n} = \int_{O} (-\nabla p + \nabla \cdot \sigma)$$

where we have used the Stokes’ formula (3.1) to establish the second equality.

The fact that $O$ is arbitrary yields

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \nabla \cdot \sigma. \quad (3.5)$$

Now we need to relate the stress tensor $\sigma$ to the velocity of the fluid: The hypothesis of Newtonian flow is a linear law relating $\sigma$ to $\nabla \mathbf{u}$:

$$\sigma = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + (\nu - \frac{2\mu}{3})\mathbf{I}\nabla \cdot \mathbf{u}$$
where \( \iota \) is the second viscosity of the fluid. For air and water the second viscosity \( \iota \) is very small. For Newtonian fluids, we assume that \( \iota = 0 \). The stress tensor becomes

\[
\sigma = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{2\mu}{3} \mathbf{I} \cdot \mathbf{u}.
\]

With this definition for \( \sigma \), equation (3.5) becomes

\[
\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \nabla \cdot [\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{2\mu}{3} \mathbf{I} \cdot \mathbf{u}]
\]

or

\[
\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p - \mu[\nabla \cdot \nabla \mathbf{u} + \nabla \cdot \nabla \mathbf{u}^T] + \frac{2\mu}{3} \nabla \cdot (\mathbf{I} \cdot \nabla \mathbf{u}) = 0.
\]

Since \( \nabla \cdot \nabla \mathbf{u} = \nabla^2 \mathbf{u} \), \( \nabla \cdot \nabla \mathbf{u}^T = \nabla(\nabla \cdot \mathbf{u}) \) and \( \nabla \cdot (\mathbf{I} \cdot \nabla \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) \), the latter equation finally yields the equation of conservation of momentum written as

\[
\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p - \mu \nabla^2 \mathbf{u} - \frac{\mu}{3} \nabla(\nabla \cdot \mathbf{u}) = 0.
\]

Taking into account the continuity equation (3.3), the equation of conservation of momentum becomes the incompressible Navier-Stokes equations

\[
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \mathbf{u}
\]

or

\[
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}
\]  
(3.6)

where \( \nu = \mu/\rho \) is the kinematic viscosity of the fluid defined in equation (2.1) and \( p \rightarrow p/\rho \) is the reduced pressure. Note that the Navier-Stokes equations are non-linear because of the term \( \mathbf{u} \cdot \nabla \mathbf{u} \) which is seen as the source of instability.

Now we write each term of (3.6) in cartesian coordinates:

\[
\mathbf{u}_t = (u_t, v_t, w_t)
\]

\[
\mathbf{u} \cdot \nabla \mathbf{u} = (u, v, w) \cdot \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} = \begin{pmatrix} uu_x + vu_y + wu_z \\ uw_x + vw_y + wv_z \\ uw_x + vw_y + wv_z \end{pmatrix}^T
\]

\[
\nabla p = (p_x, p_y, p_z)
\]

\[
\nabla^2 \mathbf{u} = \begin{pmatrix} u_{xx} + u_{yy} + u_{zz} \\ v_{xx} + v_{yy} + v_{zz} \\ w_{xx} + w_{yy} + w_{zz} \end{pmatrix}^T = \begin{pmatrix} \nabla^2 u \\ \nabla^2 v \\ \nabla^2 w \end{pmatrix}^T
\]

Thus the Navier-Stokes equations for constant density, \( \rho \), and constant viscosity, \( \mu \), are written as:

x-component

\[
u_t + uu_x + vu_y + wu_z = -p_x + \nu \nabla^2 u,
\]

(3.7)
\[ y \text{-component} \]
\[ v_t + w_x + vv_y + wv_z = -p_y + \nu \nabla^2 v, \quad (3.8) \]

\[ z \text{-component} \]
\[ w_t + uw_x + vw_y + ww_z = -p_z + \nu \nabla^2 w. \quad (3.9) \]

### 3.3 The Reynolds Number

The challenge of laminar-transition-turbulence started in 1883, when Osborne Reynolds of Manchester University (United Kingdom) made a prominent discovery that has remained a puzzle ever since. By introducing a small amount of ink into a horizontal glass pipe filled with water, he was able to check whether the flow was laminar or turbulent. Reynolds found that the transition from laminar to turbulent flow occurs spontaneously if a dimensionless quantity (see [31]), \( Re \), is larger than some critical value, about 2300. This quantity, which is known as the Reynolds number, has ever since become a quantity which engineers and scientists use to estimate if a fluid flow is laminar or turbulent. It is defined as the ratio of the inertia and viscous forces on the fluid.

Let us rewrite the Navier-Stokes equations (3.4), (3.7), (3.8), (3.9) in non-dimensional form.

Let \( U \) the characteristic velocity scale of the flow under study, \( L \) the characteristic length scale and \( T_1 \) a characteristic time (which is a priori equal to \( L/U \)), we put

\[
\begin{align*}
  u' &= \frac{u}{U}; \\
  v' &= \frac{v}{U}; \\
  w' &= \frac{w}{U}; \\
  x' &= \frac{x}{L}; \\
  y' &= \frac{y}{L}; \\
  z' &= \frac{z}{L}; \\
  t' &= \frac{Ut}{L}; \\
  p' &= \frac{p}{U^2}; \\
  \nu' &= \frac{\nu}{LU}.
\end{align*}
\]

To simplify the notation, the primes are dropped, and the non-dimensional form of the Navier-Stokes equations (3.4), (3.7), (3.8), (3.9) are respectively

\[
\begin{align*}
  u_x + v_y + w_z &= 0 \quad (3.10) \\
  u_t + uu_x + vv_y + wv_z &= -p_x + Re^{-1} \nabla^2 u \quad (3.11) \\
  v_t + uv_x + vv_y + wv_z &= -p_y + Re^{-1} \nabla^2 v \quad (3.12) \\
  w_t + uw_x + vw_y + ww_z &= -p_z + Re^{-1} \nabla^2 w \quad (3.13)
\end{align*}
\]

where the Reynolds number \( Re \) is defined as

\[ Re = \frac{UL}{\nu}. \quad (3.14) \]

It is clear that \( Re \) compares the importance of inertia \( UL \) to the effects of viscosity, characterized by \( \nu \).
4 Lie Group Treatment

This section is on the treatment of the Navier-Stokes equations using the methods of Lie. We follow the approach by Robert Eugene Boisvert [2] and R.E. Boisvert et al. [3], in reducing the Navier-Stokes equations (3.10), (3.11), (3.12) and (3.13), to the steady state. We try to establish an equivalence group of transformations of the Navier-Stokes equations, in order to find a solution and express it explicitly.

4.1 Equivalence Group of Transformations

We consider the model (3.10) to (3.13) for viscous Newtonian flow obtained previously:

\[ \nu \nabla^2 u - p_x - (u_t + uu_x + vu_y + wu_z) = 0 \] (4.1)

\[ \nu \nabla^2 v - p_y - (v_t + uv_x + vv_y + wv_z) = 0 \] (4.2)

\[ \nu \nabla^2 w - p_z - (w_t + uw_x + vw_y + ww_z) = 0 \] (4.3)

subject to the incompressibility condition

\[ u_x + v_y + w_z = 0 \] (4.4)

where \( \nu = Re^{-1} \) is the kinematic viscosity.

In order to find the Lie algebra, \( L \), admitted by these equations, we apply the second extension, \( \chi^2 \), of the generator operator, \( \chi \), of the form \( \chi = \sum_i \xi_i \frac{\partial}{\partial q_i} + \sum_j \eta_j \frac{\partial}{\partial \phi_j} \), to each equation; we therefore obtain the invariance conditions written as

\[ \chi^2 \left[ \nu \nabla^2 u - p_x - (u_t + uu_x + vu_y + wu_z) \right] = 0 \] (4.5)

\[ \chi^2 \left[ \nu \nabla^2 v - p_y - (v_t + uv_x + vv_y + wv_z) \right] = 0 \] (4.6)

\[ \chi^2 \left[ \nu \nabla^2 w - p_z - (w_t + uw_x + vw_y + ww_z) \right] = 0 \] (4.7)

subject to the incompressibility condition invariance

\[ \chi^2 [u_x + v_y + w_z] = 0 \] (4.8)

whenever (4.1) to (4.4) are verified.

In fact we look for operators \( \chi = \sum_i \xi_i \frac{\partial}{\partial q_i} + \sum_j \eta_j \frac{\partial}{\partial \phi_j} \), that take the form

\[ \chi = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \eta^3 \frac{\partial}{\partial w} + \eta^4 \frac{\partial}{\partial p} \] (4.9)
where we have considered the variables $t$, $x$, $y$, and $z$ as independent variables and $u$, $v$, $w$ and $p$ as differential variables on the space $(t, x, y, z)$. The coordinates $\xi^1$, $\xi^2$, $\xi^3$, $\chi^1$, $\eta^1$, $\eta^2$, $\eta^3$ and $\eta^4$ of the operator (4.9) are sought as functions of $t$, $x$, $y$, $z$, $u$, $v$, $w$ and $p$. The first extension, $\chi^1$, and the second extension, $\chi^2$, of $\chi$ respectively take the form:

$$\chi^1 = \chi + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_3 \frac{\partial}{\partial u_y} + \zeta_4 \frac{\partial}{\partial u_z} + \zeta_5 \frac{\partial}{\partial v_t} + \zeta_6 \frac{\partial}{\partial v_x} + \zeta_7 \frac{\partial}{\partial v_y} + \zeta_8 \frac{\partial}{\partial v_z}$$

$$+ \zeta_9 \frac{\partial}{\partial w_t} + \zeta_{10} \frac{\partial}{\partial w_x} + \zeta_{11} \frac{\partial}{\partial w_y} + \zeta_{12} \frac{\partial}{\partial w_z} + \zeta_{13} \frac{\partial}{\partial p_t} + \zeta_{14} \frac{\partial}{\partial p_x} + \zeta_{15} \frac{\partial}{\partial p_y} + \zeta_{16} \frac{\partial}{\partial p_z}$$

and

$$\chi^2 = \chi + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_3 \frac{\partial}{\partial u_y} + \zeta_4 \frac{\partial}{\partial u_z} + \zeta_5 \frac{\partial}{\partial v_t} + \zeta_6 \frac{\partial}{\partial v_x} + \zeta_7 \frac{\partial}{\partial v_y} + \zeta_8 \frac{\partial}{\partial v_z}$$

$$+ \zeta_9 \frac{\partial}{\partial w_t} + \zeta_{10} \frac{\partial}{\partial w_x} + \zeta_{11} \frac{\partial}{\partial w_y} + \zeta_{12} \frac{\partial}{\partial w_z} + \zeta_{13} \frac{\partial}{\partial p_t} + \zeta_{14} \frac{\partial}{\partial p_x} + \zeta_{15} \frac{\partial}{\partial p_y} + \zeta_{16} \frac{\partial}{\partial p_z}$$

$$+ \zeta_{17} \frac{\partial}{\partial u_{yy}} + \zeta_{18} \frac{\partial}{\partial u_{yz}} + \zeta_{19} \frac{\partial}{\partial u_{zz}}$$

