CHAPTER ONE

BACKGROUND AND OVERVIEW OF THE STUDY

1.1 Focus of the study

This study provides a qualitative examination of the role of visualization through an understanding of the thought processes that occur during visualization when learners engage in data handling and spatial tasks. In particular, it examined the thought processes of the learner during visualization while solving data handling and spatial tasks in order to obtain an understanding of the role of visualization in learners' learning of mathematics and to develop theoretical statements about these thought processes. The study starts with the description of visualization as a cognitive process of forming images and using such images effectively for mathematical discovery and understanding. Tasks are drawn from data handling and the field of spatial development both of which represent an important section of mathematics, in order to enable the study of the thought processes involved during visualization. An understanding of spatial visualization and data handling enabled the researcher to choose suitable tasks. An important component of development which governs the learner's intellectual development during the learning process is spatial thinking. By means of spatial thinking the individual orients him/herself in space (both physically and mentally), acquires knowledge and learns different forms of activities. Therefore tasks were drawn from the spatial field. Successful problem solving activities cannot be realised if the environment in which they take place is not conducive to learning. It was therefore necessary to understand the nature of the problem-centred approach so as to be able to work in a problem-centred context. It should however be noted that to realise a proper problem-centred context requires considerable time. Therefore, this study only goes as far as is possible in working in a problem-centred context. The problem-centred context provides a suitable environment in which the study is able to facilitate learning.

1.2 Relevance and importance of the investigation

1.2.1 Background of the problem

Mathematics is a subject that is concerned with objectivizing and representing abstractions from reality, and many of these representations appear to be visual, having roots in visually sensed experiences (Bishop 1989:8). We live in a society that is being exposed to graphic images on an increasing basis. This exposure is part of a trend to graphic communication which involves visualization. Kress and van Leeuwen (in Gerber, Boulton-Lewis & Bruce 1995:77) state:

> We want to treat forms of communication employing visual images more seriously than they have hitherto been treated. We have come to this position because of the overwhelming evidence of the importance of visual communication, and the staggering inability on all our parts to talk and think in any way seriously about what is actually communicated by means of images and visual design.

The importance of visualization in mathematics is illustrated by subjects which are currently integral parts of the Grade 9 curriculum, for example the field of spatial development and statistics. Furthermore, visualization not only organizes data at hand in meaningful structures, but is also an important factor guiding the analytical development of a solution. In recent literature visualization has appeared as an important aspect related to the learners' construction of mathematical concepts and problem solving processes. Mathematical visualization is the process of forming images mentally or with pencil and paper or with the aid of technology, and using such images effectively for mathematical discovery and understanding (Hershkowitz 1989:75). From the above definitions it is clear that visualization involves mental images that go through various transformations, and thus aid the understanding of problem solutions. It is therefore important that learners are encouraged to appreciate visual thinking which is important in visualization. Visual thinking has always been an important part of the thinking of mathematicians, but perhaps less so an integral part of school learners' mathematical experiences (Hadamard in Thornton 2000:251). Hershkowitz (1989:75) however recounts that during problem solving, some mathematicians avoid the use of words, algebraic or other symbols and instead use visual reasoning that incorporates spatial and other images as the basis for their intuitions, and only subsequently code them in symbolic terms. There have been some beliefs that the products of mathematical work are more important than the process of how the results are achieved, and as a result it is these symbolic representations that are often held in high esteem, with visual tools seen at most as transitory steps on the way to real mathematics (Thornton 2000:251). The result is that the setting of a mathematical agenda identifies with the symbolic and algebraic to the virtual exclusion of the visual mode of mathematical expression. The whole process that leads to a mathematical result must be appreciated.

There are still many issues concerning visualization in mathematics education, which require careful attention. In the current educational climate there are at least three reasons why the role of visualization in school mathematics needs to be re-evaluated (Thornton 2000:251). The first is that the current trend that identifies mathematics with the study of patterns, together with the ready availability of hand-held technology that can easily develop a general rule for a given pattern, has the potential to improve mathematical thinking. The second is that visualization can often provide simple, elegant and powerful approaches to developing mathematical results and solving problems, in the process making connections between different areas of mathematics. The third is the importance of recognising and valuing different learning styles, and of helping learners to develop a repertoire of techniques for looking at mathematical situations.

In a problem-centred teaching environment learners are expected to solve problems or make sense of mathematical situations for which no well-defined routines or procedures exist. It focuses on building understanding and reasoning in mathematics. An important principle for reform in mathematics includes allowing learners to make the subject of mathematics problematic (Hiebert, Carpenter, Fennema, Fuson, Murray, Olivier, Human &Wearne 1996:15). Making the subject problematic refers to allowing learners to wonder why things are, to inquire, to search for solutions, and to solve incongruities. This means that both curriculum and instruction should begin with problems, dilemmas, and questions for learners. When learners are actively looking for relationships, analysing patterns, finding out which methods work and which do not justify results or evaluating and challenging the thoughts of others, they are necessarily and optimally engaging in reflective thought about the ideas involved. This kind of reform is what is being encouraged in a problem-centred context. Although there is plenty of evidence in literature supporting the problem-centred approach about how learners should learn and should be taught mathematics, even in today’s secondary schools the following pattern seems to be followed in the majority of classrooms.

> The teacher introduces a topic by stating a rule or definition and then demonstrating it with text book examples on the chalkboard or on overhead transparencies. Learners work on exercises at their seats, with the teacher providing individual help to those experiencing difficulties. Similar exercises are completed for homework and checked as ‘right’ or ‘wrong’ at the beginning of the next class period. Homework exercises that were particularly

troublesome are worked out for the class either by the teacher or learner volunteers (Cangelosi 1996:266).

It is clear that a lot has to date been done to understand the problem-based approach (e.g., Human 1993; Bereiter 1992; Erickson 1999; Hiebert et al 1996). Grouws (1985) acknowledges that a vast amount of research has been done, but is concerned by the lack of theory-based research that systematically examines and later manipulates problem-based approaches in the classroom. The question to be answered is whether enough is being done to check the success of this approach inside the classroom.

Few or no statistics exist at most high schools in developing countries and for those countries that have now accommodated enough statistics (e.g., South Africa) in the schools, a general lack of enthusiasm and understanding of the subject prevails. Information and data play a prominent role in everyday mathematics. The National Council of Teachers of Mathematics (NCTM) of the United States of America has in the Curriculum Document (1981:324) advocated that learners learn about data both descriptively and inferentially. Data handling is a very special and important section of statistics. It is this section of statistic that takes a learner out into the real world of seeing data for him/herself, reflecting upon it individually or socially, and drawing conclusions. Data handling, according to Shaughnessy (1992:470), is describing, organizing and reducing, representing, analysing and interpretation of data. Ainley and Pratt (2001) acknowledge data handling as reflecting recent changes in the way in which the topic statistics is approached within the school curriculum, extending graphical representation within primary school age range and broadening the accessibility of statistics in secondary school. This change has been influenced by a number of factors that include technology, which makes it possible to handle large quantities of data; the increased public use and awareness of statistics, and the introduction of notions such as relevance and citizenship into school curricula. Change in school curricula is seen as offering an existing opportunity to develop learners' statistical intuitions, to foreground the mathematical concepts embedded in statistical techniques, and to create contexts in which these can be linked to broader mathematical ideas of symbol use, reasoning and logical necessity (Ainley and Pratt 2001).

Visualization is directly related to spatial ability. Spatial ability in geometry is defined as the ability to formulate mental images and to manipulate these images in the mind (McGee in

Lean & Clements 1981:270). On the other hand visualization is defined as the ability to represent, transform, generate, communicate, document and reflect on visual information Hershkowitz (1989:75). Moreover, a visual image, by virtue of its concreteness, "is an essential factor for creating the feeling of self-evidence and immediacy" (Fischbein 1987:101). Therefore, visualization "not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution." It is therefore important to realise the close relationship that exists between visualization and spatial sense. Experiences with visualizing and making pictorial representations in middle school form the foundation for a serious study of Euclidean and non Euclidean geometries (Ben-Chaim, Lappan & Huoang 1989:58). Spatial ability comprises those mental skills concerned with understanding, manipulating reorganizing or interpreting relationships visually or representing, transforming, generating and recalling symbolic, non-linguistic information (Linn & Petersen 1985:1982). Integrated within the skill of spatial ability is visual imagery that Wheatley (1991:34) has described as involving constructing an image from pictures, words or thoughts and which Presmeg (1986a:43) defines as a mental scheme depicting visual or spatial information. These spatial visualization skills are responsible for organizing information during problem solving where simultaneous processing is required. Ben-Chaim et al (1989:51) comments that a broad goal of middle school mathematics education is to develop in learners an appreciation for aesthetics of mathematical thought and an understanding of the powers of limitations of mathematical processes. This aesthetic appreciation often comes with visualizing mathematical concepts.

1.3 Statement of the problem

This investigation generated interest in the process of visualization while recognizing data handling as an important section in mathematics education which can be understood successfully if facilitated in a problem-centred context. This dissertation examined the role of visualization in mathematics through an understanding of the thought processes that occur during visualization as Grade 9 learners engage in data handling and spatial tasks. Therefore, the research question was:

> Which thought processes are involved during visualization when Grade 9 learners engage in data handling and spatial tasks in a problem-centred context which will enable us to understand the role of visualization in mathematics?

1.4 Research aim and objectives

The aim of the study was to investigate the role of visualization through the analysis of the mental processes that Grade 9 learners go through while solving data handling and spatial tasks in a problem-centred context. To achieve this aim, the research will be guided by the following objectives based on a literature and empirical study.

- To investigate the nature of visualization as a cognitive process.
- To explore the relationship between spatial sense and visualization.
- To use the above as a guiding framework in an empirical investigation to understand visualization by determining and describing the thought processes that occur during visualization when Grade 9 learners engage in problem solving via learner's responses to tasks in data handling and spatial development.
- To identify the possible effect of working in a problem-centred context.
- To investigate the nature of data handling as a section of statistics.

1.4.1 Definition of terms used

Visualization

It is the ability to represent, transform, generate, communicate, document and reflect on visual information.

Imagery

It is the occurrence of mental activity corresponding to the perception of an object, when the object is not present to the sense organ.

Visual imagery

It is described as imagery which occurs as a picture in 'the mind's eye'.

Visuality

It is described as how often learners use visualization while they were working through data handling and spatial tasks.

Spatial ability

It is the ability to form mental images and to manipulate these images in the mind.

Problem-centred approach

It is an approach which seeks to engage learners in problem solving as a mathematician's way of practice and re-invention in which the young learner must be apprenticed.

Problem-centred context

This is an environment in which the requirements of the problem-centred approach are satisfied.

Data handling

It is describing, organizing and reducing, representing, analysing and interpreting data.

1.5 Methodological organization of the study

1.5.1 Introduction of the empirical study

This study aimed to investigate the role of visualization and this was achieved by identifying and describing the thought processes that occur when Grade 9 learners engage in problem solving activities via learners' responses to tasks in data handling and the field of spatial development. The research design was of a qualitative nature and a thorough literature study was carried out followed by the empirical investigation.

1.5.2 Carrying out the investigation

Data were obtained from a group of 12 Grade 9 learners. Three days of each week were used for a total of six weeks for the investigation. Four learners working individually were provided with either a spatial or a data handling task each day. The researcher required the learner to give explanations of, and justifications for his or her solution activity. Therefore, each learner was advised and expected to show his/her problem solving procedures with explanations throughout all stages. Each learner was allowed to work through the given task for about thirty minutes with minimal intervention. The researcher provided any assistance needed as far as the understanding of the task was concerned. At the end of about thirty minutes the researcher collected the individual written responses. Each written response was marked according to pre-designed rubrics illustrated in Chapter three (Section 3.6) which are related to Kosslyn's (1994) categories of visualization in Chapter two (Section 2.2.2). The results obtained from these handwritten responses were analysed and used to help formulate questions used in the interviews which were carried out in the next stage.

The next stage involved facilitation of the interview with each of the four learners. Each learner was originally asked to explain his/her thought processes that he/she experienced while going through the problem solving process. The learner‘s initial response to the interview and his marked written response were used to formulate questions for probing further during the interview. The whole process of the interview was carried out and was recorded with the help of an audio tape recorder for each of the four learners per day. All the tape recorded data was later transcribed to be used for analysis. Data from the interview responses and written responses were assessed and analysed together qualitatively, with the aid of clearly defined categories adapted from Kosslyn (1994) given in Chapter two (Section 2.2.2) and the rubrics (Chapter three Section 3.6) designed to enable the marking and analysis of the written responses.

The goal during the analysis of data was to provide evidence that the learners had used visualization during their problem solving processes and were able to describe these thought processes using the categories and the rubrics. The goals were achieved by first analysing the data from written responses through relating them to the performance indicators in the rubrics with the mental processes in Kosslyn's (1994) categories (Section 2.2.2). For example in the geometric problem in Worksheet 3, the ability to draw a diagram well and clearly was evidence of the ability to generate an image, and the ability to describe the spatial diagram was evidence of the ability to inspect and transform an image. Labelling the sides of the spatial shape was evidence of image use. During interviews the learners explained their thought processes and by listening for evidence of the existence of the four categories of Kosslyn (1994:1-23) evidence was provided of learners having gone through visualization. Moreover, by listening to the explanations of the thought processes one was able to describe these processes in detail so as to be able to understand the role of visualization.