$$+ \zeta_{20} \frac{\partial}{\partial v_{tt}} + \zeta_{21} \frac{\partial}{\partial v_{tx}} + \zeta_{22} \frac{\partial}{\partial v_{ty}} + \zeta_{23} \frac{\partial}{\partial v_{tz}} + \zeta_{24} \frac{\partial}{\partial v_{xy}} + \zeta_{25} \frac{\partial}{\partial v_{xz}}$$

$$+ \zeta_{26} \frac{\partial}{\partial v_{yy}} + \zeta_{27} \frac{\partial}{\partial v_{yz}} + \zeta_{28} \frac{\partial}{\partial v_{zz}}$$

$$+ \zeta_{29} \frac{\partial}{\partial w_{tt}} + \zeta_{30} \frac{\partial}{\partial w_{tx}} + \zeta_{31} \frac{\partial}{\partial w_{ty}} + \zeta_{32} \frac{\partial}{\partial w_{tz}} + \zeta_{33} \frac{\partial}{\partial w_{xy}} + \zeta_{34} \frac{\partial}{\partial w_{xz}}$$

$$+ \zeta_{35} \frac{\partial}{\partial w_{yy}} + \zeta_{36} \frac{\partial}{\partial w_{yz}} + \zeta_{37} \frac{\partial}{\partial w_{zz}}$$

$$+ \zeta_{38} \frac{\partial}{\partial p_{tt}} + \zeta_{39} \frac{\partial}{\partial p_{tx}} + \zeta_{40} \frac{\partial}{\partial p_{ty}} + \zeta_{41} \frac{\partial}{\partial p_{tz}} + \zeta_{42} \frac{\partial}{\partial p_{xy}} + \zeta_{43} \frac{\partial}{\partial p_{xz}}$$

$$+ \zeta_{44} \frac{\partial}{\partial p_{yy}} + \zeta_{45} \frac{\partial}{\partial p_{yz}} + \zeta_{46} \frac{\partial}{\partial p_{zz}},$$

(4.10)

where

$$\zeta^i_j = D_j\eta^i - \sum_k \phi^k_j D_k\xi^k; \quad \zeta^i_{jk} = D_k\zeta^i_j - \sum_l \phi^l_j D_k\xi^l.$$
\[ D_j = \frac{\partial}{\partial q^j} + \sum_k \phi_j^k \frac{\partial}{\partial \phi^k} + \sum_{i,k} \phi_{ij}^k \frac{\partial}{\partial \phi_i^k} \]  \hspace{1cm} (4.11)

\((j = 1, 2, 3, 4),\)

with

\[ q = (t, x, y, z), \quad \phi = (u, v, w, p), \quad \phi_i^k = \frac{\partial \phi^k}{\partial q^i}, \quad \phi_{ij}^k = \frac{\partial \phi^k}{\partial q^i} \]

\((i, j, k = 1, 2, 3, 4).\)

After extending the determining equations (4.5) to (4.8), we find all the generators admitted by the Navier-Stokes equations (4.1) to (4.4), see Birkhoff [1] or Wilzynski [33].

Note that, in the case of two independent variables \((x, y)\), Lie group is defined (see Olver [20]), to be the group of transformations in the plane \((x, y)\) given by

\[
\bar{x} = f(x, y, a) \approx x + a\xi(x, y), \quad \bar{y} = \varphi(x, y, a) \approx y + a\eta(x, y) \hspace{1cm} (4.12)
\]

depending on a parameter \(a\), and where we have taken a linear part (in the parameter \(a\)) in the Taylor expansion of the initial transformations (called finite transformations).

As shown in the transformation (4.12) together with the generator (4.9): \( \chi = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \eta^3 \frac{\partial}{\partial w} + \eta^4 \frac{\partial}{\partial p} \), we look for the group of transformations of the forms

\[
\begin{align*}
\bar{t} &= t + \varepsilon \xi^1(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{x} &= x + \varepsilon \xi^2(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{y} &= y + \varepsilon \xi^3(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{z} &= z + \varepsilon \xi^4(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{u} &= u + \varepsilon \eta^1(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{v} &= v + \varepsilon \eta^2(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{w} &= w + \varepsilon \eta^3(t, x, y, z, u, v, w, p) + O(\varepsilon^2), \\
\bar{p} &= p + \varepsilon \eta^4(t, x, y, z, u, v, w, p) + O(\varepsilon^2),
\end{align*}
\]  \hspace{1cm} (4.13)

which leave the Navier-Stokes equations (4.1) to (4.4) invariant. Boisvert [2] proved that, this group (called the full group) is obtained by the transformations (4.13) with

\[ \xi^1 = \alpha + 2\beta t \hspace{1cm} (4.14) \]

\[ \xi^2 = \beta x - \gamma y - \lambda z + f(t) \hspace{1cm} (4.15) \]

\[ \xi^3 = \beta y + \gamma x - \sigma z + g(t) \hspace{1cm} (4.16) \]
\[
\xi^4 = \beta z + \lambda x + \sigma y + h(t) \quad (4.17)
\]
\[
\eta^1 = -\beta u - \gamma v - \lambda w + f'(t) \quad (4.18)
\]
\[
\eta^2 = -\beta v + \gamma u - \sigma w + g'(t) \quad (4.19)
\]
\[
\eta^3 = -\beta w + \lambda u + \sigma v + h'(t) \quad (4.20)
\]
\[
\eta^4 = -2\beta p + j(t) - xf''(t) - yg''(t) - zh''(t) \quad (4.21)
\]

where \( \alpha, \beta, \gamma, \lambda \) and \( \sigma \) are five arbitrary parameters and \( f(t), g(t), h(t), \) and \( j(t) \) are arbitrary, sufficiently smooth, functions of \( t \). Each of the arbitrary parameters corresponds to the well known transformation. The parameter \( \alpha \) corresponds to a translation with respect to time, \( t \); \( \beta \) represents a stretching (dilatation) in all coordinates; \( \gamma, \lambda, \sigma \) represent a space rotation. With \( f(t), g(t) \) and \( h(t) \) as constants, it is clear that translations in the various coordinate directions are also included. Moving coordinate transformations are also included as long as these changes are reflected in \( \eta^1, \eta^2, \eta^3, \eta^4 \), as shown in (4.18),(4.19),(4.20),(4.21).

From the generator (4.9): \( \chi = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} + \eta^3 \frac{\partial}{\partial w} + \eta^4 \frac{\partial}{\partial p} \), we find the infinitesimal operator associated with each parameter by setting the studied parameter equal to one, while all other parameters and arbitrary functions are equal to zero. Then we obtain the following generators:

- translation with respect to time, \( t \) (associated with \( \alpha \))
  \[
  \chi_1 = \frac{\partial}{\partial t}, \quad (4.22)
  \]

- scale (dilatation) transformation (associated with \( \beta \))
  \[
  \chi_2 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} - w \frac{\partial}{\partial w} - 2p \frac{\partial}{\partial p}, \quad (4.23)
  \]

- space rotations (associated with \( \gamma, \lambda, \sigma \))
  \[
  \chi_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}, \quad (4.24)
  \]
  \[
  \chi_4 = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} + u \frac{\partial}{\partial w} - w \frac{\partial}{\partial u}, \quad (4.25)
  \]
  \[
  \chi_5 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} + w \frac{\partial}{\partial v} - v \frac{\partial}{\partial w}, \quad (4.26)
  \]

- moving coordinates (associated with the arbitrary functions) and obtained
in the forms:

\[ \chi_{6} = f(t) \frac{\partial}{\partial x} + f'(t) \frac{\partial}{\partial u} - xf''(t) \frac{\partial}{\partial p}, \quad (4.27) \]

\[ \chi_{7} = g(t) \frac{\partial}{\partial y} + g'(t) \frac{\partial}{\partial v} - yg''(t) \frac{\partial}{\partial p}, \quad (4.28) \]

\[ \chi_{8} = h(t) \frac{\partial}{\partial z} + h'(t) \frac{\partial}{\partial w} - zg''(t) \frac{\partial}{\partial p}, \quad (4.29) \]

- pressure changes

\[ \chi_{9} = j(t) \frac{\partial}{\partial p}. \quad (4.30) \]

The operators (4.22) to (4.26) generate a finite-dimensional Lie algebra, called \( L_{5} \), which is a five-dimensional subalgebra of the infinite-dimensional algebra \( L_{\infty} \) generated by the operators (4.22) to (4.30).

### 4.2 Solutions of the Navier-Stokes equations

We are now able to find a solution of the three-dimensional Navier-Stokes equations (4.1) to (4.4) by utilizing a different subgroup of the full group (4.14) to (4.21) with, for simplicity, \( \beta = \gamma = \lambda = \sigma = 0 \) and \( \alpha = 1 \). This subgroup becomes

\[ \xi^1 = 1; \quad \xi^2 = f(t); \quad \xi^3 = g(t); \quad \xi^4 = h(t) \]

\[ \eta^1 = f'(t); \quad \eta^2 = g'(t); \quad \eta^3 = h'(t); \quad \eta^4 = j(t) - xf''(t) - yg''(t) - zh''(t) \]

and has the associated operator (4.9):

\[ \chi = \frac{\partial}{\partial t} + f(t) \frac{\partial}{\partial x} + g(t) \frac{\partial}{\partial y} + h(t) \frac{\partial}{\partial z} + f'(t) \frac{\partial}{\partial u} + g'(t) \frac{\partial}{\partial v} + h'(t) \frac{\partial}{\partial w} + \]

\[ + j(t) - xf''(t) - yg''(t) - zh''(t) \frac{\partial}{\partial p}. \]

Now we can utilize the useful result mentioned in Boisvert et al. [3] which states that:

Any steady-state solution to the three-dimensional equations can be transformed by means of

\[ \tilde{x} = x - F(t), \quad \tilde{y} = y - G(t), \quad \tilde{z} = z - H(t) \quad (4.31) \]

with

\[ u = \tilde{u}(\tilde{x}, \tilde{y}, \tilde{z}) + f(t), \quad v = \tilde{v}(\tilde{x}, \tilde{y}, \tilde{z}) + g(t), \quad w = \tilde{w}(\tilde{x}, \tilde{y}, \tilde{z}) + h(t) \]
\[ p = \tilde{p}(\tilde{x}, \tilde{y}, \tilde{z}) - xf'(t) - yg'(t) - zh'(t) + k(t), \]  

where \( F' = f \), \( G' = g \), \( H' = h \), \( k = \frac{1}{2}[f^2 + g^2 + h^2] + \int j \, dt \), into a time-dependent solution involving four arbitrary functions of time variable. Then, the transformations (4.31)-(4.32) yield:

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial \tilde{x}} &= \frac{\partial \tilde{u}}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial t} + \frac{\partial \tilde{u}}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial t} + \frac{\partial \tilde{u}}{\partial \tilde{z}} \frac{\partial \tilde{z}}{\partial t} + f'(t) \\
&= -\tilde{u}_x F' - \tilde{u}_y G' - \tilde{u}_z H' + f'(t) \\
&= -(\tilde{u}_x f + \tilde{u}_y g + \tilde{u}_z h) + f'(t), \\
\frac{\partial \tilde{p}}{\partial \tilde{x}} &= \frac{\partial \tilde{p}}{\partial \tilde{x}} - f'(t) \\
&= \tilde{p}_x - f'(t), \quad \left( \frac{\partial \tilde{x}}{\partial x} = 1 \right) \\
\frac{\partial u}{\partial x} &= \frac{\partial (\tilde{u} + f)}{\partial x} \\
&= \frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{u}}{\partial \tilde{x}} \\
&= \tilde{u}_x, \quad \left( \frac{\partial (\tilde{y} or \ \tilde{z})}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial x} = 0 \right).
\end{align*}
\]

In the same manner, \( u_y = \tilde{u}_y \), \( u_z = \tilde{u}_z \), and

\[
\nabla^2 u = (u_{xx} + u_{xx} + u_{xx}) \\
= (\tilde{u}_{xx} + \tilde{u}_{yy} + \tilde{u}_{zz}).
\]

We do the same for the \( v \)- and \( w \)-components.