1.5.3 Demographic selection of the school

The investigation was carried out at Pretoria Secondary School, in Sunnyside. The reason for the choice was that the school is in the area where the researcher resides, and most of the learners do not live far away from their homes. Pretoria Secondary School includes learners from most of the African countries. The majority are from most of the ethnic groups in South Africa.

1.5.4 Random selection of learners

Twelve learners were randomly selected from a class of thirty learners and were used for the empirical study. A purposive non-probability sampling procedure was carried out (Strydom &Venter in De Vos, Strydom, Fouché & Delport 2003:207). This was done in collaboration with the teacher of mathematics for that class. Previous class performance aided with the selection of the learners used in the empirical study. The distribution of the twelve learners was as follows: four above average learners, four average learners and four below average learners. This was done to represent the type of thinking portrayed by learners of all abilities. The respondents were drawn from the Grade 9 class. Activities that involve data handling require graphic specific skills in reading axes, comparing slopes of lines or bars, interpolating or estimating values in three dimensional spaces or estimating distances and areas and calculations for analyzing the data. Visualization also requires general skills in spatial visualization or spatial cognition. Learners who had passed through the Foundation and Intermediate phase, and had done at least Grade 8, were assumed to have gained the necessary experiences to make this empirical study possible. Facilitation of tasks involved discussions and questions written in English. The learners in Grade 9 were assumed to have acquired this language so as to enable communication.

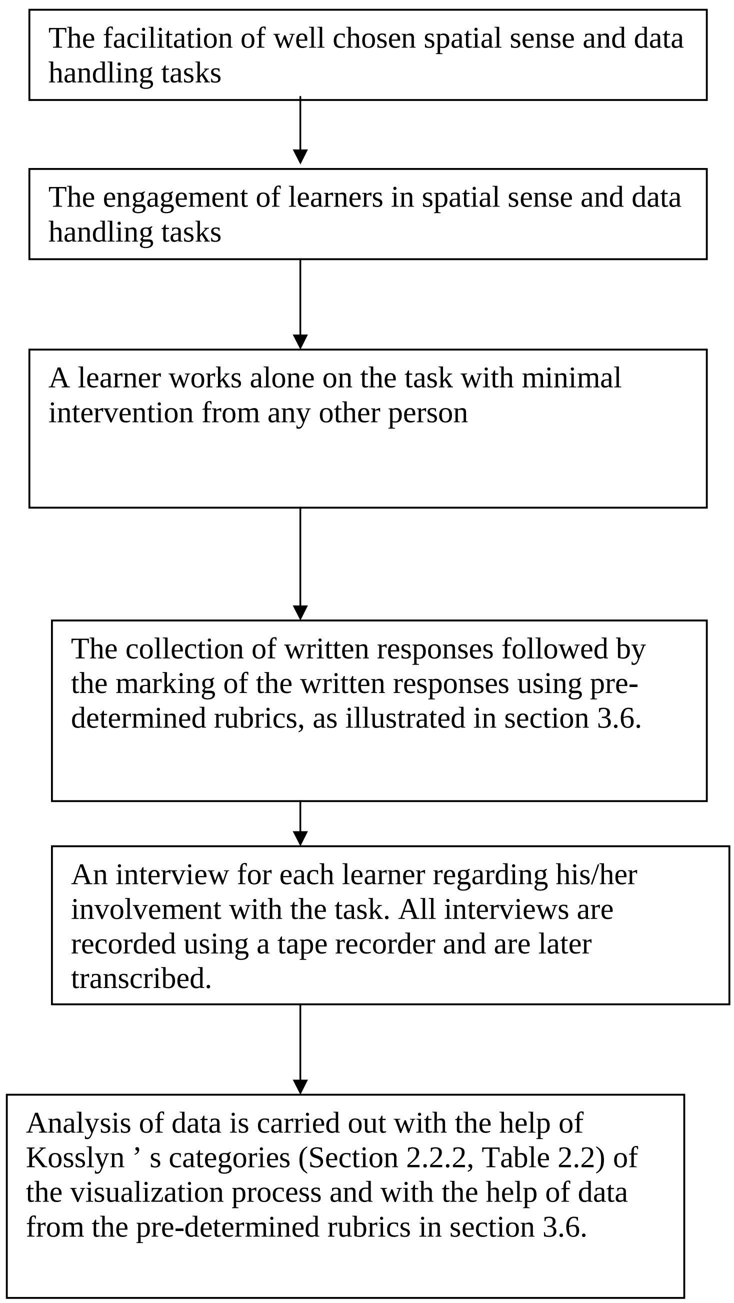
1.5.5 The nature of tasks

The tasks used were drawn from the field of spatial development and data handling and were chosen from tasks that were used during other research projects to improve visualization. The tasks were chosen to enable the researcher opportunities to retrieve data from learners as they attempted to resolve what were for them genuinely problematic situations. Data from the tasks were collected through written responses followed by interviews with the learners. The chosen activities were open-ended and involved handling of data and spatial development. The activities were intended to evoke visual imagery, providing a basis for challenging learners to reflect on and where necessary to modify concepts and images. Questions and variables were devised that included the requirements of reading the problem, understanding the problem and making a plan for the solution, interpretation of the data values by reading the data and reading between the data, and prediction of values for missing data, that is reading beyond the given values.

1.5.6 Data collection, assessment and analysis

The objectives for the analysis of data were to provide evidence of the use of visualization by the learners, and to describe the learners' thought processes that were experienced as they worked through data handling and spatial tasks. Data for this research project were collected from individual hand-written responses and from interviews with individual learners. The above objectives were realized through analysing data from the hand-written responses and the interviews that were carried out during the investigation. Sessions involved the facilitation of data handling and spatial tasks on worksheets to individual learners. Data from written responses were collected, marked and analysed using pre-determined rubrics that are shown in Chapter three (Section 3.6). All the worksheets contained the type of questions that involved visualization. The tape-recorded classroom interviews were part of the gathering of the research data, and each learners' spoken comments were transcribed and subsequently analysed with the help of categories, and subcategories adapted from Kosslyn (1994:379-407) illustrated in Chapter two (Section 2.2.2).

Figure 1.1 Stages through the investigation



1.6 Distribution of chapters

Chapter one addresses the importance of this investigation and the way in which it was carried out. The researcher also describes the focus of each chapter of the dissertation as a whole. Chapter one overall gives an overview of what the dissertation entails.

In Chapter two a thorough literature study is carried out, addressing research objectives. The visualization process is analysed and arguments from other researchers are included. Data handling is investigated and some support is given as to why it is important to the learner‘s life. The relationship between visualization and geometry is clarified through defining spatial visualization. The problem-centred context gives an idea of the kind of environment the empirical study was conducted in. It also describes the expected behaviour that should be displayed by the teacher and the learner.

Chapter three addresses the research design and methodology and provides the breakdown of the empirical study. It carefully satisfies the requirements of the research aims and objectives by explaining how selection of respondents, tasks, the collecting of data, representation of data and finally the analysing of the data were conducted.

Chapter four addresses research processing and interpretation of data, which describes the analysis of data. Data collected were analysed and made available for the required interpretation and conclusion.

In chapter five, summation, recommendations and conclusion close the dissertation. This chapter concludes by trying to answer some questions about visualization for the teacher, the learners, the curriculum developers, and for the mathematics community as a whole.

CHAPTER TWO

A LITERATURE SURVEY ON VISUALIZATION, THE PROBLEM-CENTRED CONTEXT AND DATA HANDLING

2.1 INTRODUCTION

The purpose of this chapter is to describe the views taken by different authors on the following aspects namely visualization, the problem-centred context, data handling and the relationship between visualization and spatial sense. The interest here is in constructing meaning from data through acknowledgment of the internal mental processes that take place during the process of visualization. Constructing meaning from data is used to encompass the problem solving processes (psycho/pedagogical) when learners engage in a range of activities which are seen to be essentially mathematical, involving the meaningful manipulation and interpretation of data (Ainley & Pratt 2001:2). By teaching mathematics we anticipate learners being provided with skills that can be used in solving problems from all sections of mathematics or which they can use in real life problems. In other words, we expect the learners to be able to model their mathematics so as to have meaningful solutions than they can apply.

A problem-centred instructional approach is supported by recent research on teaching and learning in mathematics. It was necessary to understand the problem-centred context, as the tasks that were used in the empirical study were facilitated in this environment. Understanding the problem-centred context should enable the mathematical community to revise their thinking of whether the problem-centred context is the best environment to work in during problem solving. Wessels and Kwari (2003:74) have indicated that the traditional way of teaching mathematics at school does not seem to be facilitating the transfer of knowledge and skills, prompting debate and research into more effective approaches to deal with this inadequacy. It is therefore necessary to look at the key players, in this approach, namely the researchers, the teachers and the learners.

Data handling in this study provides some of the tasks that allow the thought processes that occur in a learner's mind to be examined. The development of data-handling and statistics at school level has partly been influenced by the work of the International Association for Statistics Education (IASE), formed by the International Statistics Institute (ISI) in 1991

(Ainley & Pratt 2001:2). Recent changes in school curricula are seen as offering exciting opportunities to develop learners' statistical intuitions, to foreground the mathematical concepts embedded in statistical techniques, and to create contexts in which these can be linked to broader mathematical ideas of symbol use, reasoning and logical necessity. The other tasks came from the field of spatial development because visualization also includes spatial visualization which is relevant for this study.

2.2 VISUALIZATION AS A COGNITIVE PROCESS

2.2.1 Introduction

Visualization is an important aspect of mathematical understanding, insight and reasoning. This section presents the views given by some authors who have contributed to the understanding of visualization. Some influences that visualization has had on the development of mathematics are examined. Difficulties that surround the use of visualization and the factors that affect visualization are investigated. In this study visualization is characterized as both the product (visual image) and the process (Bishop 1989:9) and offers a method of seeing the unseen. In a more figurative and deeper sense seeing the unseen refers to a more 'abstract' world which no optical or electronic technology can visualize for us. What we are therefore interested in is what Pea (in Bishop 1989:11) refers to as cognitive technology which she describes as any medium that helps transcend the limitations of the mind. Mathematics, as a human and cultural creation dealing with objects and entities quite different from physical phenomena, relies heavily on visualization in its different forms and at different levels. The role of visualization is discussed in this section and is compared to the results of the investigation in Chapter four.

2.2.2 Ideas about visualization

Researchers have contributed different and useful ideas about visualization.
Visualization is described from the definition of mathematics as a science of patterns (Eisenberg in Zimmermann & Cunningham 1991:5). The mathematician seeks patterns in number, space, science, computers and in imagination. If mathematics is the science of patterns, it is natural to try to find the most effective way to visualize these patterns and learn to use visualization creatively as a tool for understanding. Supporting this idea, Presmeg (1992:40) states that visualization is an aid to understanding or a means towards an end and one can therefore speak about visualizing a concept or a problem but not a diagram. Visualizing of a concept or a problem refers to a mental image of the problem, and to

visualize a problem means to understand the problem in terms of a diagram or visual image. Hence the visualization process is one which involves visual imagery with or without a diagram, as an essential part of the method of solution (Presmeg 1985:298). The term ‘visual’ here refers to the manner in which mathematical information is presented and processed during or for problem solution. The term ‘visual processing’ does not have anything to do with vision as one of the five senses. It rather refers to the use of spatial properties and relationships of the particular representation in which a problem is presented and processed. Cobb, Yackel and Wood (1988:26-27) take on the issue of visualization as a dualism created between mathematics in learners’ minds and mathematics in their environment, present in what they call ‘the representational view of mind’ which they find to be prevalent in mathematics education today. In the representational view of mind, the overall goal of instruction is to help learners construct mental representations that correctly or accurately mirror mathematical relationships located outside the mind in instructional representations. This view of Cobb et al (1988) is in conflict with constructivist views of learning in which learning is described as a process in which learners actively construct mathematical knowledge as they strive to make sense of their own worlds. Mathematical visualization is therefore the process of forming images or constructing mental representations (mentally or with pencil and paper or with technology) and using such images effectively for mathematical discovery and understanding. Einstein wrote to (Hadamard 1954:82):

> Words and language, written or oral, seem not to play any role in my thinking. The psychological constructs which are the elements of thought are certain signs or pictures, more or less clear, which can be reproduced and combined at liberty.

Visualization incorporates those mental processes or cognitive processes that make use of or are characterized by visual imagery, visual memory, visual processing, visual relationships, visual attention and visual imagination. This is because cognitive processing embraces attending, perceiving, listening, looking, visual imaging, conceptualizing, intuitive thinking and heuristic processing. Bishop (1989:5) defines the ability for visual processing as:

> This ability involves visualization and the translation of abstract relationships and non-figural information into visual terms. It also includes the manipulation and transformation of visual representations and visual imagery. It is an ability of process and does not relate to the form of the stimulus material presented.

Bishop (1989:7) recognizes important distinctions about visualization. One way is to think of visualization in terms of the visual images that are generated in it. In his review of the literature of visualization Bishop (1989:9) stated that the whole idea of 'visual aids' was based on the understanding that such aids provided an effective introduction for the learners to the elaborate abstractions found in the study of mathematics. Manipulative material helped learners create visual images which promoted the process of visualization. These visualizations may be relatively primitive phenomena, such as imagining a particular door handle being rotated, to abstract phenomena such as imagining a right-angled triangle inscribed in a circle. A wide range of visual imagery is used by individuals even when restricted to mathematical activity. Presmeg (1985) for instance seems to locate visual imagery in the mind. She says a visual image is a mental scheme depicting visual or spatial information, and this mental scheme can exist with or without the presence of the perceptual being visualized. Presmeg (in Thornton 2000:254) lists five different kinds of visual imagery that she identified in her learners as concrete, pictorial imagery (pictures-in- the-mind), pattern imagery (pure relationships depicted in a visual-spatial scheme), memory images of formulae, kinaesthetic imagery (involving muscular activity, e.g., fingers walking) and dynamic (moving) imagery. They are illustrated in Table 2.1 below:

Table 2.1 Presmeg' s different types of visual imagery

Type of imagery	**Description of visual imagery**
Concrete or pictorial imagery	This involves a holistic image that has parts only to the extent that they are parts of everyday objects. For example, a shape that looks like a cup is just called a cup.
Dynamic imagery	Shapes can change into new related shapes. Dynamic imagery is a means of linking different concrete images to one concept as well as a means of linking different concepts.

Pattern Imagery	This imagery allows learners to remember configurations and important features of configuration. It also allows learners to abstract important notions. In spatio-mathematics, pattern imagery is used when there is a conscious recognition of some of the properties of the concrete images and their relationships often in the form of patterns. For example a large triangle may be seen as a composite of four small triangles (e.g., making squares with matches).
Action or kinaesthetic imagery	This is imagery were a learner pictures him/herself doing the moving objects in an activity. In geometry the learner is aware of how the pieces were to be joined or transformed for a particular purpose, but generally did not know the whole procedure for making a specific configuration.
Procedural imagery	After learners have, made a shape, they are usually able to repeat the procedure of putting the pieces together to form the composite shape.

Such categorization is a tool for drawing attention to the diversity, richness and variety of uses of visual imagery. Memory images of formulae and pattern images are two types of imagery which provide a quick means of recall of abstract general principles and procedures; the former in a concrete image which encapsulates a procedure, the latter in a more schematic image which stresses regularities. In this case a shape is inspected in terms of its properties. Pictorial or concrete imagery and dynamic imagery are imagery involved during transformation since they involve generalizing about the nature of a shape or changing shapes into new related shapes. In this kind of imagery a ‘U’ can be called a cup, or a square can be seen as a parallelogram. In procedural imagery a learner is able to repeat a procedure and is in this way able to use originally generated images to make other shapes. Learners would normally use different visual imagery in different situations. The range of visualizations generated by individuals is therefore an important factor to keep in mind when considering mathematics teaching.

Visualization on another note attends to the process, the activity, the skill, the ‘how’ of visualization. Visualization is a complex process of constructing, re-presenting, and transforming mental images (Wheatley 1998:10-11). Being a mental picture, it is difficult to study directly, and as a result has been much debated. Research, however, has shown that mental images do exist and that many operations may be performed by individuals on these images in transformations such as rotating and folding. This research has led to the conclusion that "images are internal representations that ‘stand in’ for (re-present) the corresponding objects" (Kosslyn 1994:3). In Kosslyn (1980:29)'s psychometric model visualization is defined in terms of what happens inside one's head in the Cognitive Neuroscience direction. Kosslyn is of the opinion that the mind generates and operates upon analogical representations that preserve the spatial properties of visual stimuli. Structures and processes are distinguished. Two types of structures are identified, one being a visual buffer or form of short term memory which contains an array of cells that are activated whenever an image is formed. The second structure is the type of information stored in the long-term memory. This may include both propositional information about the parts of objects or an array of an object's literal appearance. In addition to the structures, he suggests the existence of processes that operate upon the different structures. One process that operates upon the visual buffer is called *regenerate*. This process activates or refreshes the representation which otherwise would fade with time. Other processes include r*otate, scan, zoom,* and *translate*. Each involves some transformation of the image resulting in a modification of the representation in the visual buffer. *Find* and *resolution* processes inspect and classify information in the representation.

Apart from the psychometric approach Kosslyn (1994:1-23) describes visualization as a cognitive process involving visual imagery in which images involved are either generated, inspected, transformed or used for mathematical understanding. Kosslyn (1994:379-407) proposes that visualization involves four specific steps: *image generation, image inspection, image transformation* and *image use.* There are micro-strategies within each of these general steps. Image generation occurs when a person brings forth an image from more abstract representation in long-term memory, and involves Action or Kinaesthetic imagery. In this process the learner pictures him/herself in an activity in which he/she is doing the moving of pictures in his/her mind. Concrete or pictorial imagery and dynamic imagery occur during image transformation and are important for changing shapes into other related shapes, and to generalize shapes into other known objects. Pattern imagery occurs during image inspection and results when one examines an image in order to answer questions about it and is therefore important for remembering shapes. Shapes during this process are recognized in terms of their properties, and pattern imagery is also used by learners in estimating sizes, in creating, recognizing and naming shapes. In spatio-mathematics it is used when there is a conscious recognition of some of the properties of the concrete images and their relationships, often in the form of patterns. In this way pattern imagery connects visual images and abstract conceptualizations by seeing, looking for and describing patterns as basic forms of mathematical thinking. Pattern imagery is also an important means for learners to remember configurations, important features of configurations and also allow learners to abstract important notions. Image transformation is when one changes or operates upon an image. Dynamic imagery occurs during image transformation, when shapes are changed into new related shapes. Decker and Elshout-Mohr (1998:310) describe this transformation as reconstruction of one's work. To reconstruct one's work means to criticize one's work and that may happen internally or externally. For example if a learner writes and rewrites his/her response to a problem this is evidence of a reconstruction and hence an image transformation. Use of image occurs when an image is employed in the service of some mental operation and this includes comparing properties of images or answering questions about an image. Procedural imagery in image use allows one to be able to repeat procedures. Each kind of imagery is important and it is wrong to regard one kind of imagery as at a higher level than another (Owens & Clements 1998:210). The mental processes in Kosslyn's categories were also described by Owens and Clements (1998: 204) in terms visual memory, visual processing, visual relationships, visual attention, and visual imagination. It must be remembered that Bishop defined the ability for visual processing as the ability that involves visualization and the translation of abstract relationships and non-figural information into visual terms. It also includes the manipulation and transformation of visual representations and visual imagery. It is an ability of process and does not relate to the form of the stimulus material presented (Bishop 1989:8). The detailed analysis of the image processes are given in Table 2.2.

Table 2.2 Description of Kosslyn' s categories of mental processes

PROCESS	DESCRIPTION OF MICRO-STRATEGIES

Image generation	• Occurs when a person recalls a picture or visual mental representation from long term memory and placing that image in a central location within working memory. • Multiple uses of materials. (Incorporating new experiences)
Image inspection	• Involves a focused mental scanning of the qualities of the image within working memory. • Results when one examines an image in order to answer questions about it (classify, scrutinize). • Noticing similarities and differences in shapes (connection). • Selectively attending to input, feedback from existing schemata.(heuristic processes): - establishing meaning of the problem - developing tactics - self-monitoring
Image transformation	• It is when one changes or operates upon an image: - rotate - zoom - translate - scan • Movement of images of shapes into related shapes (e.g., square to hexagon)
Image use	• It occurs when an image is employed in the service of some mental operation. • Mental involvement in making the shape (refinement) • Making, using, changing and storing images, concepts, understanding and schemata.

It is important to note that the image processes in Kosslyn's categories are hierarchical. A learner has to generate an image first, to be able to inspect it, transform it and then be able to use it. In some cases learners can generate an image and then fail to inspect it in a way that would aid them with their problem solving processes and this results in what is called an uncontrollable image. Another case is where a learner generates an image, but the image proves to be useless to that particular problem. In this case there is need for one to regenerate another image, and take it through all the processes of visualization if he/she is to progress, or else the learner fails to solve the task. Since visualization involves the use of visual imagery, visual imagery is therefore utilized in the visualization process to formulate the sub-categories in Kosslyn's categories. The definition of visualization is summed up in this study by putting together the ideas from different authors. Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (Bishop 2003:217; Zimmermann & Cunningham 1991:3; Hershkowitz et al 1989:75. Kosslyn's categories of the visualization process accommodate all the above ideas of what visualization is. For example, in image generation a product of creation is required and one goes through a process in order to create it and thereafter transformation interpretation and reflections are carried out upon the image. It must, however, be noted that the visual imagery processes involved within these categories are not hierarchical and can occur at the stages when they are required. They are only involved when they are needed.

There is increasing recognition that visualization plays a role in the learning of mathematics. Ben-Chaim et al (1989:49) recognized the role of visualization in developing inductive, deductive and proportional reasoning. The process of forming a conjecture by generalizing from a pattern of observations made in particular cases is the main component of inductive reasoning. This is the idea that is needed while solving tasks on patterns in Worksheet 4 and 5.Testing conjectures by constructing either a supporting logical argument or a counter example is the essence of deductive reasoning. This is an idea where visualization is a central component of many processes for making transition from the concrete to the abstract modes of thinking. Wheatley (1991) concluded that imagery based processes play an important role in all levels of mathematical problem solving and are used to represent mathematical ideas and information. Representing of mathematical ideas is important for all problem solving and good problem solvers are those who utilize the visualization process. Ben-Chaim et al (1989:50) indicate that visualization is a good predictor of problem solving performance. Problem solving is another aspect in which visualization is the key. It plays a central role in inspiring a whole solution, beyond the merely procedural. In another way it serves as a support and illustration of essentially symbolic results by in effect providing a proof in its own right and on another note it is used as a possible way of resolving conflict between correct symbolic solutions and incorrect intuition. It helps to re-engage with and recover conceptual underpinnings which may be easily bypassed by formal solutions. Davis (1984:35) describes a phenomenon which he calls visually-mode related sequences (VMS). It relates to an experience of thinking in one's mind if one wants to get to a place he has been before. In VMS one is able to see some key landmark, and get help from a visual reminder that will enlighten one's directions. In this case visualization may function as a tool to extricate one from situations in which one may be uncertain about how to proceed. As such visualization is linked, in this case not so much to concepts and ideas but rather to perceptions which lead to procedural decisions. Visualization at the service of problem solving may also play a central role to inspire a whole solution. For example, when one requires generalizing from a pattern of numbers or shapes visualization identifies gnomes as sub-structures of the whole in which a clear pattern can be established. Visualization makes use of visual representation of the problem statement. For example, a solver can imagine a visual story, impose it on the problem and derive from it the solution.

2.2.3 The quality of visualizations

The quality of the images generated during visualization also appears to vary in a marked way. Some images generated are clear while others are not so clear. This is referred to by Richardson (in Presmeg 1986:44) as the 'vividness of images' that helps learners particularly in memory situations. Some positive qualities, associated with visual imagery are its integrative power, its exemplary use, its concretization of abstract ideas, and its sometimes sudden illuminative aspect (Presmeg 1986:45). There are qualities that are unique to those learners who have higher visualization skills. If we call them qualities then they can be instilled or trained to the learners through the way we teach them. Krutetskii(1976:325) says of the gifted visualizers in his study:

> …the graphic schemes used by these learners are a unique synthesis of concrete abstract…Supporting thought by, and even binding it to such a generalized visual image cannot prevent generalized thinking. In such a case this image is in a certain sense the bearer of sense and content of an abstract concept.

Other qualities of visualization often appear in a negative frame relating more to the obstacles which they can create, one of which is called "geometric rigidity" (Hoz in Presmeg 1985:295). Some kinds of imagery can be controllable or uncontrollable and can occur in one or more of the five modalities: visual, gustatory, olfactory, tactile and auditory (Presmeg 1992:596). Uncontrollable images are caused by a tendency for thought to be riveted to an image which can sometimes not be useful in the solution of that particular problem. This is evident in a child who is unable to see a diagram in a different way. He/she sees the diagram in one simple way, and is often a way that does not help him/her to solve his/her mathematical problem. If a teacher is aware of the causes of this 'geometric rigidity' then surely many ways can be found that will help reduce this problem. It must be remembered that it was earlier indicated that visualization not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution.

2.2.4 Relevance in mathematics education

The research findings of the different authors cited above have very important implications for mathematics education. The educational features of visualization include the procedures, the role of the material and social environment, and how the individual interacts with that environment. An understanding of the precise limitations of children's imagery could inform one on how to use imagery in teaching. For example, if learners' images are static, in that learners cannot transform their images (see them meld into a new shape), they may still be able to make use of "blink" transformations (that is erasing the first image and imaging the object anew in some altered way) (Bishop 1989:8). If so, then it would make sense to shift from trying to teach the child the rules of transformation to teaching him the rules of formation. Learning how to break up an initial image and see it flow into a new shape is different from learning how to form a second image that is related to the first in some way. Similarly, if young children are able to form images easily, while they have difficulty in maintaining them, it would be better to try to teach them by inducing a series of rather simple images. In addition, if one knew the specific limitations of children's imagery one might be able to use the symbolic mode of imagery to teach fairly 'abstract' ideas. Van Niekerk and Lampen (1997:28) stress the following:

- The mental processes that underlie the experience of an image are similar to those that underlie the perception of objects or pictures.
- An image is a coherent integrated representation of a scene or object from a particular viewpoint and is open to a perceptual-like process of scanning.
- An image can be subject to apparently continuous mental transformations, such as rotations in which intermediate states correspond to intermediate views of an actual object undergoing the corresponding physical transformation.
- Images represent not only objects, but also interrelationships between an object's component parts and other objects. That is "the functional relations among objects as imagined must to some degree mirror the functional relations among those same objects as actually perceived".