After substituting into the time-dependent Navier-Stokes equations (4.1) to (4.4), we find that, the functions \( \tilde{u}, \ \tilde{v} \) and \( \tilde{w} \) satisfy the steady Navier-Stokes equations:

\[
\begin{align*}
\tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}} + \tilde{w}_{\tilde{z}} &= -\tilde{p}_x + \nu[\tilde{u}_{\tilde{xx}} + \tilde{u}_{\tilde{yy}} + \tilde{u}_{\tilde{zz}}], \\
\tilde{u}_{\tilde{y}} + \tilde{v}_{\tilde{y}} + \tilde{w}_{\tilde{z}} &= -\tilde{p}_y + \nu[\tilde{v}_{\tilde{xx}} + \tilde{v}_{\tilde{yy}} + \tilde{v}_{\tilde{zz}}], \\
\tilde{w}_{\tilde{x}} + \tilde{u}_{\tilde{y}} + \tilde{w}_{\tilde{z}} &= -\tilde{p}_z + \nu[\tilde{w}_{\tilde{xx}} + \tilde{w}_{\tilde{yy}} + \tilde{w}_{\tilde{zz}}], \\
\tilde{u}_x + \tilde{v}_y + \tilde{w}_z &= 0.
\end{align*}
\]  

Another interesting result of the transformation (mentioned in the same Boisvert et al. [3]), is that, different subgroups of the reduced, (time-independent) full group may now be used to study (4.33), and transform it into a system of ordinary differential equations. Consequently, from the full group (4.14) to (4.21), it follows that the dilatation
where \( \Gamma, \Lambda, \Phi, \Omega \) satisfy the partial differential equations; and leading to the invariants

\[
\eta^1 = -u; \quad \eta^2 = -v; \quad \eta^3 = -w; \quad \eta^4 = -2p
\]

and has the associated operator (4.9):

\[
\chi = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} - w \frac{\partial}{\partial w} - 2p \frac{\partial}{\partial p}.
\]

We obtain the invariants, \( I \), of this subgroup, by integrating the associated \( \chi I = 0 \). Then the characteristic equations (Boisvert [3]) are given by

\[
\frac{d\tilde{x}}{\tilde{x}} = \frac{d\tilde{y}}{\tilde{y}} = \frac{d\tilde{z}}{\tilde{z}} = \frac{d\tilde{u}}{\tilde{u}} = \frac{d\tilde{v}}{\tilde{v}} = \frac{d\tilde{w}}{\tilde{w}} = \frac{d\tilde{p}}{-2\tilde{p}}
\]

leading to the invariants

\[
\eta_1 = \tilde{y}/\tilde{x}, \quad \eta_2 = \tilde{z}/\tilde{x}
\]

and

\[
\tilde{u} = \tilde{x}^{-1}\Gamma(\eta_1, \eta_2), \quad \tilde{v} = \tilde{x}^{-1}\Lambda(\eta_1, \eta_2)
\]

\[
\tilde{w} = \tilde{x}^{-1}\Phi(\eta_1, \eta_2), \quad \tilde{p} = \tilde{x}^{-2}\Omega(\eta_1, \eta_2)
\]

where \( \Gamma, \Lambda, \Phi, \Omega \) satisfy the partial differential equations;

\[
\begin{align*}
-\Gamma^2 - \eta_1 \Gamma \eta_1 - \eta_2 \Gamma \eta_2 + \Lambda \Gamma \eta_1 + \Phi \Gamma \eta_2 - 2\Omega - \eta_1 \Omega \eta_1 - \eta_2 \Omega \eta_2 \\
-\nu(2\Gamma + 4\eta_1 \Gamma \eta_1 + 4\eta_2 \Gamma \eta_2 + \Gamma \eta_1 + \Gamma \eta_2 + \eta_1^2 \Gamma \eta_1 + 2\eta_1 \eta_2 \Gamma \eta_1 + \eta_2^2 \Gamma \eta_2) = 0,
\end{align*}
\]

\[
\begin{align*}
-\Gamma \Lambda - \eta_1 \Gamma \Lambda \eta_1 - \eta_2 \Gamma \Lambda \eta_2 + \Lambda \Lambda \eta_1 + \Phi \Lambda \eta_2 + \Omega \eta_1 - \nu(2\Lambda + 4\eta_1 \Lambda \eta_1) \\
+4\eta_2 \Lambda \eta_2 + \Lambda \eta_1 + \Lambda \eta_2 + \eta_1^2 \Lambda \eta_1 + 2\eta_1 \eta_2 \Lambda \eta_1 + \eta_2^2 \Lambda \eta_2 = 0,
\end{align*}
\]

(4.34)

\[
\begin{align*}
-\Gamma \Phi - \eta_1 \Gamma \Phi \eta_1 - \eta_2 \Gamma \Phi \eta_2 + \Lambda \Phi \eta_2 + \Phi \Phi \eta_2 + \Omega \eta_2 - \nu(2\Phi + 4\eta_1 \Phi \eta_1) \\
+4\eta_2 \Phi \eta_2 + \Phi \eta_1 + \Phi \eta_2 + \eta_1^2 \Phi \eta_1 + 2\eta_1 \eta_2 \Phi \eta_1 + \eta_2^2 \Phi \eta_2 = 0,
\end{align*}
\]

(4.35)

However, no further group reduction is possible (Boisvert et al. [3]). But by setting \( \eta = \eta_1 - \eta_2 \), the system (4.34) is reduced to the system of ordinary differential equations;

\[
\begin{align*}
-\Gamma^2 - \eta \Gamma \eta + \Lambda \eta - \Phi \eta - 2\Omega - \eta \Gamma \eta - \nu(2\Gamma + 4\eta \Gamma \eta + 2\Gamma \eta + \eta^2 \Lambda \eta) = 0,
\end{align*}
\]

(4.36)

\[
\begin{align*}
-\Gamma \Lambda - \eta \Gamma \Lambda \eta + \Lambda \Lambda \eta + \Phi \Lambda \eta + \Omega \eta - \nu(2\Lambda + 4\eta \Lambda \eta + 2\Lambda \eta + \eta^2 \Lambda \eta) = 0,
\end{align*}
\]

(4.37)

\[
\begin{align*}
-\Gamma \Phi - \eta \Gamma \Phi \eta + \Lambda \Phi \eta - \Phi \Phi \eta - \Omega \eta - \nu(2\Phi + 4\eta \Phi \eta + 2\Phi \eta + \eta^2 \Phi \eta) = 0,
\end{align*}
\]

(4.38)
The last of these is satisfied when
\[ \Lambda - \Phi = \eta \Gamma - c_1 = 0, \] (4.39)
where \( c_1 \) is an arbitrary constant. The substitution of (4.39) into (4.35) yields
\[ -\Gamma^2 - c_1 \Gamma \eta - 2\Omega - \eta \Omega_\eta - \nu(2\Gamma + 4\eta \Gamma_\eta + 2\Gamma_\eta \eta + \eta^2 \Lambda_\eta \eta) = 0 \] (4.40)
The substitution of (4.39) into (4.36) and (4.37) and then subtracting yields
\[ -\eta \Gamma^2 + 2\Omega_\eta - \nu(6\eta \Gamma + 4\Gamma_\eta + 6\eta^2 \Gamma_\eta + 2\eta \Gamma_\eta \eta + \eta^3 \Gamma_\eta \eta) = 0. \]
Solving the latter equation for \( \Omega_\eta \) yields
\[ \Omega_\eta = \frac{1}{2}[\eta \Gamma^2 + \nu(6\eta \Gamma + 4\Gamma_\eta + 6\eta^2 \Gamma_\eta + 2\eta \Gamma_\eta \eta + \eta^3 \Gamma_\eta \eta)] \] (4.41)
and replacing it into (4.40), gives
\[ \Omega = \frac{1}{2}[-\Gamma^2 - \frac{1}{2}\eta^2 \Gamma^2 - c_1 \Gamma \eta - \nu(2\Gamma + 3\eta^2 \Gamma + 6\eta \Gamma_\eta \\
+ 3\eta^3 \Gamma_\eta + 2\Gamma_\eta \eta + 2\eta^2 \Gamma_\eta \eta + \frac{1}{2}\eta^4 \Gamma_\eta \eta)]. \] (4.42)
The differentiation of equation (4.42) with respect to \( \eta \) and setting it equal to (4.41) implies that
\[ 2\eta \Gamma^2 + 2\Gamma \Gamma_\eta + \eta^2 \Gamma \Gamma_\eta + c_1 \Gamma_\eta \eta + \nu(12\eta \Gamma + 12\Gamma_\eta + 18\eta^2 \Gamma_\eta \\
+ 12\eta \Gamma_\eta \eta + 6\eta^3 \Gamma_\eta \eta + 2\Gamma_\eta \eta \eta + 2\eta^2 \Gamma_\eta \eta \eta + \frac{1}{2}\eta^4 \Gamma_\eta \eta \eta) = 0 \] (4.43)
One solution of (4.43) is
\[ \Gamma = -6\nu. \] (4.44)
The corresponding values for \( \Lambda \) and \( (\Lambda - \Phi) \) from (4.42) and (39) are
\[ \Lambda = -12\nu. \] (4.45)
\[ \Lambda - \Phi = -6\nu \eta - c_1. \] (4.46)
Substitution of (4.44), (4.45), (4.46) into (4.37) results in
\[ 4\Phi - \left[ \frac{c_1}{\nu} + 4\eta \right] \Phi_\eta - (\eta^2 + 2) \Phi_\eta \eta = 0, \]
whose general solution, for the case \( c_1 = 0 \), is
\[ \Phi = c_2 \nu \eta - c_3 \nu \left[ \frac{1}{4} + \frac{1}{8} \eta^2 (\eta^2 + 2)^{-1} - \frac{3\eta}{8\sqrt{2}} \arctan \left( \frac{\eta}{\sqrt{2}} \right) \right]. \] (4.47)
and the substitution into (4.46) yields

\[ \Lambda = -6\nu \eta + c_2 \nu \eta - c_3 \nu \left[ \frac{1}{4} + \frac{1}{8} \eta^2 (\eta^2 + 2)^{-1} - \frac{3\eta}{8\sqrt{2}} \arctan \left( \frac{\eta}{\sqrt{2}} \right) \right]. \]  

(4.48)

Using these last expressions, together with the relations (4.31)-(4.32), we rewrite the sought solutions \( u, v, w \) and \( p \) in the original variables, which leads to the solution of the unsteady three-dimensional Navier-Stokes equations (4.1) to (4.4). We will also use \( \eta \) as follows:

\[ \eta = \eta_1 - \eta_2 = \frac{\tilde{y}}{\tilde{x}} - \frac{\tilde{z}}{\tilde{x}} = \tilde{x}^{-1} (\tilde{y} - \tilde{z}) = (x - F(t))^{-1} (y - F(t) - z + F(t)) = (x - F(t))^{-1} R \]

with \( R = y - F(t) - z + F(t) \).