What is interesting for the study is what can be learnt from research about how the educational process relates to visualization, and what further research can do to improve the visualization skills of all learners. If learners are placed in specific material and social environments and are required to carry out certain tasks, it has been found that children create and use very different kinds of imagery, and they operate with those visualizations in very different ways. Whereas imagery is necessarily individualistic in the sense that an image resides in a particular person's mind, different people can have more or less common images in the same way that we say people have a shared understanding of a concept. This can occur when images have developed partially from the use of the same physical phenomena, bodily movements and interactions.

Some researchers however argue that visualization is not trainable. Clements and Del Campo (1989:27) state that psychologists have talked about training to improve imagery but they also agree that there is little evidence of progress at either the theoretical or pedagogical level regarding this training. Overall encouragement to develop visual images and to promote general visualization awareness is attainable in teaching, but the promotion of visualization in learners is not an easy matter. Leon (in Bishop 1989:11) concludes, "The evidence…indicates that these various skills (involved in visualization) are trainable given the appropriate experiences". This can be done as it has been widely demonstrated, by teaching learners about different kinds of figures and images, by showing the conventions used, by giving practice in representation according to those conventions, and generally by developing their visual vocabulary. There is a need to choose appropriate tasks for the learners to work on. Moreover, there is need for a very relevant and conducive environment, which in this case is the problem-centred context. The following suggestions (Thornton 2000:255) were given as starting points for teachers to help learners become more effective visual thinkers.

- Be sensitive to the possibility of finding visual solutions or representations of a given result.
- Encourage concrete pictorial imagery by asking learners to picture themselves as part of the situation.
- Encourage pattern imagery by connecting results in number and algebra with models such as area and length.
- Encourage dynamic imagery by using software such as Cabri or Geometer's Sketchpad.
- Promote discussion of alternative ways of thinking, and particularly of the transition from visual to symbolic.

- Encourage learners to look at problems holistically instead of breaking them into parts.

2.2.5 Difficulties around visualization

While there is a general support for visualization many learners do not like it, preferring to use algorithms. One of the main reasons for this reluctance is situated in the fact that thinking visually makes higher cognitive demands on the learner (Eisenberg & Dreyfus 1989:2). One needs high cognitive skills to visualize properly. Larkin and Simon (in Presmeg 1986:44) said that diagrammatical representations are not immediately intelligible to the uninitiated. It takes cognitive processing to make sense of diagrammatic representations. Diagrams are useful only to those problem-solvers who can construct a good diagram and the appropriate computational processes for taking advantage of them or otherwise learners experience difficulty interpreting diagrams or even constructing useful diagrams. Some of the problems experienced in the resistance against visualization are philosophical in nature. According to tradition, mathematics was said to be non-visual regardless of whether or not a visual representation is at the root of an idea. In addition, visualization methods often require more time and more space to think.

Sword (in Eisenberg &Dreyfus 1991) describes the problems experienced by highly capable visual thinkers who may be at risk in the school system because their learning style is not recognised. She maintains that traditional teaching techniques are designed for auditory-sequential learners, and hence disadvantage visual-spatial learners. Materials introduced in a step by step manner, carefully graded from easy to difficult, with repetition to consolidate ideas is not only unnecessary for the visual-spatial learner, but by failing to create links in a holistic manner, actively works against such learners progressing to their potential. As a result gifted visual-spatial learners often exhibit characteristics such as lack of motivation, inattentiveness, weaknesses in basic calculations and disorganization. The common advice to learners is to read problems carefully, break them down into manageable steps, formulate algebraic expressions representing the situation, and then solve this formalized version of the original problem. This could prove quite counter-productive for some learners. Seldom do we ask our learners to step back from the problem, to look at it holistically and to try to visualize the situation in its entirety.

The difficulties around visualization were classified by Eisenberg and Dreyfus (1991:32) under the title “some unseens we are beginning to see”, and were put into three categories: cultural, cognitive and sociological. A cultural difficulty refers to the beliefs and values held about what mathematics and doing mathematics could mean, what is legitimate or acceptable, and what is not. Controversy within the mathematics community and statements such as ‘This is not mathematics’ (Sfard 1994:45) by its most prominent representatives, are likely to permeate through to the classroom, via curriculum materials, and teacher education and shape their emphasis and spirit. Presmeg (1997:310) calls this attitude ‘devaluation of visualization’, and it leaves little room for classroom practices to incorporate and value visualization as an integral part of doing mathematics. When visualization acts upon conceptually rich images, the cognitive demand is certainly high. Besides, reasoning with concepts in visual settings may imply that there are not always procedural ‘safe’ routines to rely on (as may be the case with more formal symbolic approaches). Consciously or unconsciously, such situations may be rejected by learners (and possibly teachers as well) on the grounds of being too ‘slippery’, ‘risky’ or ‘inaccurate’. Another cognitive difficulty arises from the need to attain flexible and competent translation back and forth between visual and analytic representations of the same situation, which is at the core of understanding much of mathematics. Learning to understand and be competent in the handling of multiple representations can be a long-winded, context-dependent, non-linear and even tortuous process for learners (Schoenfeld, Smith & Arcavi in Arcavi 2003:220). This in itself can deter learners from preferring to use visualization.

Under sociological difficulties Eisenberg and Dreyfus (1991:32) under what they considered as “issues of teaching” suggested that teaching implies “didactical transposition” which briefly stated means, the transformation knowledge inexorably undergoes, when it is adapted from its scientific/academic character to the knowledge as it is to be taught. It is claimed that this process by its very nature linearizes, compartmentalizes and possibly also algorithmetizes knowledge, stripping it (at least in the early stages) from many of its rich interconnections. As such, many teachers may feel that analytic representations that are sequential in nature seem to be more pedagogically appropriate and efficient. Another socio-cultural problem is the tendency of schools in general and mathematics classrooms in particular, to consist of learners from various cultural backgrounds. Some learners may come from visually rich cultures, and therefore for them visualization may counteract possible deficits.

With the idea that visualization involves images that are constructed, re-presented, transformed and maintained, Presmeg (1985) describes learners' experience with what she calls uncontrollable images. These images appear to arise from the tendency for thought to be riveted to an image which is inappropriate or which prevents mathematical generalization. Twnman (in Leon & Clements 1981:274) commented that the creation of an image can introduce difficulties associated with decoding the image. For example, the image might possess irrelevant details which distract the problem solver from the main elements in the original problem stimulus, and make it difficult for him/her to formulate necessary abstractions. Krutetskii (1976:29) provides a classification of the difficulties experienced with images:

- The one-case concreteness of an image or diagram may tie thought to irrelevant details, or may even introduce false data;
- An image of a standard figure may induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram;
- An uncontrollable image may persist, thereby preventing the opening up of more fruitful avenues of thought, a difficulty which is particularly acute, if the image is vivid;
- On the other hand, imagery that is vague needs to be coupled with rigorous analytical thought process if it is to be helpful.

Whilst it is tempting to believe that all visualizations must necessarily play a useful role in mathematical activity, there clearly is need for research which helps to understand more about which of the features contribute significantly to the role in a given mathematical situation. It is sometimes said that the most able mathematicians can think entirely in abstractions and have no need of diagrammatic props. MacFarlane Smith(in Presmeg, Aspinwall & Shaw 1997:303) believe that it could be that these gifted individuals have their own internal blackboards and can visualize complicated structures without being aware that they are doing so.

2.2.6 Factors that affect visualization

2.2.6.1 Spatial sense in relation to visualization

In the field of mathematics some mathematicians have claimed that all mathematical tasks require spatial thinking. The world around us is partly geometric and activities such as perception, manipulation and navigation all require a spatial understanding of our environment and everything in it. The ability to create a mental image of an object and then to manipulate it mentally has significant practical application in fields such as mathematics (Robichaux 2000: 1). Spatial visualization involves mentally manipulating images of visually presented objects in a holistic rather than piece by piece fashion (Battista & Wheatley 1989:18). According to Kosslyn (1980:89) spatial sense must be thought of in terms of imagery, and yet visualization is also thought of in terms of imagery. Imagery involves the construction, representation and transformation of self-generated images. Images are constructed from viewing objects, reading a passage or by just reflecting and are influenced by what we know. Once an image is constructed it must later be called up when needed and we do this by representing the images. It is what you do if you are asked to determine the number of windows on your house. Transforming images is a dynamic process, for example transforming a rhombus into a square. So, in general, spatial sense involves using mental images and one can gain a picture of the kind of skills necessary for this interaction. Several researchers have hypothesized that the mathematical behaviours exhibited by individuals as they solve problems or learn mathematics may depend on the more fundamental or ‘primary’ mental abilities. Bishop (in Battista & Wheatley 1989:18) suggests the abilities to be spatial visualization and formal reasoning. It was therefore suggested that mathematical ability could be developed not by teaching mathematics but also by suitably emphasizing and developing those primary abilities.It is generally agreed that persons with well developed spatial skills should be capable of imagining spatial arrangement of objects from different points of view and of manipulating visual images which, according to mathematics, allows a learner to gain mathematical power. Mathematical power involves the capacity to make connections, both between mathematical objects and concepts and between mathematics and the physical world. Visualization, whether in the form of concrete images, pattern images, dynamic images or many other images, has a key role to play in the development of learners’ mathematical powers.

In spatial visualization the viewer mentally moves the representation of the indicated object by either turning it as in mental rotation or rearranging it as in mental transformation. Kesh and Cook (in Fennema, Carpenter & Peterson 1991:29) describe this as mental rotation or mental transformation. Visualisation involves visual imagery of objects and movement or change in the objects or their properties. In other words objects or their properties must be manipulated mentally. Spatial orientation is the other category of spatial skills. Here nothing needs to move except the perception of the person taking the test, or some change that has taken place between two representations. Spatial orientation requires that the subject mentally readjust his/her perspective to become consistent with a representation of an object presented visually. It means‘re- seeing’ it, seeing it from a different angle, recognising it, or making sense out of the object. This means that the development of learners’ spatial abilities encourages the visualization process of these learners. Yakimanskaya (1991) defines spatial ability as the ability of children to manipulate mental images in their thoughts. It is however generally agreed that persons with well developed spatial skills should be capable of imagining spatial arrangements of objects from different points of view and of manipulating visual images (Clements & Battista 1992). Therefore the concept of image plays a central role in the study of spatial ability. Moses (in Lean & Clements 1981:271) concluded that spatial ability is a good predictor of problem solving and visuality. Visuality refers to how often a learner uses visualization during problem solving. It is therefore important to develop activities in such a way that the child is able to move from 3-D representations to 2-D representations and vice versa.

2.2.6.2 The computer and material environment in relation to visualization

The term visualization according to Hiebert et al (1996:15) is described as a process of producing or using mental or geometrical or graphical representations of mathematical concepts, principles or problems, whether hand-drawn or computer generated. Computer graphics has greatly expanded the scope and power of visualization in every field. One of the areas in which the above is clarified is in “Computer visualization as a mathematical tool” Instead of producing and understanding only the mental images, the computer can help with this clarification of images. For example, one is able to see a 3-D object on the computer from within and is therefore is able to transform and use the images. There is evidence that computer graphics have played an increasingly important role in both core and applied mathematics and the opportunities for utilization are enormous. There are more technologies like the video-tape film and interactive video-dice, which further expand the kinds of images that can be used in visualization. Visualization technology is expanding at a terrific rate.

Today the micro-computer is increasing the range of aids to visualization enormously, and its presence in the mathematics classroom is stimulating a great deal of research and development in this area. On the other hand there is evidence from a variety of studies that the power and accessibility of computer-generated images can have a stimulating influence on learners’ visualizations. In relation to geometric ideas, the studies of Gallou-Dumiel Olive, Langenau, and Osta (in Bishop 1989:13) all demonstrate in different ways how an interactive computer environment, particularly when dynamic visual images are employed, can encourage and to some extent develop the learners' visualization abilities. The computer is therefore a very powerful tool in the area of visualization development. Overall, computers have proven to be a great aid to the creative human activities of visualization. Computer generated images have provided access to a variety of data which may have remained impenetrable by pre-graphical techniques.

The material environments seem to interact one way or another with visualization. Earlier research (e.g. Bishop 1989; Ben-Chaim et al 1989) proves that structured and manipulative materials can help to encourage the creation of visualizations and thus the visualization process itself. Learners learn visual skills by involving themselves in a rich combination of drawing, manipulating, imagining, paper-folding, building, and using computerized enhancements or simulations of any or all of these. The activities may be physical, but the essential skills are mental and must be complimented by reflection on what they have done. Relevant activities that have been found to be significant include drawing perspective views of block buildings or matching perspective drawings with a building. Isometric dot grids are used for the drawings. There exists an amazing computer tool like the geometer's sketch-pad for drawing perspective views of block buildings. This tool requires only mouse clicks to draw either whole cubes, any single face of a cube or just lines.

2.2.6.3 The social environment in relation to visualization

The role played by the social environment is also important. For many learners, visualization and visual thinking serve as the first opportunity to participate in open ended mathematical activities, and reduces an old equity problem. These visualization skills can be found through informal experiences at home, such as building models, manipulating structured visual materials like blocks or Legos, and taking things apart and putting them back together. Therefore, out of school experiences with such materials may help build the foundations for mathematical achievement. Henningsen and Stein (1997:529) distinguish between visualization and visual thinking. By visualization they mean bringing inherently visible things to mind (spatial visualization), and the term visual thinking to refer more broadly to a visual rendering of ideas that are not inherently spatial (for example visual thinking about multiplication). In this study visualization refers to both descriptions given above and therefore is able to accommodate the tasks from data handling.