Then,

\[ u = \tilde{u} + f(t) \]
\[ = \tilde{x}^{-1} \Gamma + f(t) \]
\[ = (x - F(t))^{-1} \Gamma + f(t) \]
\[ u = -6\nu (x - F(t))^{-1} + f(t) \]  

(4.49)

\[ v = \tilde{v} + g(t) \]
\[ = \tilde{x}^{-1} \Lambda + g(t) \]
\[ = (x - F(t))^{-1} \Lambda + g(t) \]
\[ \begin{aligned}
\end{aligned} \]

\[ = x - F(t)^{-1} \left\{ -6\nu (x - F(t))^{-1} R + c_2 \nu (x - F(t))^{-1} R - c_3 \nu \left[ \frac{1}{4} \right. \right. \]
\[ + \frac{1}{8} (x - F(t))^{-2} R^2 ((x - F(t))^{-2} R^2 + 2)^{-1} - \frac{3(x - F(t))^{-1} R}{8\sqrt{2}} \arctan \left( \frac{(x - F(t))^{-1} R}{\sqrt{2}} \right) \left] \right\} + g(t), \]

\[ v = \nu \left\{ (c_2 - 6)(x - F(t))^{-2} R - c_3 \left[ \frac{1}{4} (x - F(t))^{-1} \right. \right. \]
\[ + \frac{1}{8} (x - F(t))^{-3} R^2 ((x - F(t))^{-2} R^2 + 2)^{-1} - \frac{3}{8\sqrt{2}} (x - F(t))^{-1} R \]
\[ \times \arctan \left( \frac{(x - F(t))^{-1} R}{\sqrt{2}} \right) \left] \right\} + g(t), \]  

(4.50)

\[ w = \tilde{w} + h(t) \]
\[ = \tilde{x}^{-1} \Phi + h(t) \]
\[ = (x - F(t))^{-1} \Phi + h(t) \]
\[
= x - F(t)^{-1} \left\{ c_2 \nu(x - F(t))^{-1} R - c_3 \nu \left[ \frac{1}{4} \right] \right.
+ \frac{1}{8}(x - F(t))^{-2}R^2((x - F(t))^{-2}R^2 + 2)^{-1} - \frac{3(x - F(t))^{-1}R}{8\sqrt{2}} \arctan \left[ \frac{(x - F(t))^{-1}R}{\sqrt{2}} \right] \left\} + g(t),
\]

\[
w = \nu \left\{ c_2(x - F(t))^{-2}R - c_3 \left[ \frac{1}{4} (x - F(t))^{-1} \right. \right.
+ \frac{1}{8}(x - F(t))^{-3}R^2((x - F(t))^{-2}R^2 + 2)^{-1} - \frac{3}{8\sqrt{2}}(x - F(t))^{-1}R
\times \arctan \left[ \frac{(x - F(t))^{-1}R}{\sqrt{2}} \right] \left\} + h(t) \right. (4.51)
\]

and

\[
p = \bar{p} - xf'(t) - yg'(t) - zh'(t) + k(t)
= \ddot{x}^{-2} \Omega - xf'(t) - yg'(t) - zh'(t) + k(t)
= (x - F(t))^{-2}\Omega - xf'(t) - yg'(t) - zh'(t) + k(t)
\]

\[
p = -12\nu^2(x - F(t))^{-2} - xf'(t) - yg'(t) - zh'(t) + k(t). \quad (4.52)
\]

In terms of the Reynolds number, these solutions become:

\[
u = -6 \frac{1}{Re} (x - F(t))^{-1} + f(t) \quad (4.53)
\]

\[
v = \frac{1}{Re} \left\{ (c_2 - 6)(x - F(t))^{-2}R - c_3 \left[ \frac{1}{4} (x - F(t))^{-1} \right. \right.
+ \frac{1}{8}(x - F(t))^{-3}R^2((x - F(t))^{-2}R^2 + 2)^{-1} - \frac{3}{8\sqrt{2}}(x - F(t))^{-1}R
\times \arctan \left[ \frac{(x - F(t))^{-1}R}{\sqrt{2}} \right] \left\} + g(t) \right. (4.54)
\]

\[
w = \frac{1}{Re} \left\{ c_2(x - F(t))^{-2}R - c_3 \left[ \frac{1}{4} (x - F(t))^{-1} \right. \right.
+ \frac{1}{8}(x - F(t))^{-3}R^2((x - F(t))^{-2}R^2 + 2)^{-1} - \frac{3}{8\sqrt{2}}(x - F(t))^{-1}R
\times \arctan \left[ \frac{(x - F(t))^{-1}R}{\sqrt{2}} \right] \left\} + h(t) \right. (4.55)
\]

\[
p = -12 \left( \frac{1}{Re} \right)^2 (x - F(t))^{-2} - xf'(t) - yg'(t) - zh'(t) + k(t) \quad (4.56)
\]

All these solutions of the equations governing the flow are expressed in their explicit forms, and one can see that the Reynolds number, \( Re \), clearly appears. This proves the fact that the Reynolds number influences the three types of fluid flow’s regimes observed.
experimentally. As example, the time-dependent evolutions of the \( u \)- and \( v \)-velocity components are presented by Persillon and Braza [24], in both two and three-dimensional case, for Reynolds number 200 and 300. Their drawings (in the sixth part of the same article [24]) show the quasi-periodic character of the studied flow, and one can see that the amplitudes of the oscillations increase with the Reynolds number.

## 5 Finite Element Treatment

This section is on the treatment of the Navier-Stokes equations using the method of finite elements. We now consider the domain \( \Omega \) defined in the first chapter and the established model for viscous Newtonian flow (3.6), subject to body forces in this case, and which is given by the Navier-Stokes equations

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{Re}{\nabla^2} \mathbf{u} = \mathbf{f} & \quad \text{in } \Omega \times (0,T], \\
\nabla \cdot \mathbf{u} = 0 & \quad \text{in } \Omega \times (0,T], \\
\mathbf{u} = 0 & \quad \text{on } \Gamma_{\text{rigid}} \times (0,T], \\
\mathbf{u}^{\text{in}} = \mathbf{u} & \quad \text{on } \Gamma_{\text{in}}, \\
\mathbf{u}(x,0) = \mathbf{u}_0(x) & \quad \text{in } \Omega,
\end{align*}
\]

where \( \mathbf{f} \) is the body force per unit mass (note that we will assume that \( \mathbf{f} = 0 \)). \( \Gamma_{\text{rigid}} \) and \( \Gamma_{\text{in}} \) are the rigid part and the inflow part of the boundary \( \Gamma \), respectively. We assume that \( \Omega \) does not change in time. Before going further, let us outline some useful forms, norms and function spaces.

### 5.1 Function Spaces, Norms, and Forms

The finite element discretization of the Navier-Stokes problem is based on the variational formulation, and the use of Sobolev spaces is needed for the mathematical treatment of the variational formulation of the model. We use sub-spaces of the usual Hilbert space

\[
L^2(\Omega) = \left\{ f : \int_{\Omega} |f|^2 \, dx < \infty \right\}
\]
of square-integrable functions on $\Omega$, where integration is in the sense of Lebesgue.

\[ L^2_0(\Omega) = \{ f : f \in L^2(\Omega), \ (f, 1) = 0 \} , \]

and the corresponding inner products and norms

\[ (f, g) = \int_{\Omega} fg \, dx, \quad \|f\|_0 = (f, f)^{1/2} . \]

Next, for any non-negative integer $k$, we define the Sobolev space

\[ H^k(\Omega) = \{ f : f \in L^2(\Omega), \ D^s f \in L^2(\Omega), \ \text{for } s = 1, \ldots, k \} \]

of square integrable functions, all of whose derivatives of order up to $k$, are also square integrable, where $D^s$ denotes any and all derivatives of order $s$. $H^k(\Omega)$ comes with the norm

\[ \|f\|_k = \left( \|f\|_0^2 + \sum_{s \leq k, \ s \neq 0} \|D^s f\|_0^2 \right)^{1/2} . \]

The following definitions can now be stated:

\[ H^0(\Omega) = L^2(\Omega) \]

\[ H^1(\Omega) = \{ f : f \in L^2(\Omega), \ \partial_i f \in L^2(\Omega), \ 1 \leq i \leq 3 \} \]

\[ \|\nabla f\|_0 = (\nabla f, \nabla f)^{1/2} \]

\[ \|f\|_1 = \left( \|f\|_0^2 + \|\nabla f\|_0^2 \right)^{1/2} = \left( \|f\|_0^2 + \sum_{i=1}^{3} \left\| \frac{\partial f}{\partial x_i} \right\|_0^2 \right)^{1/2} . \]  \tag{5.6}

Of particular interest is the subspace of $H^1_0(\Omega)$ of $H^1(\Omega)$ defined by

\[ H^1_0(\Omega) = \{ f : f \in H^1(\Omega), \ f = 0 \ \text{on } \Gamma \} \]

whose elements vanish on the boundary $\Gamma$.

For functions belonging to $H^1(\Omega)$, the semi-norm

\[ |f|_1 = \left( \sum_{i=1}^{3} \left\| \frac{\partial f}{\partial x_i} \right\|_0^2 \right)^{1/2} \]  \tag{5.7}

defines a norm equivalent to (5.6). The proof of this statement is not our aim in this article, but it can be found in Dietrich Braess [4]. Thus for such functions, (5.7) may be used instead of (5.6).
We denote by $H^{-1}(\Omega)$ the dual space consisting of bounded linear functionals on $H^1_0(\Omega)$, i.e., $f \in H^{-1}(\Omega)$ implies that $(f, w) \in \mathbb{R}$ for all $w \in H^1_0(\Omega)$. A norm for $H^{-1}(\Omega)$ is given by

$$\|f\|_{-1} = \sup_{0 \neq w \in H^1_0(\Omega)} \frac{(f, w)}{|w|_1}$$

Since the velocity field $\mathbf{u} = \mathbf{u}(u, v, w) = (u_i)_{i=1,2,3}$ is a vector valued function, we use the spaces

$$H^k(\Omega) = H^k(\Omega)^3 = \{ \mathbf{u} : u_i \in H^k(\Omega) \text{ for } i = 1, 2, 3 \},$$

$$H^1_0(\Omega) = H^1_0(\Omega)^3 = \{ \mathbf{u} : u_i \in H^1_0(\Omega) \text{ for } i = 1, 2, 3 \},$$

and

$$H^{-1}(\Omega) = H^{-1}(\Omega)^3 = \{ \mathbf{u} : u_i \in H^{-1}(\Omega) \text{ for } i = 1, 2, 3 \},$$

For $k \geq 0$, $H^k(\Omega)$ is equipped with the norm

$$\|\mathbf{u}\|_k = \left( \sum_{i=1}^{3} \|u_i\|_k^2 \right)^{1/2}.$$

Alternatively, for functions belonging to $H^1_0(\Omega)$, we may use

$$|\mathbf{u}|_1 = \left( \sum_{i=1}^{3} |u_i|_1^2 \right)^{1/2}.$$

The inner product for functions belonging to $L^2(\Omega) = H^0(\Omega) = L^2(\Omega)^3$ is also given by

$$(\mathbf{u}, \mathbf{w}) = \int_{\Omega} \mathbf{u} \cdot \mathbf{w} \, dx.$$
2. If a solution exists, is it unique?