2.2.6.4 Intuitive thinking

Intuitive thinking is a process of thinking that does not subject itself to analytical thinking. Fischbein (in Carpenter & Franke 1996:4) defines intuition as a global guess or mental product for which an individual can or cannot give written justification. It should be noted that intuitive thinking is a personal matter, since what can be accepted as true by one person might not be accepted as true by another person. Intuitive thinking can neither be taught nor learnt, it can be acquired by mastering the art of the subject (discipline). In other words the more experience one has in a subject, the more one can think intuitively. Experience is vital for the development of mathematical intuition (Fischbein 1987; Carpenter & Franke 1996). Mental visualizing is similar to intuitive thinking and as such also requires a lot of experience in the subject. Tall (1991:14) mentions that:

> Intuition is the product of concept images of the individual. The more educated in logical thinking, the more likely the individual‘s concept imagery will resonate with a logical response.

An important factor in intuitive thinking is the level of sustained attention to one‘s work (Bishop1988: 183).

2.2.6.5 Reflective thinking

Reflective thinking is a meta-cognitive process. The meta-cognitive processes that are characteristic of problem solving include self-monitoring, self-regulating and self-assessment (Ramnarain in Colet 1999: 31). Reflective thinking is a process whereby one keeps on checking whether one is solving a problem correctly or not. In other words reflection does not only take place after one has resolved a problem, but it also takes place during the process of solving. Persons who reflect on their activities or solutions have control over their thinking and they are able to pursue several paths to the solution (Wheatley in Zimmerman & Cumming 1991:6). Gagatsis (in Carpenter & Franke 1996:33) argues that if one is aware of the processes involved in solving a problem, one is able to describe and explain it to others. It follows that in order to be successful in visualization, learners should be encouraged to use meta-cognitive strategies. Initial stages of reflective thinking are observation, noticing things and asking questions. Reflective thinking is often associated with clever learners in the classroom, but the truth is all learners can make use of reflective thinking.

2.2.7 Summary

The definition of visualization is achieved by looking at it in three ways. Visualization is defined in terms of it being an object, and this brings us in this study to look at the visual images generated by individuals. Visualization is also defined in terms of it being a cognitive process. This idea is supported by Nemirovsky and Noble (1997:255) who describe visualization as the means of traveling between external representation and the learner's mind. It is what goes on in an individual's mind when he/she is solving a problem. Lastly visualization is defined in terms of its educational features. Visualization should not be reduced to vision but should make visible all that is not accessible to vision. There is a gap between visual perception and visualization. Visual perception needs exploration through physical movements because it never gives a complete apprehension of the object. On the contrary visualization can provide a complete picture of any organization of relations.

The above findings are important when one is trying to investigate the activities that are relevant for enriching learners' visualization processes. This is because the visualization process involves the learner constructing some kind of images and using them appropriately.

Different tasks will stimulate different images. Thus the tasks need to be 'appropriate' visualization tasks. How much of this activity is required must depend entirely on the interaction between the stimulus and the individual concerned. Each individual must interpret a task in his/her own way, for there is never a unique way to carry out a task. What then happens will depend on one's preference, on one's memory of visualization, on one's ability to recall or to generate appropriate visualizations, or to choose appropriate ones, and then on one's ability to operate appropriately with the chosen visualizations. The implication of this analysis is clearly that visualization and the use of visual imagery is a very personal matter. Because the visualization process is creative in that it generates images, it therefore makes sense to expect differences in the way people deal with it. Therefore, individual learners need to be given time to develop their own images, and there seems little value in teachers expecting identical images to result from such a personal process. As a teacher one must anticipate, and, to some extent, encourage the possibility of diversity in visualization.

For the purpose of this study the mental processes that one goes through during problem solving, the role of the material and social environment and the way in which the individual interacts with the environment were investigated using data handling and spatial tasks in a problem-centred environment. Visualization is a good predictor of problem performance, and provides the learners with additional strategies potentially enriching their problem-solving repertoire (Ben Chain et al 1989:51). Teaching strategies using visualization are needed especially for the verbalizer and those of mixed mode to develop spatial skills (Ben Chain et al 1989:50). The other implication of the above analysis concerns the research itself. If we want to understand more about the visualization process we need to study it in a variety of task and stimulus contexts, and we thus need to move away from mere problem solving. Visualization is trainable provided learners are involved in activities that will allow them to use senses other than vision, for example touching, manipulating constructing and drawing. In data handling learners should be allowed to become involved in all types of activities that include either drawing, seeing, reading from the data and making primary analysis. According to Robichaux (2000: 2), spatial visualization topics and activities should be taught well throughout the mathematics curriculum, particularly in the middle grades so as to allow for the development of visualization in learners.By ignoring visualization, curricula not only fail to engage a powerful part of learners' minds in service of their mathematical thinking, but also fail to develop learners' skills at visual exploration and argument. The development of visualization skills is not supposed to be the goal of mathematics education only, as courses in the design of clothing, or stage sets, or furniture could do that as well.

2.3 THE PROBLEM-CENTRED CONTEXT

2.3.1 Introduction

The problem solving part of the empirical part of the study was carried out in a problem-centred context. A problem-centred context is an environment in which the problem-centred approach is implemented. Therefore in order to understand the problem-centred context in which the research was carried out, an understanding of the problem-centred approach was important. What prompted the different authors in this field to pursue the problem-centred approach was that the mathematics educationists had raised doubts about the traditional teacher-centred approaches. In the traditional approach, learners were taught how to solve problems as a section of the mathematics curriculum, instead of integrating it into the whole curriculum. This meant that learners could often solve a specific problem using mathematical knowledge, but as soon as a different problem was posed, they could no longer solve it. The problem-centred approach aims to teach all mathematics through problem solving, allowing learners to make connections between the mathematics concept which they are developing and a wide range of problems. It is therefore in the problem-centred context that the study was carried out.

2.3.2 Defining the problem-centred context

The problem-centred approach is perceived by Murray, Olivier, and Human (1998:272) as 'Learning mathematics through problem solving' as opposed to the problem solving approach which refers to "teaching mathematics for problem solving" which is regarded from the learners' point of view as learning to solve problems. Hiebert et al (1996:15) advocate a problem-centred environment in which the subject is allowed to be problematic as one dimension of a problem-centred approach to the mathematics education curriculum. In the problem-centred context, learners are allowed to wonder why things are, to inquire, to search for solutions and resolve incongruities. This implies that in this context learners should be allowed to discover mathematics for themselves by finding their own unique solutions to various problems, without having the teacher dictate to them how they should do it. The learners need to be able to construct their mathematical knowledge by engaging themselves in actually solving problems instead of merely rote learning concepts and procedures taught by the teacher. Therefore in a problem-centred context, instruction should begin with problems, dilemmas and questions for learners, inquiry oriented involved in creating, conjecturing, exploring, testing and verifying.

The problem-centred context in mathematics also comprises the establishment and maintenance of a classroom culture that promotes active thinking by learners. Active thinking (reflection) by learners, a reflective attitude to learning, and an insistence on understanding, are absolute prerequisites for sound understanding and the realisation of personal mathematical potential. In this context, problems serve as the centre for the study as a whole. The whole learning is problematic to effect the construction of understanding. These problems offer opportunities for using multiple solution strategies, multiple representations of the concept, and mathematical communication that includes explanation and justification, what is called reflective inquiry. This context advocates for a situation where learners are engaged in and are part of the activities. Real life problems are provided and this way the problem-centred context attends to the range of learners by making learners make sense of the mathematics in their own way, bringing to the problems only the skills and ideas that they own. The problem-centred environment lets the learners know that as they do activities and solve problems in class, the teacher will look at their work, listen to their explanations and provide them with feedback in terms of a rubric rather than as a letter grade or a percentage.

2.3.3 The nature of the task and the classroom culture in a problem-centred context

The nature of the task has a significant role to play in the problem-centred context. There are certain desirable features of the tasks that are required in a problem-centred environment. Erickson (1999:517) summarizes them using a problem solving approach. The features are genuine problems that reflect the goals of school mathematics, motivating situations that consider learner' interests and experiences, local contexts, puzzles and applications, interesting tasks that have multiple solution strategies, multiple representations and multiple solutions, rich opportunities for mathematical communication, appropriate content considering learners' ability levels and prior knowledge and reasonable difficulty levels that challenge yet do not discourage.

Although the content of the tasks is important, the culture of the classroom determines how the tasks are to be treated by learners. Effective problem-centred mathematics education requires radical changes in classroom organization and classroom culture. Among other things it involves that all learners will not have the expectation that the teacher will show or guide them towards methods of computation, that learners will wholeheartedly accept the obligation to do mathematics purely through their own efforts at solving problems, reflection about their efforts and the exchange of ideas with other learners. Human (1993:3) proposes that learners think actively only when they are faced with something that makes thinking worthwhile, when they are given the freedom and time to think, when they feel safe to follow the course their thinking takes and when they are obliged to think. In this culture justification of answers and methods, clear and systematic articulation of methods, respect for each other's learning opportunities and contributions are the central norms are the instruments the teacher must value to promote the effective learning of mathematics. Lampert (in Erickson 1999:519) suggests that the teachers must implement a code of behaviour that encourages learners to take risks by offering their ideas, conjectures, and conclusions without fear of reprisal, ridicule or embarrassment.

2.3.4 The teacher and the learner in a problem-centred context

In a problem-centred context learners are encouraged to value mistakes since they illuminate the problem situation or bring to light possible misconceptions about the mathematics. In this context learners are able to realise that mistakes are just as important as correct answers in understanding the mathematics involved. They must be held accountable for thoughtful, reasoned responses. Learners should be motivated by the solutions to the problems, and do not rely on the appreciation of the teacher for judging success. Competition and reward must not be the aim, as learners must act as their own judges. To meet the above challenge successfully the teacher must let learners struggle together towards solutions without suggesting procedures, yet providing sufficient scaffolding or guidance to keep learners interested and on the task (Clark in Erickson 1999:519). Scaffolding may involve additional mathematics instruction for unknown content when necessary, individual handheld strategies, prompt cards, and hint cards. Additionally, teacher-questioning strategies to help learners clarify their own thinking are important for success in this approach. Teachers press learners to give meaningful explanations, demand accountability both for completing the task and learning the concepts and processes embedded in the task, and expect learners to answer questions about mathematical assumptions and generalizations and justify their reasoning and conclusions.

The nature of the teacher's role, obligations and actions in the classroom, and the way in which the teacher supports learners are quite different from that of transmission teaching. Instead of telling and showing learners how they should do mathematics and checking that they follow these instructions, the teacher presents the learners with challenging problems to be solved and deliberately selects and sequences these problems in such a way that it would facilitate learner's progress from existing knowledge to new knowledge. From the above it is clear that the teacher needs to take an active role in selecting and presenting tasks. To select appropriate tasks the teacher must draw on two resources: knowledge of the subject to select tasks that encourage learners to wrestle with key ideas, and knowledge of learners' thinking to select tasks that link up with the learners' experience and for which learners can see the relevance of the ideas and skills they already possess. Careful minimal intervention also encourages learners to stretch their existing knowledge as far as possible and thus extends and enriches their conceptual network of facts, relationships and their implications rather than to assume that each new twist of the situation requires an additional element of instruction. Reflection and discussion, with the learners expressing their perceptions of the situation in as many different ways as possible, help to connect the new knowledge firmly to the old.

2.3.5 Summary

A problem-centred mathematics curriculum is characterized by an approach that gives learners opportunities to construct their own mathematical ideas. That way meaning or making sense of what is being learnt and retention are not imposed but acquired through the active involvement of the learner.There is need for learners to be acquainted with the ‘know how’ to deal with problems around them. A situation where learners are engaged in and are part of the activities is advocated. Learners should be empowered so as to feel that they are creators or builders of their knowledge. Initiating and sustaining mathematical development through posing problems that learners have to work on, is found to be a successful way of learning mathematics, but only if the problems are well designed and well-sequenced, and the classroom culture in its full complexity supports learning. In other words it is advocated that mathematics must be learnt in the context of solving problems. So the starting point in a problem-centred context is to problematize the subject. This means that learners are given problems to solve and from which to use and generate as much mathematics as possible with the teacher’s role being that of a guide. Research has shown that learners who have learnt to work in a problem-centred context display increased problem solving, decision-making, modelling and reasoning-process skills. They also exhibit enhanced conceptual understanding of specific mathematics content such as multiple representations for learning functions.

2.4 Data handling

2.4.1 Introduction

We live in a data rich society, and therefore the ability to comprehend the meanings embedded in data and to work with data to answer specific questions is an important educational outcome. As learners manipulate data sets and examine how to represent relationships through a variety of graphical forms in data handling, they develop a strong conceptual understanding of the foundations required for further study of statistics. Even more important is the fact that learners need to be able to distinguish what questions are amenable to investigating with data and how to collect, organize and interpret that data. This is in line with the specific outcomes of the National Council of Teachers of Mathematics. (NCTM) standard for data analysis and probability which states:

> ‘The learners are to formulate questions that can be addressed with data and collect, organize and display relevant data to answer them, select and use appropriate statistical methods to analyze data, develop and evaluate inferences and predictions that are based on data, and to understand and apply basic concepts of probability (NCTM 1981:324).