First, we have to clarify the notion of a solution of (5.1)-(5.5). There exists several concepts of the notion of a solution of the above system, the most important of which are the classical solution and the weak solution.

**Definition 5.1. (classical solution)**

A pair \((u, p)\) is called a classical solution of the Navier-Stokes (5.1)-(5.5) if:

1. \((u, p)\) satisfies the Navier-Stokes problem (5.1)-(5.5).
2. \(u\) and \(p\) are infinitely many times differentiable with respect to space and time variables.

Then, according to J. Volker and S. Kaya [29], the existence of a classical solution of (5.1) – (5.5) cannot yet be proven, but if a classical solution exists, it is unique.

To define a weak solution, we first need to transform (5.1) into a weak form by
- multiplying (5.1) with a suitable vector valued function \(\varphi\) (test function),
- integrating over \(\Omega \times (0, T]\),
- applying integration by parts (Green's theorem).

The last step is possible only if there are some restrictions on the domain. For the test function \(\varphi\), one requires
- \(\varphi \in C^\infty_0, \text{div}(\Omega)\) for each time \(t\), where \(C^\infty_0, \text{div}(\Omega) = \{f : f \in C^\infty_0(\Omega), \nabla \cdot f = 0\}\),
- \(\varphi\) is infinitely differentiable with respect to time,
- \(\varphi(\cdot, T) = 0\).

This gives the weak formulation of the Navier-Stokes equations

\[
\int_0^T \left[ -(u, \varphi_t) + (u \cdot \nabla u, \varphi) + Re^{-1}(\nabla u, \nabla \varphi) \right] dt = \int_0^T (f, \varphi) dt + (u_0, \varphi(\cdot, 0)).
\] (5.8)

which has the following features:
- There is no time derivative of \(u\)
- There is no second order spatial derivative with respect to \(u\)
- The pressure vanishes, since the Green's formula yields

\[
(\nabla p, \varphi) = \int_{\partial \Omega} p \varphi \cdot n \ ds - (p, \nabla \cdot \varphi) = 0
\]

because \(\varphi \cdot n = 0\) on \(\partial \Omega\) and \(\nabla \cdot \varphi = 0\).

**Definition 5.2. (weak solution)**

A function \(u\) is called weak solution of the Navier-Stokes equations if:
1. \( u \) satisfies (5.8) for all test functions \( \varphi \) with the properties on \( \varphi \) given above,

2. \( u \) has the following regularity

\[
\mathbf{u} \in L^2(0, T; H^1_{0, \text{div}}(\Omega)) \cap L^\infty(0, T; L^2_{\text{div}}(\Omega)),
\]

where the subscript \( \text{div} \) means space of divergence-free functions; for instance

\[
C^\infty_{0, \text{div}}(\Omega) = \{ f : f \in C^\infty_0(\Omega), \nabla \cdot f = 0 \}
\]

and

\[
L^2(0, T; H^1_0(\Omega)) = \left\{ f(x, t) : \int_0^T \| f \|_0^2 dt < \infty \right\}.
\]

More generally

\[
L^q(t_0, t_1; X) = \left\{ f(x, t) : \int_{t_0}^{t_1} \| f \|_X^q dt < \infty \right\} \quad \text{for any } q \in [1, \infty),
\]

is the space of strongly measurable maps \( f : [t_0, t_1] \to X \), such that

\[
\| f \|_{L^q(t_0, t_1; X)} = \left( \int_{t_0}^{t_1} \| f \|_X^q dt \right)^{1/q} < \infty \quad \text{for } q \in [1, \infty)
\]

and \( X \) is a Banach space. Furthermore

\[
L^\infty(t_0, t_1; X) = \left\{ f(x, t) : \text{ess sup}_{t_0 \leq t \leq t_1} \| f \|_X < \infty \right\}
\]

with

\[
\| f \|_{L^\infty(t_0, t_1; X)} = \text{ess sup}_{t_0 \leq t \leq t_1} \| f \|_X < \infty \quad \text{for } q = \infty.
\]

It is obvious that all these spaces are needed for the weak formulation given in the next section.

The existence of a weak solution of (5.1)-(5.5) was proved in 1934 by Jean Leray [13].

The weak solution is unique if every other weak solution satisfies an additional regularity assumption, Serrin’s condition, see J. Serrin [28], or J. Volker and S. Kaya [29]. But it is not known in 3-D if every weak solution possesses such additional condition.

According to the same article [29], the existence of a weak solution of the Navier-Stokes equations can be proven in arbitrary domains, but the uniqueness cannot yet be proven.

The answer to the question of uniqueness of the weak solution in 3-D, or existence of a classical solution in 3-D is one of the major mathematical challenges of this century (J. Volker and S. Kaya [29]). There is a prize of one million US-Dollars for people who can answer these questions.
5.3 A Galerkin-Type Weak Formulation

We introduce the bilinear forms

\[ a(u, w) = Re^{-1}(\nabla u, \nabla w) = Re^{-1} \int_{\Omega} \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \quad \text{for all } u, w \in H^1(\Omega) \] (5.9)

\[ b(p, u) = -(p, \nabla \cdot u) \quad \text{for all } u \in H^1(\Omega) \text{ and } p \in L^2(\Omega) \] (5.10)

and the trilinear form

\[ c(u, v, w) = (u \cdot \nabla v, w) = \int_{\Omega} \sum_{i,j=1}^{3} u_j \frac{\partial v_i}{\partial x_j} w_i, \quad \text{for all } u, v, w \in H^1(\Omega). \] (5.11)

In addition to the above spaces, we will need to use the space

\[ H = \{ u \in L^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega; \ u = 0 \text{ on } \Gamma \}, \]

which consists of (weakly) divergence free functions, i.e. functions whose divergence vanishes almost everywhere.

Recall that \( \partial \Omega = \Gamma = \Gamma_{\text{rigid}} \cup \Gamma_{\text{in}}, \) then, following the same procedure mentioned earlier of defining the weak solution, the weak (variational) formulation of the Navier-Stokes equations (5.1)-(5.5), reads as follows:

Given

\[ f \in L^2(0, T; H^{-1}(\Omega)) \text{ and } u_0 \in H, \]

find functions \( u \in L^2(0, T; H^1_0(\Omega)) \cup L^\infty(0, T; H) \) and \( p \in L^2[0, T; L^2_0(\Omega)] \) such that

\[
\begin{aligned}
\left( \frac{\partial u}{\partial t}, v \right) + a(u, v) + c(u, u, v) + b(p, v) &= (f, v) \quad \text{for all } v \in H^1_0(\Omega) \\
b(q, u) &= 0 \quad \text{for all } q \in L^2_0(\Omega) \\
\quad u(0, x) &= u_0(x) \quad \text{for } x \in \Omega
\end{aligned}
\] (5.12)

where the first two equations of (5.12) hold on \((0, T), \) in the sense of distributions.

5.4 Spacial Discretizations

To discretize the above problem with the spatial variables, we introduce the triangulation, named \( T_h, \) of \( \overline{\Omega}, \) with width \( h \) into (closed) cells \( K \) (tetrahedra) such that the following regularity conditions are satisfied:
• \(\Omega = \bigcup \{ K \in \mathbb{T}_h \}\).
• Any two cells \(K, K'\) only intersect in common faces, edges or vertices.
• The decomposition \(\mathbb{T}_h\) matches the decomposition \(\partial \Omega = \Gamma = \Gamma_{\text{rigid}} \cup \Gamma_{\text{in}}\).

On the finite element mesh \(\mathbb{T}_h\), one defines spaces of "discrete" trial and test functions with the following constructions:

For each \(h\), let \(W^h\) and \(Q^h\) be two finite-dimensional spaces such that

\[
W^h \subset H^1(\Omega), \quad Q^h \subset L^2(\Omega)
\]

and throughout this chapter we assume that \(Q^h\) contains the constant functions.

We set

\[
\begin{align*}
V^h_0 &= W^h \cap H^1_0(\Omega) = \{ v^h \in W^h : v^h = 0 \text{ on } \Gamma \} \\
S^h_0 &= Q^h \cap L^2_0(\Omega) = \{ q^h \in Q^h : \int_\Omega q^h dx = 0 \}
\end{align*}
\]

(5.13)

There are many pairs of these finite element spaces. Some of them are stable, others are not. Naturally, one would like to know which are best. It is generally thought that elements which at least yield elementwise mass conservation, are superior. This judgement is largely based on the examination of graphical representations of solutions, e.g. streamline plots. For details on the choice of pairs of finite element spaces, consult Gunzburger [8].

With these spaces, the finite element approximation of the problem (5.12) is given by:

Find a pair \((u^h, p^h) \in V^h_0 \times S^h_0\) such that

\[
\left( \frac{\partial u^h}{\partial t}, v^h \right) + a(u^h, v^h) + c(u^h, u^h, v^h) + b(p^h, v^h) = (f, v^h) \quad \text{for all } v^h \in V^h_0 \text{ and } t \in (0, T)
\]

(5.14)

\[
b(q^h, u^h) = 0 \quad \text{for all } q^h \in S^h_0 \text{ and } t \in (0, T)
\]

(5.15)

\[
u^h(0, x) = u^h_0 \in V^h_0 \quad \text{for } x \in \Omega,
\]

(5.16)

where \(u^h_0\) is an approximation to the initial function \(u^h(0, x)\).

In order that (5.14)-(5.16) is a stable approximation of (5.12) as \(h \to 0\), it is crucial that we relate the continuous and discrete spaces by the following hypotheses (for a complete and rigorous analysis of these approximations, refer to Girault and Raviart [6]):

**Hypothesis H1** (Approximation property of \(V^h_0\))

There exists an operator \(r^h \in \mathcal{L}(H^2(\Omega) \cap H^1_0(\Omega)^2; V^h_0)\) and an integer \(l\) such that

\[
\| \varphi - r^h \varphi \|_1 \leq Ch^n \| \varphi \|_{m+1} \quad \text{for all } \varphi \in H^{m+1}(\Omega), \ 1 \leq m \leq l.
\]

(5.17)

**Hypothesis H2** (Approximation property of \(Q^h\))

There exists an operator \(s^h \in \mathcal{L}(L^2(\Omega); Q^h)\) such that

\[
\| q - s^h q \|_0 \leq Ch^n \| q \|_m \quad \text{for all } q \in H^m(\Omega), \ 0 \leq m \leq l.
\]

(5.18)
Hypothesis H3 (Uniform inf-sup condition)
For each \( q^h \in S_0^h \), there exists a \( v^h \in V_0^h \) such that
\[
b(q^h, v^h) = \|q^h\|_0^2
\]
\[
|v^h|_1 \leq C\|q^h\|_0
\]
(5.19)
where the constant \( C > 0 \) is independent of \( h, q^h \) and \( v^h \); \( \mathcal{L}(Y, W) \) is the space of linear operators from \( Y \) to \( W \); \( \| \cdot \|_0 \) and \( \| \cdot \|_m \) are the standard norms in \( L^2(\Omega) \) and \( H^m(\Omega) \) respectively; \( | \cdot |_1 \) is the standard semi-norm in \( H^1(\Omega) \).

Now we may choose specific bases for \( V_0^h \) and \( S_0^h \) which are both finite-dimensional in such a way that the system (5.14)-(5.16) becomes equivalent to a system of non-linear ordinary differential equations with linear algebraic constraints. Indeed if \( \{q_j(x)\}_{j=1}^J \) and \( \{v_k(x)\}_{k=1}^K \) denote bases for \( S_0^h \) and \( V_0^h \), respectively, we can then write
\[
p^h(t, x) = \sum_{j=1}^J \alpha_j(t)q_j(x) \quad \text{and} \quad u^h(t, x) = \sum_{k=1}^K \beta_k(t)v_k(x).
\]
The system (5.14)-(5.16) is therefore equivalent to the system of ordinary differential equations
\[
\sum_{k=1}^K (v_k, v_l) \frac{d\beta_k}{dt} + \sum_{k=1}^K a(v_k, v_l)\beta_k(t) + \sum_{k,m=1}^K c(v_m, v_k, v_l)\beta_k(t)\beta_m(t)
\]
\[
+ \sum_{j=1}^J b(v_l, q_j(t))\alpha_j(t) = (f, v_l) \quad \text{for} \ l = 1, \ldots, K,
\]
(5.20)
with initial data \( \beta_k(0), \ k = 1, \ldots, K \) satisfying
\[
\sum_{k=1}^K v_k\beta_k(0) = u_0^h
\]
(5.21)
and are subject to the linear algebraic constraints
\[
\sum_{k=1}^K b(v_k, q_i)\beta_k(t) = 0 \quad \text{for} \ i = 1, \ldots, J.
\]
(5.22)
The system of ordinary differential equations (5.20), or equivalently (5.14)-(5.16), may now be discretized with respect to time. In this regard, it is convenient to rewrite the semi-discrete system (5.20) as
\[
\left( \frac{\partial u^h}{\partial t}, v^h \right) = F(f, u^h, p^h; v^h) \quad \text{for all} \ v^h \in V_0^h
\]
(5.23)
where the linear functional \( F(\ldots, ; v^h) \) is defined, for any \( u^h \in V_0^h \) and \( p^h \in S_0^h \) and any \( f \), by
\[
F(f, u^h, p^h; v^h) = (f, v^h) - a(u^h, v^h) - c(u^h, u^h, v^h) - b(p^h, v^h) \quad \text{for all} \ v^h \in V_0^h.
\]
(5.24)
5.5 Time Discretizations

There are many time discretization algorithms that fall into one of four classes of methods, (see Max D. Gunzburger [8]), namely single-step and multistep methods of both fully implicit and semi-implicit type. We will focus on the Crank-Nicolson scheme which is a single-step fully implicit method. Explicit methods are not in common use for time discretizations of the Navier-Stokes equations, because of their severe stability restriction. We apply the Crank-Nicolson extrapolation scheme to the time discretization of the system (5.20).

We subdivide the time interval \([0, T]\) into \(M\) intervals of uniform length \(\delta = \frac{T}{M}\). Throughout, \(u^m\) and \(p^m\), \(m = 0, \ldots, M\), will respectively denote approximations to \(u^h(m\delta, x)\) and \(p^h(m\delta, x)\) where \(u^h\) and \(p^h\) denote the solution of (5.14)-(5.16). Likewise, for \(m = 0, \ldots, M\), \(k = 1, \ldots, K\), and \(j = 1, \ldots, J\), \(\alpha^m_j\) and \(\beta^m_k\) denote approximations to \(\alpha_j(m\delta)\) and \(\beta_k(m\delta)\), respectively, where \(\alpha_j, j = 1, \ldots, J\), and \(\beta_k, k = 1, \ldots, K\), denote the solution of (5.20) and (5.22). Also, throughout, \(f^m = f(m\delta, x)\).

Given \(u^0\) (which may be chosen to be \(u_0\)), \(\{u^m, p^m\}\) for \(m = 0, \ldots, M\), are determined from

\[
\frac{1}{\delta}(u^m - u^{m-1}, v^h) = F(f^m_\theta, u^m_\theta, p^m_\theta; v^h) \quad \text{for all } v^h \in V^h_0 \tag{5.25}
\]

and

\[
b(q^h, u^m) = 0 \quad \text{for all } q^h \in S^h_0 \tag{5.26}
\]

where

\[
u^m_\theta = \frac{u^m - u^{m-1}}{2} \quad \text{and} \quad p^m_\theta = \frac{p^m - p^{m-1}}{2} \tag{5.27}
\]

and likewise for \(f^m_\theta\).

The scheme (5.25)-(5.26) requires an initial condition for the pressure since \(p^1\) is obtained from \(p^0_\theta\) and \(p^0\) through (5.27). Due to the fact that we are interested in the velocity field only, there is no need to compute \(p^1\), since the latter will not be used in the velocity computation at the next time level, i.e., \(t = 2\delta\). However, one may use Taylor’s theorem \(p^1 = p^0_\theta + O(\delta/2) + O(\delta^2)\) and therefore take \(p^1_\theta\) as the pressure approximation at \(t = \delta\), and again never need \(p^0\). This procedure results in a loss of accuracy in the pressure.

Clearly, (5.25)-(5.26) is a system of non-linear algebraic equations. In order to minimize the cost of computing each pair \((u^m, p^m)\), one should solve the equivalent problem

\[
\frac{2}{\delta}(u^m_\theta, v^h) - F(f^m_\theta, u^m_\theta, p^m_\theta; v^h) = \frac{2}{\delta}(u^{m-1}, v^h) \quad \text{for all } v^h \in V^h_0 \tag{5.28}
\]

and

\[
b(q^h, u^m_\theta) = \begin{cases} 
(1/2) b(q^h, u^0) & \text{if } m = 1 \\
0 & \text{if } m > 1
\end{cases} \quad \text{for all } q^h \in S^h_0 \tag{5.29}
\]
and then set
\[ u^m = 2u_{m}^0 - u^{m-1} \quad \text{and} \quad p^m = 2p_{m}^0 - p^{m-1}. \]  
(5.30)

Substituting (5.24) into (5.28) yields
\[ a(u_{m}^0, v^h) + c(u_{m}^0, u_{m}^0, v^h) + b(p_{m}^0, v^h) + \frac{2}{\delta}(u_{m}^0, v^h) = \]

\[ (f_{m}, v^h) + \frac{2}{\delta}(u_{m-1}^0, v^h) \quad \text{for all} \quad v^h \in V_0^h. \]  
(5.31)

When \( \delta \) is small, a good starting guess for any iterative method for solving (5.29) and (5.31) is the solution \( u^{m-1} \) at the previous time step. On the other hand, due to the fully implicit character of the scheme (5.25)-(5.27), a different non-linear system has to be solved for each \( m \).

As has been noted, the above scheme is second-order accurate with respect to \( \delta \), i.e., \( O(\delta^2) \). This observation is with respect to both the \( L^2(\Omega) \) and \( H_0^1(\Omega) \) - norms of the differences \( u(m\delta, .) - u^m, \ m = 1, ..., M \).

Keeping in mind the above scheme of the finite element method applied to the non stationary Navier-stokes equations, we may investigate the stability of the model being studied, in order to understand how a laminar flow may develop into a turbulent flow. Most of the traditional theory for fluid flow is of qualitative nature, based on eigenvalue criteria through a hydrodynamic stability argument. We investigate the case associated with energy stability analysis.

6 Energy Stability Analysis

In the global theory, energy methods have an important place. These methods lead to a variational problem for the first critical Reynolds number (or viscosity) of the energy theory, and to a definite criterion which is sufficient for the global stability of the (basic) flow.

Bear in mind that the procedure which follows, has already been mentioned in the introduction of this report. We consider the flow \( u \) in the domain \( \Omega \), which is mathematically represented by the non stationary Navier-stokes equations (5.1)-(5.5). For simplicity, we denote \( u(x, 0) = u_0(x) \) by \( U(x) \) and thus at \( t = 0 \), the flow \( u \) has the velocity field \( u(x, 0) = U(x) \). Suppose that at this instant, we perturb the flow with a perturbation \( w(x, 0) \). The subsequent departure of the perturbed flow from the given flow is denoted by \( w(x, t) \), so that the perturbed flow is henceforth given by \( u(x, t) + w(x, t) \), where
\( u \) denotes the subsequent unperturbed flow. Nothing is assumed concerning the size of the initial disturbance \( w(x, 0) \) relative to the size of the given flow \( U(x) \). We assume that both unperturbed flow \( u \) and the disturbed flow \( u + w \) satisfy the unsteady Navier-Stokes equations (5.1)-(5.5), and have the same, possibly homogeneous, values at the boundary \( \Gamma \). Thus we have

\[
\nabla \cdot w = 0, \quad in \; \Omega \times (0, T) \; and \; w = 0 \; on \; \Gamma_{\text{rigid}} \times (0, T)
\]

However, due to the non-linear convection term, \( w \) does not satisfy the unsteady Navier-Stokes equation (5.1).

To assign a definite meaning to the word "(un)stable", the \textit{average energy of the disturbance}

\[
E(t) = \frac{1}{2} \int_{\Omega} w \cdot w \, dx
\]

is introduced, where we assume again that all variables have been non-dimensionalized, so that the kinematic viscosity \( \nu \) can be taken as the inverse of the Reynolds number \( Re \).

\textbf{Definition 6.1.} We say that the given flow \( u \) is stable in the energy sense, (see Joseph [10] or Gunzburger [8]), if

\[
E(t) \to 0 \; \text{as} \; t \to \infty.
\]

On the basis that both unperturbed flow \( u \) and the disturbed flow \( u + w \) satisfy the unsteady Navier-Stokes equations (5.1)-(5.5) and \( \nabla \cdot w = 0, \; in \; \Omega \times (0, T) \; and \; w = 0, \; on \; \Gamma_{\text{rigid}} \times (0, T) \), we are led to

\[
\frac{dE(t)}{dt} = \frac{1}{2} \frac{d}{dt} \int_{\Omega} w \cdot w \, dx
\]

\[
= \int_{\Omega} w \cdot \frac{dw}{dt} \, dx
\]

\[
= \int_{\Omega} w \cdot \left( \frac{\partial w}{\partial t} + U \cdot \nabla w \right) \, dx, \; since \; \nabla w = 0
\]

\[
= \int_{\Omega} w \cdot \left( -w \cdot \nabla U - w \cdot \nabla w + Re^{-1} \nabla^2 w \right) \, dx, \; since \; u \; and \; u + w \; satisfy \; (5.1)
\]

\[
= \int_{\Omega} w \cdot \left( -\nabla U \cdot w + Re^{-1} \nabla^2 w \right) \, dx, \; since \; \nabla w = 0
\]

\[
= -\int_{\Omega} w \cdot \nabla U \cdot w \, dx + Re^{-1} \int_{\Omega} w \cdot \nabla^2 w \, dx
\]

\[
= -\int_{\Omega} w \cdot \nabla U \cdot w \, dx - Re^{-1} \int_{\Omega} \nabla w \cdot \nabla w \, dx,
\]
where we have applied the divergence theorem to the second integral and the fact that \( w = 0 \) on \( \Gamma_{\text{rigid}} \times (0, T] \). Then,

\[
\frac{dE(t)}{dt} = -\int_\Omega \left[ w \cdot D(U) \cdot w + \Re^{-1} \nabla w \cdot \nabla w \right] \, dx
\]  

(6.1)

with \( \nabla U = (\partial_i u_j)_{ij} = (D_{ij}[U])_{ij} = D(U) \), where

\[
D(U) = \frac{1}{2}(\nabla U + (\nabla U)^T)
\]

is the rate of strain or the rate of deformation tensor of the given flow \( U \), and \( (\nabla U)^T \) the transpose of \( \nabla U \).