Children are encountering more and more graphic representations of data in their learning and everyday life. Much of this data occurs in quantitative forms as different forms of measurement are incorporated into the graphics during their construction. In their formal education, children are required to learn to use a range of these quantitative representations in subjects across the school curriculum. These calls for reform have advocated a broader approach to the study of data handling, one that includes describing, organizing, representing and interpreting data. Interpreting and analyzing data are problem solving processes that are essential for dealing with information presented in many different forms, including but not limited to graphs and tables. Recognizing trends, extracting patterns and extrapolating from data are high-order problem solving components of data interpretation and analysis (Curcio & Artzk 1997:125).

2.4.2 Defining data

Data broadly refer to both descriptive and numerical data. The data may come from a variety of sources collected from experiments or surveys carried out by learners themselves from mathematical investigation, invented data provided by teachers or text book authors or extracted from real-world sources such as government statistics. Data can be treated as objects independent of the existence of that which they represent. Data can be manipulated and conclusions can be drawn about these manipulations independent of actions in the world (Lehrer & Romberg 1996:70). For example, by manipulating data, new questions can be posed about relations among elements of the data structure. Hancock and Kaput (in Lehrer & Romberg 1996:70) suggests that the objectification of data gives rise to a deductive quality. Action taken on data can generate new data. Data can externalize the processes of weighing evidence, reasoning and reaching conclusions. Weighing evidence and reaching conclusions spread the concept of data to include terms such as chance and inference.

2.4.3 Data handling

Data handling, according to Shaughnessy (1997), includes describing, organizing and reducing, representing, analysing and interpretation of data. In response to the vital role that information and data play in our technological society, there have been worldwide calls for reform in statistics education at all grade levels (Australian Education Council (AEC) 1991:178; NCTM 2000:324). These calls for reform have advocated a broader approach to the study of data handling, one that includes describing, organizing, representing and interpreting data. This broadened perspective has created a need for further research and understanding of data handling. From Curcio and Artzk's (1997:123) research describing data involve direct reading of data displays, finding information explicitly stated in the display and recognizing graphical conventions between the original data and the display. Organizing and reducing data incorporate mental actions such as ordering, grouping, and summarizing data. It therefore involves characterizing data using measures of centre and spread. Representing data involves the construction of visual representation of data including representations that exhibit different organization of data. For example, any graph can be thought of as a visual language for understanding, expressing and communicating quantitative and qualitative data. Analysing and interpreting data focus on recognizing patterns and trends in data and on making inferences and predictions from the data. Reading beyond the data involves making predictions and inferences from the data by tapping existing schemata (i.e., background knowledge) for information that is neither explicitly nor implicitly stated in the graph.

2.4.4 Ideas about data handling

In data handling there is need to understand how learners construct meaning from data during the process of visualization. Constructing meaning from data is described clearly by Ainley and Pratt (2001:2) as encompassing the psychological, epistemological and pedagogical processes when learners engage in a range of activities that we see as essentially mathematical, involving the meaningful manipulation and interpretation of data. On a psychological level it is interesting to investigate how learners, when in possession of data, mentally transform that set of numbers, characters of figures or figures into meaningful information. The theoretical framework is based on the idea that children already possess intuitions that would support the constructions of meaning for data and there is interest in how these intuitions operate and how they might further develop. Epistemologically this study aimed to provide tasks that produce results from data, with the special focus being the processes employed to produce such results. Pedagogically the approaches used encourage exploration, problem solving and reasoning through the use of meaningful data. Since data handling involves organizing and reducing, representing, analysing and interpretation of data, understanding the different natures of these can aid better understanding of data handling.

Creating data

Learners are involved in creating data. McClain and Cobb (in Ainley & Pratt 2001:3) state that the data creation process provides a means for constructing initial meanings for the numbers themselves, but that the construction of meanings for data must extend well beyond appreciation at that level.

Data representation

For more than one hundred years, mathematics educators have been interested in the visual and figural representation of mathematical ideas both in the work of individuals and in the process of teaching about those ideas. Bishop (1989) is of the opinion that the whole notion of 'visual aids' is based on the knowledge that such visual presentations offer a powerful introduction to the complex abstractions of mathematics and that 'manipulatives', 'concrete embodiments' and 'intuitional devices' are part of the up-to-date teacher's resources. Data representation involves the graphical presentation of information, with the goal of providing the viewer with a qualitative understanding of the information content. Information may be data, processes, relations, or concepts. Graphical presentation may entail manipulation of graphical entities (points, lines shapes, images, text) and attributes (colour, size, position, shape). Understanding may involve detection, measurement, and comparison, and is enhanced via interactive techniques and providing the information from multiple views and with multiple techniques. Data representation leads learners to concept formation through a meaningful synthesis of diagrams and visual images in addition to verbal definitions and analyses on the other (Van Niekerk 1997:91).

Statistical diagrams cannot just be drawn at random as in the case when a child is playing with spatial or geometric shapes. Graphs require drawing skills and higher cognitive skills in identifying the axes (as in histograms), the percentages (as in pie charts) or simply the higher cognitive arrangements (as in stem and leaf) and the numerical calculation techniques (as in box and whiskers or ogives). By giving the learners a chance to get the results themselves by visually seeing the problem we are allowing the learners an opportunity to construct knowledge by themselves.

Analysis

At the heart of data handling is exploratory data analysis which is the discovery of meaning in data sets. Rather than merely developing statistics or computing such parameters as averages or spreads, exploratory data analysis develops graphical displays that reveal patterns and identifies subgroups within a data set. Shaughnessy et al (1996) suggest that data analysis studies patterns, centres, clusters, gaps, spreads, and variations in data, and in essence can be simplified to look at the data (preliminary analysis), look between the data (comparisons), look beyond the data (informal inference) and look behind the data and thus create a mathematical model. In data handling the learners observe the data and search for patterns or relationships while they endeavour to give meaning to their representation. They sort out useless information which places the learners at the descriptive level. The eyes and the mind work closely together, until finally the reality and explanation of the problem become clear.

2.4.5 Modelling, visualization and data handling

Mathematical modelling is the mathematizing of reality in order to solve real world problems by applying appropriate mathematical techniques and then translating the mathematical results in the real world. Modelling is a process to come up with a mathematical model. Tall (1991: 34) defines a model as a simplified representation of reality, which employs mathematical concepts and/or symbols. Being a simplified representation, modelling mirrors only some aspects of reality and overlooks others. Mathematical models include formulae, equations, graphs, tables etc. Modelling is the construction of these models to describe, represent or idealize a certain practical situation. The implication of the definitions is that the modelling process is intended to translate a situation from one form which appears to be complex into another which facilitates visualization and is easy to understand. If the situation is described verbally, a diagram may help to create a mental picture that is easy to analyze. Sometimes a symbolical model helps to conjure up a similar situation and consequently a familiar solution process or one that requires slight modifications. Modelling activities might involve making drawings, diagrams or tables, or it could involve developing informal mathematical notation or using conventional mathematical notation (De Lange 1987: 61). The construction and use of data is also referred to as data modelling and this process plays a crucial role in data handling. Data construction and analysis provide an opportunity to involve learners in the important enterprise of mathematical modelling (Lehrer & Romberg 1996:90). The construction and use of data are connected with the development of mathematical models. The idea of data entails a separation between the world and a representation of that world, and this begins the process of modelling. Creation of a symbolic structure existing in correspondence with the world, but not in that world is an essential characteristic of models (Hesse in Lehrer & Romberg 1996:70).

2.4.6 Summary

Data handling is an important section of statistics and the learners’ involvement in it serves as the beginning of the understanding of the world around the learner. Children learn data handling by tackling novel problems independently, by evaluating their own answers critically against the requirements of problem contexts, and by comparing their answers with one another and by explaining their methods to one another. Explaining the methods to one another serves the important purpose of inducing learners to reflect upon their methods, upon what they have done to solve problems and to perform calculations. Apart from consolidating knowledge and the correction of mistakes, this rethinking often induces important refinements in methods.

2.5 Conclusion

The idea that visualization is a cognitive process involving mental representations leads one to consider that the representations are conceptual organizations of actions built up by solvers in problem solving situations and serving as interpretive tools for understanding to aid their solution activity. This study focuses on the cognitive activity of the learners with particular emphasis to the ways they elaborate, re-organize, and re-conceptualize their solution activity while engaged in mathematical problem solving. This supports Fischbein (1987:101)'s idea that visualization not only organizes data at hand in meaningful structures but it is also an important factor guiding the analytical development of a solution. Problem-solving activities engage learners' cognitive processes in such a way that representations of concepts are developed. Tall (1991:31) suggests that to be successful in mathematics, it is desirable to have rich mental representations of concepts. A representation is rich if it contains many linked aspects of that concept. A representation is poor if it has too few elements to allow for flexibility in problem-solving. It should be noted that mental representations arise from concrete ones. Several competing mental representation of a concept may co-exist in somebody's mind and can be utilized. Different mental representations may be called upon for considering different mathematical situations. However different mental representation may also enter into conflict or several mental representations for the same concept may complement each other and eventually may be integrated into a single representation of that concept. As a result of this process one has available what is best described as multiple-linked representations, a state that allows one to use several of them simultaneously and efficiently switch between them at appropriate moments as required by the problem or situation one is investigating.

An important purpose of mathematics in the Further Education and Training band (FET) in South Africa is the establishment of proper connections between mathematics as a discipline and the application of mathematics in the real-world contexts. Data handling provides the above needed connection. It leads learners to concept formation through a meaningful synthesis of diagrams and visual images in addition to verbal definitions and analyses on the other. Data handling provides the tasks or problems, which sometimes appear in concrete form, for instance in graph, numerical or spatial objects form. Concrete materials play a significant part in the development of imagery, in focusing attention and in assisting links in concepts. It must, however, be noted that concrete materials are not used to transmit visual images or spatial concepts to learners but are vehicles to encourage problem solving, knowledge construction and the development and use of different kinds of imagery (Vinner in Owens & Clements 1998:215).

If the learners' visualization skills are to be well developed, the learners must be given the freedom and time to use their own initiative to solve meaningful problems. In a problem-centred context as opposed to a method-centred context, this ability of children is utilised as the basic source for the development of data representation knowledge and skill. In such a context children learn data handling by tackling novel problems independently, and by evaluating their own answers critically against the requirements of problem contexts. A problem-centred environment is a viable and effective alternative only when it is utilised consistently. This occurs when it is used only as an auxiliary to direct demonstration and mathematical guidance or when learners get opportunities to solve novel problems independently. Problem solving skills, conceptual understanding of specific mathematics content and the ability to apply mathematics to difficult problems are some of the outcomes achieved by implementing the problem-based strategies in the classroom. It also allows learners to think differently about mathematics, feel confident about tackling interesting and difficult problems and take more mathematics courses. Presmeg (in Clements & Del Campo 1989:32) suggest that teachers should create mathematics learning environments which enable children to link their verbal knowledge, their visual imagery and relevant episodes in which they have been previously engaged and this will result in long term learning taking place.

CHAPTER THREE
METHODOLOGICAL ORGANIZATION OF THE STUDY

3.1 Introduction: An overview of the empirical study

The purpose of this dissertation is to investigate the role of visualization in a problem-centred context through the analysis of the mental processes that Grade 9 learners go through while solving data handling and spatial tasks. The present study wishes to establish that learners go through visualization when they engage in the chosen data handling and spatial tasks. In addition, this study wishes to describe the thought processes that the learners are involved in during visualization so as to understand it. The objective of the present chapter is to present a description of the research design, the respondents, sample and sampling procedures, research instruments as well as the data collection and data analysis procedures, used in this study. This objective sets the general structure of this chapter. The stages in the empirical investigation include facilitating data handling and spatial problems to individual learners, collecting data from hand-written and interview responses, marking hand-written responses according to the suggested rubrics for each task, relating interview responses to the suggested Kosslyn's categories, and deriving conclusion from the recorded results. Reasons for the nature of tasks and the nature of Kosslyn's categories and the pre-determined rubrics used for marking written responses are described. The objectives of the analysis of data are mentioned followed by how the investigation achieves these objectives.

Rubin and Babbie (in De Vos, Strydom, Fouché, & Delport 2003:138) define a research design as all the decisions made in planning the study. This includes not only the type of design to use but also sampling, sources and procedures for collecting data, measurement issues and data analysis plans. The research design employed in this study is qualitative and in cases where quantitative data was used it was to complement the qualitative data. The research is qualitative in the sense that written responses and interviews are used to collect data. Quantitative data is used in the sense that collected data is quantified and analysed using simple statistical methods so as to use it to support the qualitative data.

3.2 Literature study

A literature study was done on three major elements namely visualization, the problem-centred context, data handling and spatial sense in relation to visualization. The aim of this literature study was to identify the relevant literature and research projects that had been conducted in similar areas of visualization, the problem-centred context, data handling and the spatial field in geometry. Therefore, relevant theoretical orientations endorsed by different authors and their influence on empirical research that intend to improve our understanding of the learning of data handling were discussed. During analysis the goal was to relate the data to existing literature through the use of Kosslyn's categories. The use of these categories permitted a possible analysis of the processes involved during visualization.

3.3 Sampling

3.3.1 The geographical position of the school

The investigation was carried out at Pretoria Secondary School in Sunnyside. The reason for the choice was that it is in the area where the researcher resides, and most of the learners do not live far away from their homes. Pretoria Secondary School includes learners from most of the African countries. The majority are from most of the ethnic groups in South Africa.