In equation (6.1), the term \( \int_\Omega \nabla w \cdot \nabla w \, dx \) truly represents the average dissipation, and the term \( \int_\Omega w \cdot D(U) \cdot w \, dx \) represents the production integral which couples the given flow \( U \) (with stretching tensor \( D(U) \)) to the disturbance \( w \).

If the right-hand side of (6.1) is negative (i.e., the derivative of \( E \) is less than zero), then \( E \) will decrease as \( t \) increases, characterizing the stability of the flow according to the previous definition. Now let

\[
\frac{1}{\Re} = \tilde{\nu} = \max_v \left( \frac{-\int_\Omega \nabla \cdot D(U) \cdot v \, dx}{\int_\Omega \nabla v \cdot \nabla v \, dx} \right),
\]  

(6.2)

where the maximum is sought over all the vector fields \( v \) satisfying \( \nabla \cdot v = 0 \) in \( \Omega \), and \( v = 0 \) on \( \Gamma_{\text{rigid}} \). The allowed perturbation \( w \) satisfies these two constraints so that (6.2) implies that

\[
-\frac{\int_\Omega w \cdot D(U) \cdot w \, dx}{\int_\Omega \nabla w \cdot \nabla w \, dx} \leq \tilde{\nu}
\]

or

\[
-\int_\Omega w \cdot D(U) \cdot w \, dx \leq \tilde{\nu} \int_\Omega \nabla w \cdot \nabla w \, dx
\]

combining with (6.1) yields

\[
\frac{dE(t)}{dt} \leq -\left( \nu - \tilde{\nu} \right) \int_\Omega \nabla w \cdot \nabla w \, dx
\]  

(6.3)

or

\[
\frac{dE(t)}{dt} \leq -\left( \frac{1}{\Re} - \frac{1}{\Re} \right) \int_\Omega \nabla w \cdot \nabla w \, dx
\]  

(6.4)

so that if the solution \( \frac{1}{\Re} = \tilde{\nu} \) of the maximization problem (6.2) satisfies \( \tilde{\nu} < \nu \) (or equivalent to \( \Re < \tilde{\Re} \)), then \( \frac{dE(t)}{dt} < 0 \) and the flow is stable.
From equation (6.4), we can say that there is a critical value of the Reynolds number for which the transition from a stable state to an unstable state occurs. This is in concordance with the energy stability theorems developed in the book by D. D. Joseph [10]. We are led to the problem of energy stability limit. This limit is defined by

$$\frac{1}{\tilde{Re}} = \max_v \left( -\frac{\int_{\Omega} \mathbf{v} \cdot D(U) \cdot \mathbf{v} \, d\mathbf{x}}{\int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{v} \, d\mathbf{x}} \right),$$

(6.5)

where $\tilde{Re}$ is seen as the critical value.

One may actually understand a mathematical explanation of the fact that laminar flows break down, their subsequent development, and trigger their eventual transition to turbulence as the Reynolds number becomes large. This theoretical result is in good agreement with experiments (O. Reynolds [26], W. Orr [21]) concerning the critical Reynolds number at which the first bifurcation occurs. This bifurcation triggers the beginning of the instability of the flow.

### 6.1 The Energy Eigenvalue Problem

In this, we want to convert (6.5) into an eigenvalue problem, to characterize the set of eigenvalues with respect to completeness, and to show that $\tilde{Re}$ defined by (6.5) can also be found as the principal eigenvalue of a differential equation. For simplicity we use the notation

$$\langle \cdot \rangle = \int_{\Omega} (\cdot) d\mathbf{x}.$$

The following fundamental lemma of the calculus of variations, for vector fields (see D. D. Joseph [10]) proves to be useful:

**Lemma 6.1.** If a fixed function $F(x) \in C^1(x)$ and if $\langle F \cdot \phi \rangle = 0$ for all vectors fields $\phi \in C^3(\Omega)$ such that $\phi \cdot \mathbf{n} = 0$ on $\partial\Omega$, then there exists a single-valued potential $s = s(x)$ such that

$$F = -\nabla s.$$

Let us consider a slightly more general problem than (6.5), i.e.

$$\frac{1}{\tilde{\varrho}} = \max_v \left( \frac{\mathcal{F}}{\mathcal{D}} \right),$$

(6.6)

where

$$\mathcal{F} = -\langle \mathbf{v} \cdot D(U) \cdot \mathbf{v} \rangle, \quad \mathcal{D} = \langle 2D(v) : D(v) \rangle.$$
Suppose that the maximum of (6.6) is attained when \( v = \vec{v} \). Consider the values of \( \mathcal{F}/\mathcal{D} \) when \( v_i = v_i + \varepsilon \eta_i \) where \( \eta_i = \left. \frac{\partial v}{\partial \varepsilon} \right|_{\varepsilon=0} \) is an arbitrary vector (satisfying (6.5)). For each fixed \( \eta_i \) we have

\[
\frac{1}{\varrho(\varepsilon)} = \frac{\mathcal{F}(\varepsilon)}{\mathcal{D}(\varepsilon)}. \tag{6.7}
\]

Clearly \( 1/\varrho(\varepsilon) \) is a maximum when \( \varepsilon = 0 \). Then

\[
\varrho(\varepsilon) \mathcal{F}(\varepsilon) - \mathcal{D}(\varepsilon) = 0
\]

and

\[
\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} [\varrho(\varepsilon) \mathcal{F} - \mathcal{D}] = \varrho(0) \frac{d\mathcal{F}}{d\varepsilon} - \frac{d\mathcal{D}}{d\varepsilon} = 0. \tag{6.8}
\]

Using equation (6.6), we may write (6.8) as

\[
\varrho \left< \vec{v} \cdot \mathbf{D}(\mathbf{U}) \cdot \frac{\partial v}{\partial \varepsilon} \right> + 2 \left< \mathbf{D}(\vec{v}) : \frac{\partial \mathbf{D}(v)}{\partial \varepsilon} \right> = 0. \tag{6.9}
\]

Here all quantities are evaluated at \( \varepsilon = 0 \) (then \( \mathbf{v} \simeq \vec{v} \)) and, we have used the symmetry of \( \mathbf{D} \) to write

\[
\left< \frac{\partial v}{\partial \varepsilon} \cdot \mathbf{D}(\mathbf{U}) \cdot \vec{v} \right> = \left< \vec{v} \cdot \mathbf{D}(\mathbf{U}) \cdot \frac{\partial v}{\partial \varepsilon} \right>.
\]

Equation (6.9) may be regarded as Euler’s functional equation. It holds for every vector field \( \partial v/\partial \varepsilon \) such that \( \nabla \cdot (\partial v/\partial \varepsilon) = 0 \) in \( \Omega \) and \( \partial v/\partial \varepsilon = 0 \) on \( \partial \Omega \). To convert this equation into an eigenvalue problem for a system of differential equations, we note that

\[
2 \left< \mathbf{D}(\vec{v}) : \frac{\partial \mathbf{D}(v)}{\partial \varepsilon} \right> = 2 \left< \nabla \cdot \left( \mathbf{D}(\vec{v}) \cdot \frac{\partial v}{\partial \varepsilon} \right) \right> - \left< \frac{\partial v}{\partial \varepsilon} \cdot \nabla^2 \vec{v} \right>.
\]

Therefore, the equality (6.9) becomes

\[
\varrho \left< \vec{v} \cdot \mathbf{D}(\mathbf{U}) \cdot \frac{\partial v}{\partial \varepsilon} \right> = -2 \left< \nabla \cdot \left( \mathbf{D}(\vec{v}) \cdot \frac{\partial v}{\partial \varepsilon} \right) \right> + \left< \frac{\partial v}{\partial \varepsilon} \cdot \nabla^2 \vec{v} \right>. \tag{6.10}
\]

But the divergence theorem (\( \int_{\Omega} \nabla \cdot F \, dx = \int_{\partial \Omega} F \cdot n \, ds \)) yields

\[
-2 \left< \nabla \cdot \left( \mathbf{D}(\vec{v}) \cdot \frac{\partial v}{\partial \varepsilon} \right) \right> = -2 \int_{\Omega} \nabla \cdot \left( \mathbf{D}(\vec{v}) \cdot \frac{\partial v}{\partial \varepsilon} \right) \, dx
\]

\[
= -2 \int_{\partial \Omega} \mathbf{D}(\vec{v}) \cdot \frac{\partial v}{\partial \varepsilon} \cdot n \, ds
\]

\[
= 0, \quad since \quad \frac{\partial v}{\partial \varepsilon} = 0 \ on \ \partial \Omega.
\]

Then, (6.10) becomes

\[
\left< \left( \varrho \vec{v} \cdot \mathbf{D}(\mathbf{U}) - \nabla^2 \vec{v} \right) \cdot \frac{\partial v}{\partial \varepsilon} \right> = 0.
\]
If we set $F = \varrho \nabla \cdot D(U) - \nabla^2 \nabla$, then $F$ satisfies the conditions of the fundamental lemma 6.1. Applying the lemma to $F = \varrho \nabla \cdot D(U) - \nabla^2 \nabla$, we obtain

$$\varrho \nabla \cdot D(U) - \nabla^2 \nabla = -\nabla s,$$

which is the same as

$$\nabla \cdot D(U) - \frac{1}{\varrho} \nabla^2 \nabla = -\nabla s. \quad (6.11)$$

Finally, we obtain the Euler equations corresponding to the maximization problem (6.5) and given by

$$\lambda \nabla^2 w - \nabla s = w \cdot D(U) \quad \text{in } \Omega, \quad (6.12)$$

$$\nabla \cdot w = 0 \quad \text{in } \Omega, \quad (6.13)$$

and

$$w = 0 \quad \text{on } \partial \Omega, \quad (6.14)$$

where we have set $\frac{1}{\varrho} = \frac{1}{\tilde{R}e} = \lambda$ and where $s(x)$ is seen as Lagrange multiplier associated with the constraint $\nabla \cdot w = 0$. Given the velocity field $U(x)$, the system (6.12)-(6.14) is a self-adjoint linear eigenvalue problem for the triple $w(x) \neq 0$ (the perturbation), $s(x) \neq 0$, and $\lambda \in \mathbb{R}$.