3.3.2 Random selection of learners

Powers (in De Vos et al 2003:198) define a population as a set of entities in which all the measurements of interest to the researcher are represented. Seaberg (in De Vos et al 2003:198) agrees with the above definition which he describes a population as the total set from which the individuals or units of the study are chosen. In accordance with this, the population of this study was made up of Grade 9 learners from one secondary school. A sample can be considered to be a small population but comprises the elements of the population considered for actual inclusion in the study (Arkava & Lane in De Vos et al 2003:199). In this study the sample was twelve Grade nine learners who were randomly selected from a class of thirty learners. A purposive non-probability sampling procedure was carried out by the researcher (Strydom &Venter in De Vos et al 2003:207). This was done in collaboration with the teacher of mathematics for that class. Previous class performance aided the selection of the learners used for the empirical study. The distribution of the twelve learners was as follows: four above average learners, four average learners and four below average learners. This was done so as to be able to represent the type of thinking during visualization portrayed by learners of all abilities. Activities that involve data handling require graphic specific skills in reading axes, comparing slopes of lines or bars, interpolating or estimating values in high dimensional spaces or estimating distances and areas and calculations for analysing the data. Visualization also requires general skills in spatial visualization or spatial cognition. Learners who had passed through the Foundation Phase, Intermediate phase, and had completed at least Grade 8 were assumed to have gained the necessary experiences to make this empirical study possible. Facilitation of tasks involved discussions and questions written in English. The learners in Grade 9 were assumed to have acquired English so as to enable communication.

3.4 Description of stages in the empirical investigation

3.4.1 Facilitating learners with data handling and spatial tasks

Data was obtained from a group of twelve Grade 9 learners. Three days of each week were used for a total of six weeks for the investigation. Four learners working individually were facilitated with a task each day. The researcher required the learner to give explanations of, and justifications for his/her solution activity. That is each learner was advised and expected to show all his/her problem solving procedures with an explanation at all stages. Each learner was allowed to work through the given task for about thirty minutes with minimal intervention. The above was done so as to focus on building detailed accounts of learners' development of mathematical concepts as they work alone. At the end of about thirty minutes the researcher collected the individual written responses. Each written response was marked according to pre-designed rubrics for the tasks which are illustrated in Table 3.3 to Table 3.6 (Section 3.6.1). The results obtained from these handwritten responses were analysed and used to help formulate questions used in the interviews carried out in the next stage. The next stage involved facilitation of the interview to each of the four learners with the help of a tape recorder. Each learner was originally asked to explain his/her problem solving process. The learner's response to questions during the interview and his/her marked written response were used to progress with the interview. The whole process of the interview was carried out with the help of an audio tape recorder, for each of the four learners per day. All the tape recorded data were later transcribed ready to be used for analysis. The investigation generated a model that encapsulated learners' thinking processes and their responsiveness during tasks. Learners' responses were systematically categorized and analysed using rubrics, taking into account the type of explanations associated with them. The categorization scales were the result of a careful evaluation of responses in order to authentically reflect learners' types of reasoning and disposition. In most cases answers involved an open response with a reason why. Learners were probed to get a more accurate description of their mental activity after going through the tasks

3.4.2 The nature of tasks

Tasks were administered with two objectives which were to obtain data from written responses and to structure the interview process so as to obtain valuable information about the learners' thought processes during visualization. The tasks chosen for this research were selected since they had been previously used to develop spatial visualization and visualization in general and could easily be adapted for use in the Grade 9 school classroom. For example, the problems from Worksheet 1 to Worksheet 3 were adapted from Robichaux (2000:22-25) on her study of the improvement of Spatial Visualization and from Fischbein (1987) on his study of the nature and role of visualization and imagery in teaching. Worksheet 4 and Worksheet 5 tasks were adapted from Thornton (2000: 251) on his study of 'A picture is worth a thousand words: the mathematical power of visual thinking'. Thornton also wrote a paper on 'Promoting visual thinking in the classroom'. Included in the problems were some problems represented by dots in geometric configuration (Appendix A, Worksheet 4-5). Learners needed to visualise these configurations, generate numerical data from them, organize the data, search for patterns and formulate conjectures about the problem and their supporting arguments. For this dissertation Colet's (1999:1) choice of high-level tasks was adapted and used in this study. Colet describes the difference between high-level tasks and low-level tasks. High-level tasks were used in this study for visualization to take place. Low-level tasks are such tasks in which information is directly observable from the presentation. For low-level tasks, cognitive processing is primarily of a perceptual nature, that is seeing and looking is all that is required. In contrast to low-level tasks, high-level tasks are tasks in which information has to be mentally processed in some way via either remembering or through generation, integrating with other information and /or involving judgements. In addition high-level processing typically requires comparisons, mental computations or transformations of information held in human memory, and this is what is required during visualization.

The activities were intended to evoke visual imagery, providing a basis for challenging learners to reflect on and where necessary to modify existing concepts, images and skills. Lean and Clements (1981:274) point to some variables which influence the amount of visual imagery a person uses when performing a task and one of which is the characteristic of the task. Usually a task which requires thinking about familiar physical objects evokes more visual imagery than one that does not involve physical objects.

3.5 Data collection instruments and techniques

3.5.1 Collection of data through handwritten responses

During a session data were always collected from individual learners' written responses. All learners were involved in the individual writing of responses to tasks on data handling and the field of spatial development. Learners put their names on their scripts and handed in their written responses to the researcher who marked the scripts using pre-determined rubrics while formulating questions to be used for the interviews that would follow. The method for formulating rubrics used was adapted from Moses (in Bishop 1989:10). Moses used written responses to collect data from facilitated tasks to check learners' 'degree of visuality'. He noted the prevalence of pictures, diagrams, graphs, lists and tables in order to determine the 'degree of visuality' present in each learner's problem solving processes. Presmeg (in Bishop 1989:10) modified Moses's and other methods to include interviews which provide much richer data. In this study the written responses were also followed by interviews which are described in the next section of 3.5.2. Some of the diagrams or interesting explanations within the written responses were used for supporting the final report that was compiled about the role of visualization. Learners' written responses to the questions in the tasks were analysed and then used to carry out meaningful interviews.

3.5.2 Collection of data through interviews and the tape recorder

Interviewing was the main method used for information collection in qualitative research. De Vos et al (2003:292) assert that interviews, when used with care and skill, are an incomparably rich source of data. The researcher agreed with this and was motivated to use the interview because it was flexible and respondents' answers could be followed up for more information and vague statements could be clarified. One of the stages of the investigation involved collecting data through interviewing individual learners about their problem solving processes. After writing a written response which was analysed and marked according to pre-determined rubrics the interview was carried out. The interviews were tape recorded and later transcribed to enable the analysis of data. The objective during the collection of interview data was to be able to provide evidence of learners' use of visualization and to be able to make meaningful descriptions of the thought processes during visualization that the learner encountered while they engaged with data handling and spatial tasks. An interview allowed the researcher to enter the learners' minds in order to retrieve their thought processes.

The following checklist was suggested while interviewing: movement of images of shapes into related shapes, mental involvement in making the shape, noticing similarities and differences in shape, multiple uses of material, selectively attending to inputs and to feedback from existing schemata and making, using, changing and storing images, concepts, understandings and schemata and how often visualization in the form of image generation, image inspection, image transformation and image use was involved. Learners commented on the strategies they had used or discovered, or how they had carried out their activities in general. All interviews were fully tape recorded and replayed later to extract the important data. The aim at extracting various kinds of information concerning visualization from interviews carried out in data handling and spatial tasks was achieved through the use of Ben-Zvi and Arcavi's (2001: 50) suggested questioning skills. These included asking descriptive questions (What is?), predictive questions (What will be?), prescriptive questions (What can be done about it?) and the causal questions (Why?). Data from thought processes were obtained more fully from the interviews than from only the written responses. This is because the learners were able to communicate their problem solving processes clearly, provided they were given the right environment.

3.6 The analysis of data

3.6.1 Describing categories and rubrics

The categories for the empirical study were adapted from Kosslyn (1994). Presmeg (in Owens & Clements 1998:204) aided with the analysis of each and every category through her study of different types of visual imagery briefly illustrated in Chapter two (Table 2.1 Section 2.2.2). The categories were used in order to relate the data to existing literature through the use of prior categories. Kosslyn (1994:1-23) specifies four classes of cognitive image processes during visualization: image generation, image inspection, image transformation, image use given with their sub-categories in Table 2.2. An explanation was provided on how the four categories that were used in this study were related to the rubrics used to mark the tasks that were given to the learner. Table 3.2 is a framework for analysing the first three spatial tasks on Worksheet 1 to 3 and Table 3.1 is a framework for analysing the data handling tasks on Worksheet 4 to 6.

Table 3.1 Description of Kosslyn's categories in relation to the data handling tasks

PROCESS	DESCRIPTION OF THE PROCESS
Image generation	• Occurs when a person brings forth an image from more abstract representation in long-term memory • Incorporating new experiences The above sub-categories occur when learners - Invoke and use some previous knowledge to deduce a pattern. - Make comments that reflect attempts to realise or identify a pattern or a trend from the given data. - Make a particular type of representation for the given data.
Image inspection	• Results when one examines an image in order to answer questions about it (classify, scrutinize). • Noticing similarities and differences in shapes (connection). • Selectively attending to input, feedback from existing schemata (heuristic processes): - establishing meaning of the problem - developing tactics - self-monitoring The above sub-categories occur when learners: - Show that they have read the problem when they verbalize their understanding of what they have read. - Make statements about how they proceed with the problem-solving process. - Use a diagram, a flow chart, a substitution or calculations to solve a problem. - Plan which type of graph they will use to represent data.
Image transformation	• It is when one changes or operates upon an image. - Rotate, zoom, translate and scan The above sub-categories occur when learners: - Look at the patterns from the given data and can notice a trend in the change from one period to another. - Show that there is evidence of trying and retrying through more than one diagram, representation or method used to solve the problem. - Indicate a desire to redo the problem during the interview. - Indicate how they should have solved the problem as compared to what they have on paper.
Image use	• It occurs when an image is employed in the service of some mental operation. • Mental involvement in making the shape (refinement). • Making, using, changing and storing images, concepts, understanding and schemata. The above sub-categories occur when learners: - Make future predictions using the trends in the data.

	- Read between and beyond the data to make meaningful conclusions. - Can think in their minds of the resulting picture of earlier or future happenings. - Make calculations of the nth term successfully.

Table 3.2 Description of Kosslyn's categories in relation to spatial problems

PROCESS	DESCRIPTION OF THE PROCESS
Image generation	• Occurs when a person brings forth an image from more abstract representation in long-term memory. • Incorporating new experiences. The above sub-categories occur when learners: - Invoke and use some previous knowledge in order to draw a shape. - Show that they have read the problem when they verbalize their understanding of what they have read. - Explain the meaning of the words in the question in order to identify the shape to be drawn.
Image inspection	• Results when one examines an image in order to answer questions about it (classify, scrutinize). • Noticing similarities and differences in shapes (connection). • Selectively attending to input, feedback from existing schemata (heuristic processes): - establishing meaning of the problem - developing tactics - self-monitoring These sub-categories result when learners: - Answer questions about their descriptions and are able to compare and contrast between figures. - Make statements revealing that they are trying to simplify, reformulate or analyse the problem. - Describe their diagrams if they are to convince a second person about the shapes they have. - Description is also in terms of clarity about which sides or faces are opposite each other etc. - The clear description of the shape is written down in terms of sides, angles directions etc. - Can identify similarities and differences by the way they describe or illustrate shapes. - Learners should be able to realise special types of shapes (e.g., that a square is a parallelogram).

Image transformation	• It is when one changes or operates upon an image: - rotate - zoom - translate - scan • Movement of images of shapes into related shapes (e.g., square to hexagon). These sub-categories result when learners: - Put drawings in different positions in order to describe them better. - Move stretch or shear the diagram to come up with what they want. - Realise their short-comings as regards the nature of their diagrams when they communicate with someone else. - Can make more than one shape to represent the wanted shape.
Image use	• It occurs when an image is employed in the service of some mental operation. • Mental involvement in making the shape (refinement) • Making, using, changing and storing images, concepts, understanding and schemata. These sub-categories result when learners are able to deduce that: - A rectangle is a parallelogram. - A parallelogram is not always a rectangle. - A square is a parallelogram. - A parallelogram is not always a square. Also when learners retrieve properties of the shape in order to use them for describing the shape.

Pre-determined rubrics were prepared for all tasks from Worksheet 1 to Worksheet 6. A rubric is a scale to judge performance on a single task not a series of exercises. It uses performance indicators which are task specific statements that describe what performance looks like at each level of the rubric. These pre-designed rubrics were used to analyse data from the written responses. The performance indicators described the variations in the ability of the learners to interpret the range of problems in data handling. In describing such variations, it was necessary to focus on different aspects of the interpretive task, for example how learners search for information from the task and how they construct meaning for data on particular types of problems. The tables showing the rubrics used are presented in Tables 3.3 to 3.6. The performance indicators in the rubrics were related to the mental processes in Kosslyn's categories. It must be noted that the criteria and characteristics in the rubric tables were not hierarchical because they could be identified at any stage in the learner's problem solving process.