The solution $\tilde{\nu} = \frac{1}{\tilde{R}e}$ of the maximization problem (6.5), is then given by the largest eigenvalue (M. D. Gunzburger [8]) of the system (6.12)-(6.14). The existence of a non-negative $\tilde{\nu}$ follows from the fact that the trace $[D(U)] = \nabla \cdot U = 0$. Then if $\nu > \tilde{\nu}$ (or $\tilde{R}e < \tilde{R}e$), the given flow $U$ is stable.

Even for simple flows $U$ in simple domains $\Omega$, it is not possible to determine the eigenvalues of the system (6.12)-(6.14), except through numerical procedures. Thus in the following section, we investigate the finite element approximations of the eigenvalues of the system (6.12)-(6.14).

### 6.2 Finite Element Approximations of the Eigenvalues

In order to define such approximations, with the help of the general principles stated in section 5.3, one first recasts the system (6.12)-(6.14) into the following weak form:

Given $U \in H^r(\Omega)$ for some positive integer $r$, find $w \in H^1_0(\Omega), w \neq 0, \ s \in L^2_0(\Omega), s \neq 0, \ and \ \lambda \in \mathbb{R}$ such that

$$\lambda \tilde{a}(w, v) + b(s, v) = d(U; w, v) \quad \text{for all } v \in H^1_0(\Omega) \quad (6.15)$$

and

$$b(q, w) = 0 \quad \text{for all } q \in L^2_0(\Omega) \quad (6.16)$$
where \( \tilde{a}(., .) = \text{Re} \ a(., .) \) and the bilinear forms \( a(., .) \) and \( b(., .) \) are defined in (5.9) and (5.10), respectively, and where

\[
d(U; \ w, \ v) = - \int_\Omega \ w \cdot D(U) \cdot v \ dx.
\]

Therefore we are interested in finding an approximation for \( \tilde{\nu} = \frac{1}{\text{Re}} \tilde{\nu} \), which now denotes the largest eigenvalue of the system (6.15)-(6.16). We denote by \( m \) the algebraic multiplicity of the eigenvalue \( \tilde{\nu} = \frac{1}{\text{Re}} \tilde{\nu} \) and by \( \mathcal{R}_\tilde{\nu} \) the space spanned by the eigenvectors \( \{s, w\} \) of (6.15)-(6.16) corresponding to the eigenvalue \( \tilde{\nu} \). Due to the fact that (6.15)-(6.16) is self-adjoint, \( m \) is also the geometric multiplicity of the eigenvalue \( \tilde{\nu} \) and hence the dimension of the eigenspace \( \mathcal{R}_\tilde{\nu} \).

Now, following the same principles as in Section 5.4, the finite element formulation is given by the following problem:

Given \( U \in \mathbf{H}'(\Omega) \) for some positive integer \( r \), find \( w^h \in V_0^h \subset H_0^1(\Omega) \), \( w^h \neq 0 \), \( s^h \in S_0^h \subset L_0^2(\Omega) \), \( s^h \neq 0 \) and \( \lambda^h \in \mathbb{R} \) such that

\[
\lambda^h \tilde{a}(w^h, v^h) + b(s^h, v^h) = d(U; w^h, v^h) \quad \text{for all } v^h \in V_0^h \tag{6.17}
\]

and

\[
b(q^h, w^h) = 0 \quad \text{for all } q^h \in S_0^h. \tag{6.18}
\]

We assume that the bilinear forms \( a(., .) \) and \( b(., .) \) and the approximating subspaces \( V_0^h \) and \( S_0^h \) satisfy all the hypotheses \( \mathbf{H1}, \mathbf{H2}, \mathbf{H3} \) (Section 5.4), required for suitable approximations. We also assume that for the given velocity field \( U \), the bilinear form \( d(U; w, v) \) is continuous for all \( w, v \in H_0^1(\Omega) \); this assumption is valid whenever, e.g., \( U \in H_0^1(\Omega) \). Of interest here, are the following results (for more details, see the PhD thesis by Peterson J. [25] or see Max D. Gunzburger [8]): Firstly there are exactly \( m \) eigenvalues of (6.17)-(6.18), counted according to the multiplicity, which as the discretization parameter \( h \to 0 \), converge to the eigenvalue \( \tilde{\nu} = \frac{1}{\text{Re}} \tilde{\nu} \) of the system (6.15)-(6.16). Thus if we denote these \( m \) eigenvalues by \( \{\tilde{\nu}_j^h\} \), \( j = 1, \ldots, m \) then we have

\[
\tilde{\nu}_j^h \to \tilde{\nu} \quad \text{as}, \quad h \to 0. \tag{6.19}
\]

In addition, we also have the error estimate: For \( h \) sufficiently small, there exists a constant \( C \) such that

\[
|\tilde{\nu} - \tilde{\nu}_j^h| \leq C(\xi^h)^2 \quad \text{for, } j = 1, \ldots, m \tag{6.20}
\]

where

\[
\xi^h = \sup_{(s, w) \in \mathcal{R}_\tilde{\nu}} \inf_{\|s\|_0 + |w|_1 = 1} \left( |w - v^h|_1 + \|s - q^h\|_0 \right). \tag{6.21}
\]
Globally and monotonically stable

Globally stable

Conditionally stable

E(0)

Unstable

Figure 1. Stability limits for the basic flow. \( R_E, R_G \) and \( R_L \) are the critical values of the Reynolds number, depending on the type of instability.

From the latter equations, we see that the usual situation concerning eigenvalue approximations by the finite element methods, is obtained in the present case; namely that the error in the eigenvalue is the square of the error for the eigenfunction, the latter being measured in the ”natural” norm sense. In this way we calculate an approximation of the critical Reynolds number,

\[ \tilde{Re} = \frac{1}{\tilde{\nu}} \]

for which the transition from laminar to turbulent flow occurs. Figure 1 shows the stability limits for the basic flow, and the different types of (in)stability. It clearly shows different zones of (in)stability which change with values of the Reynolds number.

Finally, we have used the finite element theory to show that the Reynolds number is once again at the core of the stability of a fluid motion. We have shown a way to compute the critical Reynolds number at which the first bifurcation (appearance of the alternating vortex pattern), occurs. This critical Reynolds number varies according to the type of the flow and physical condition imposed on it.
7 Conclusion

The present study contributes, among other numerical and analytical studies to the knowledge of physical phenomena related to the transition from laminar to turbulence in Newtonian fluid flow. The strategy adopted in this work consists of investigating the governing equations, for the case of incompressible viscous Newtonian fluid flow. These equations are the continuity and the Navier-Stokes equations.

Experimental studies carried out by many authors proved the existence of three different regimes of the flow, which are laminar, transition and turbulent regimes. Experimental studies also showed that this situation depends on the Reynolds number. Our main objective in this work has been to explain this dependence mathematically.

Therefore, we have used two effective methods: Lie group theory, and the finite element method, to explain why the Reynolds number influences the different regimes of Newtonian fluid flows:

By Lie group theory, we have solved analytically the Navier-Stokes equations, using the symmetry approach. Finally, we have succeeded to express explicitly a solution; the $u$-, $v$-, and $w$-velocity components as well as the pressure $p$. We found them to be functions of the Reynolds number, even though there are other analytic functions appearing in their expressions. This explains, for example, the figures drawn in the sixth part of the article [24], which show that the time evolutions of velocity components become more perturbed as the Reynolds number increases.

Secondly, we have used the finite element method to show a way to compute the critical Reynolds number at which the first bifurcation occurs. The stability theory has helped us to prove that the stability of a flow motion is proportional to the Reynolds number. One can now understand how a laminar flow may develop into a turbulent flow, through the critical Reynolds number, for which the transition regime occurs.

The results obtained from both methods are almost the same: Laminar flow occurs at low Reynolds numbers, where viscous forces are dominant, and is characterized by smooth, constant fluid motion, while turbulent flow, on the other hand, occurs at high Reynolds numbers and is dominated by inertial forces, producing random eddies, vortices and other flow fluctuations. The transition between laminar and turbulent flow is indicated by a critical Reynolds number, which depends on the exact flow configuration.

Reynolds numbers are of extreme importance in the study of Newtonian fluid flows. As our focus here is the transition flow only, it follows that the occurrence of transition is expressed in terms of the Reynolds number mainly, though there are other factors, not taken into consideration in the present work. Accordingly, it seems easier to comprehend the experiments carried out by many scientists: Firstly, the one of pipe flow
done by Osborne Reynolds [26], where intermittent flashes of turbulence could be seen as the Reynolds number increased beyond a critical value. Secondly, the experimental studies of the wake formation behind bluff bodies, pointed out by Roshko [27], who first observed the existence of a transition regime in the wake of the cylinder and found distinct irregularities in the wake velocity fluctuation. He showed that there exist three different regimes of the flow at low moderate Reynolds numbers, namely laminar, transition and irregular turbulent regimes. In the transition regime, he reported that the low-frequency irregularities obtained experimentally are related to the pressure of three-dimensionalities in the flow, which lead to the development of turbulent motion further downstream. Thirdly, in the same type of flow, Williamson [32] observed the existence of two modes of formation of streamwise vorticity in the near wake, each occurring at a different range of Reynolds numbers, and both being related to the three-dimensional transition between Reynolds numbers from 180 to 260. The first mode occurs beyond Reynolds number 180 and is characterized by a continuous change in the wake formation, as the primary vortices become unstable and generate large-scale vortex loops. The second, beyond Reynolds number 260, corresponds to the appearance of small-scale streamwise vortex structures.

From a numerical point of view, a large number of numerical studies have been devoted to the analysis of unsteady flow around a circular cylinder in the low and moderate Reynolds number regime. But these studies are only two-dimensional simulations. Reliable three-dimensional numerical simulations of this category of flow have only very recently appeared, due to the increased capacities and evolution of supercomputing technology. Karniadakis and Triantafyllou [12] have computed the three-dimensional flow around a circular cylinder in the Reynolds number range of 200-500, by using the spectral-element method by Patera [23]. In the same way, Hélène Persillon and Marianna Braza [24] succeeded to compute the three-dimensional flow around a circular cylinder in the Reynolds number range of 100-300.

Thanks to the direct observations of fluids, like those studied by Reynolds, researchers know that the profile of a fluid during laminar flow is parabolic. This can also be seen by solving the Navier-Stokes equations. The non-dimensional form of the Navier-Stokes equations (3.10)-(3.13), clearly shows that the Reynolds number is the only parameter of the fluid flow. The Navier-Stokes equations generally have more than one solution, unless the data satisfies very stringent requirements, as we saw in the subsection 5.2 of this report.

Even though we have chosen simplified hypotheses throughout the present work, others may be inspired to give relevant meanings to all these experimental observations. However, transition to turbulence remains complex, and its study is far from being fully achieved. In this work, we tried to provide only basic explanations. In the study of transition to turbulence, there are many other elements that still have to be taken into consideration: For example, the prediction of the frequency modulation, and the forma-
tion of a discontinuity region delimited by two frequency steps within a given Reynolds number range. Another example is the birth of streamwise vorticity and the kinetic energy distribution in the studied region, where the similarity laws do not always hold.

It is encouraging to know that great works in the field, are still in progress all over the world.

References