Table 3.3 Rubric for the geometry problem of Worksheets 1&2

Visualization process	CRITERIA	CHARACTERISTICS	Possible Mark	Task Mark
Image generation	Drawing a diagram	• Correct diagram also showing the parallel sides or even correct angles (5) • Some sort of a correct sketch (3) • Attempted a wrong figure (1) • Drew nothing (0)	5	
Image inspection	Clarifying the diagram	• Showing of parallel lines, equal angles or equal sides (5) • Showing of directions in order to clarify the diagram (5) • Was able to draw using similar shapes (3) • No clarification of the diagram (0)	5	
Image transformation	Evidence of trying and re-trying	• Changing from one shape to another (5) • Changing from one description to another (5) • One shape or one description is done but result shows choices had to be made (3)	5	
Image use	Describing the diagram for someone else	• Using angles or directions to instruct how to describe the shape e. g. horizontal/vertical, north east etc (5) • Using measurement to instruct how to draw this shape (5) • Attempted an unclear description (2) • No description was done (0)	5	
	TOTAL		**20**	

Table 3.4 Rubric for the geometry problem of Worksheet 3

Visualization process	**CRITERIA**	**CHARACTERISTICS**	**Possible Mark**	**Task Mark**
Image generation	Drawing a diagram	• Correct diagram also showing the parallel sides or even correct angles (5) • Some sort of a correct sketch (3) • Attempted a wrong figure (1) • Drew nothing (0)	5	
Image inspection	Clarifying the diagram	• Showing of parallel lines, equal angles or equal sides (5) • Using directions in order to clarify the diagram (5) • Was able to draw using similar shapes (3) • No clarification of the diagram (0)	5	
Image transformation	Evidence of trying and re-trying	• Changing from one shape to another (5) • Changing from one description to another (5) • Searching for the correct way to put labels (2)	5	
Image use	Identifying the shape	• Correct labels on all sides (5) • Some pairs of opposite labels are correct (one mark for each correct pair of faces numbered correctly) (3) • Naming the shape of the 3-dimensional figure with names other than cube (2)	5	
	TOTAL		**20**	

Table 3.5 Rubric for the pattern problems on Worksheets 4&5

Visualization process	**CRITERIA**	**CHARACTERISTICS**	Possible Mark	Task Mark
Image generation	Meaningful pattern	• Deduced a correct clear meaningful pattern (4) • Some clear but not correct pattern generalization by formulae or by observation (3) • Some unmeaningful pattern generalization was made (1)	4	
Image inspection	Checking the process of the problem solving	• Organized exploration towards working out the problem (3) • Changing from one idea to another (3) • Using diagram, formulae, substitution, flow-chart or a calculation to aid solution of the problem (3) • Reason given for choice of prediction (2) • Unorganized exploration towards a solution (0)	3	
Image transformation	Evidence of trying and re-trying	• More than one method is used to arrive to an answer (6) • More than one representation is used to arrive at an answer (6) • One way towards a solution but it shows there was reasoning in it (3) • No method shown (1)	6	
Image use	Prediction in data	• Correct prediction or calculation of nth term using a correct pattern (2) • Correct prediction from wrong method (1) • Some incorrect deduction was made (0)	2	
	TOTAL		**15**	

Table 3.6 Rubric for representation problem on Worksheet 6

Visualization Process	**Criteria**	**Possible mark**	**Your mark**
Image generation	o Representation clear and visible (4) o A representation was done but not so clear or not correct (2) o No representation done (0)	**4**	
Image inspection	o Organized planning towards drawing a shape (4) o A reliable scale or ratio or calculations used (4) o Evidence of an explanation towards drawing the representation (2)	**4**	
Image transformation	o Explanation of why that particular representation (2) o Evidence of trying and re-trying (2) o One way followed but shows that choices had to be made (2)	**2**	
Image use	o Evidence of method used to show different sections of representation (2) o Ability to discuss about the diagram (2)	**2**	
	TOTAL	**12**	

3.6.2 The analysis of data through categories and rubrics

The responses were assessed and analysed qualitatively, with the aid of clearly defined categories adapted from Kosslyn (1994) given in Table 3.1 and 3.2 together with the rubrics designed to enable the marking of the written responses given in Table 3.3 to Table 3.6. The categories used were adapted to suit the different tasks used. The aim of the study was to investigate the role of visualization in a problem-centred context through the analysis of the thought processes involved during visualization by Grade 9 learners while solving data handling and spatial field tasks. Therefore, the objective of the analysis of data was to provide evidence of the learner' use of visualization by identifying through learner' thought processes, the processes of visualization aided by Kosslyn' s categories, and to be able to describe these processes so as to derive conclusions about the role of visualization.

The above objectives were achieved through stages. One stage involved the analysis of the written responses through the use of rubrics and their performance indicators. Pre-determined rubrics were prepared for each task from Worksheet 1 to Worksheet 6 and are illustrated above in Table 3.3 to Table 3.6. The performance indicators in the rubrics are related to the mental processes in Kosslyn's categories. This is because from the literature study the learner's ability to perform certain tasks can justify the learner's having generated an image, inspected an image, transformed an image or used an image. The tasks were marked using the rubrics in order to check evidence of learners' use of visualization by determining the learner's use of the four Kosslyn categories. Using the pre-determined rubrics and performance indicators in relation to Kosslyn' s categories of visualization, the data from the written responses were collected and represented in tabular form. Absolute frequencies were used to summarize the findings based on Borg and Gall (1983:520) who assert:

> By far the most common method of summarising content analysis data is through the use of absolute frequencies such as the number of specific incidents found in the data.

In this study the data was then simplified to a percentage table in terms of visuality. 'Visuality' refers to how often a learner uses visualization during problem solving. The percentage tables were used to support evidence of use of visualization. The percentage of the learner who fell in each category was recorded. This process involved the presentation and analysis of data to provide evidence of learners' use of visualization in terms of visuality in percentage, and to provide a further understanding of the role of visualization by comparing the percentage visuality among chosen groups of learners. The results from these written responses were later related with those from the interviews to come up with an overall percentage visuality of the learner which enabled the researcher to make conclusions about the role of visualization.

The second stage involved analysing data from interviews. The goal during the analysis of data from the interview was to produce evidence of the learner' use of visualization and then to produce detailed descriptions of the thought processes which were related to the process of visualization so as to understand the role of visualization. A tape recorder was used to collect data during all interviews. During the listening of the tape recorded data from the interviews, a tally table was produced for each learner per each task, and the results were simplified and recorded into percentage form so as to check for the ‘visuality’. The percentage visuality enabled one to deduce that the learner had used visualization by checking the visualization process using Kosslyn’ s categories. Data from the tally table were also represented in tabular form. The next process involved the identification and description of the processes of visualization involved by learners while working through data handling and spatial tasks with the aid of Kosslyn’ s categories so as to draw conclusions about the role of visualization. Together with the data from written responses, percentage visuality tables were established and conclusions were drawn from them. The data in percentage form from the written responses could be related to the data in percentage form from the interviews.

The final stage involved relating data from the written responses to those of the interview responses. The goal of all the data collected was first to provide evidence of the use of visualization by checking the existence of the four Kosslyn categories which are image generation, image inspection, image transformation and image use in written and interview responses. The second goal was to describe the nature of the thought processes that occur during visualization with examples of these processes derived from the learners’ thought processes. In order to achieve these goals the study was aided by the results that were obtained from the learners’ responses. Visuality is recorded in the following ways. Tallies were done for interview data. The totals were recorded and categorized per learner and later converted to percentages. The data from written responses were also summed up, recorded and categorised per learner and later converted to a percentage. By putting together the percentage results from the interview data and from written responses percentage tables were obtained by finding the mean of this data. The data from the percentage tables were used to support the qualitative data of the evidence of use of visualization during problem solving. The interpretation of the data was done and results recorded. The above goals were used as a guiding framework in the detailed description of mental processes that the learners used during visualization. Description of the thought processes used were generalized and described within the categories adapted from Kosslyn. The percentage data in the Tables were also used to support these descriptions where it was relevant.

3.7 Carrying out a pilot study

A pilot study was conducted to check the validity and the reliability of the tasks on the worksheets and to find ways of improving the researcher's methods of handling hand written responses, questioning skills during interviews and the way to report on children's thought processes. Five learners from Pretoria Secondary school were involved during the pilot study. Having gained permission to do the research at this school from the Department of Education it was easier to choose learners from the same school. Five learners from Grade 9 were chosen but were not the same learners who were used for the actual investigation. The same procedures for the investigation were done with the five learners. Written responses of the same questions used for the investigation were used, and they went through the same procedure of marking and analysis as explained in section 3.5.1. Interviews were carried out, transcribed and analysed. The researcher was able to pick up errors during this stage, and modify problem tasks if necessary. The pilot study also gave a chance to check the learners' understanding of English as they were English second language speakers.

3.8 Validity and reliability

Validity is defined in two parts: in terms of instruments that actually measure the concept in question and whether the concept involved is measured accurately. Content validity is used to check whether the instrument is really measuring the concept we assume it is measuring and also to check whether the instrument provides an adequate sample of items that represent that concept (Bostwick & Kyte in De Vos et al 2003:166). According to De Vos et al (2003:166) content validity is judgemental and it depends on the judgement of the researcher or any other practitioner. Content validity was achieved through the researcher's choice of tasks to be used in the study. Tasks were chosen from problems that had been used in the study of visualization by experienced authors. The tasks chosen for this research were selected because they had been previously used to develop visualization during studies done by researchers in middle school and could therefore be adapted for use to the Grade 9 learner. For example, the problems from Worksheet 1 to Worksheet 3 were adapted from Robichaux (2000:22-25) on her study of the improvement of Spatial visualization in middle school and also from Fischbein (1987) on his study of the nature and role of visualization and imagery in teaching. Worksheet 4 and Worksheet 5 tasks were adapted from Thornton' s (2000: 251) paper which suggested that visual thinking should be an integral part of learners' mathematical experiences. In this study it was the researcher's objective to take care of validity by obtaining confirmation of results from two data sources. This included the use of written responses as well as using a tape recorder to collect data from interviews. This way the researcher used triangulation which is the use of two or more methods of data collection in the study of human behaviour (Cohen & Manion 1994:241). In this study the richness and complexity of mental processes were recorded fully from more than one position. Anderson (1990:175) agrees with the above authors in that the major safeguard on validity is to obtain information from as many sources as possible. It was with this view that written responses and interviews were used as data collection instruments. Criterion validity involves multiple measurements, and is established by comparing scores on an instrument with an external criterion known to or believed to measure the concept trait or behaviour being studied (Boswick & Kyte in De Vos et al 2003:168). In this study scores on a rubric are compared to performance indicators which are in turn compared to the Kosslyn categories. The pilot study that was carried out for this research also served as a validity and reliability check instrument. The performance indicators in the rubrics went through a lot of change during the pilot study until such a time when the marking of the written responses showed less deviation from the suggested measurement.

Reliability refers to accuracy or precision of an instrument (Boswick & Kyte in De Vos et al 2003:168). This in general refers to the extent to which independent administration of the same instrument consistently yields the same results under comparable conditions. In this study an independent person helped with the collection of data by identifying the processes that occur during visualization while learners engage in problem solving.

3.9 Results of the pilot study

The pilot study that was carried out for this investigation, made it possible to choose the most relevant tasks for this study as it was clear that, with some tasks the learners were not obliged to involve visualization. In cases where tasks did not require the learner to use visualization, learners would quickly give a correct or wrong answer without finding a reason for it. Through the pilot study it was discovered that the process of interviewing had to be improved in two respects. Information was easily obtained from the discussions that were done privately. The interviews had to be done in private in the sense that the researcher had to be apart from the other learners. The individual learner being interviewed wanted a sense of privacy and a guarantee that what he/she said was not going to be communicated to anyone else. Learners did not want the other learners to listen to what they were thinking about. During the analysis of data an independent person, together with the researcher listened to the tape recordings in order to ensure accuracy of data. Where a difference in subcategory classification occurred, discussions took place until stronger bases for classification and for development of a problem-solving story line were established. Conjectures were made about how the various categories seemed to interact. Patterns of relationships between sub-categories were developed from cross tabulations of categories, from immersion in the data and from deliberate attempts to identify and check links.

3.10 CONCLUSION

This chapter described the research methodology under the headings research design, subjects, research instruments, data collection procedures and data analysis techniques. Some of the definitions of the research design that are found in literature were cited and discussed briefly. The research design for this study was identified as being mainly qualitative. The quantitative approaches were employed in order to support the qualitative data. The sampling procedures used in this study were discussed and validity and reliability were ensured through the choice of tasks that had been proved useful during some studies of visualization, the use of a video tape recorder to collect data during interviews, the use of pre-determined rubrics for the marking of written responses, involvement of a second person during the analysis of data and the use of quantitative data to support the qualitative data.

This investigation allowed one to appreciate the process of visualization as it did not check the end product only, but the whole process that the child engages in from the moment that the task is started until the completion of the problem-solving process. The categories adapted from Kosslyn (1994) fitted best with this study because they enabled the researcher to analyse the collected data so as to understand the role of visualization. The various forms of visual imagery were used to describe the different sub-categories that exist in Kosslyn’ s visualization categories. It should be noted that this research was developmental. The researcher tried to make sense of what was going on in the classroom against the background of the thought processes that preceded the instructional activities. Lijnse (in van Niekerk 1997) said the aim of developmental research is not aimed at building grand theories, for example, understanding the human mind, but at understanding and developing good teaching practice.

CHAPTER FOUR

DATA ANALYSIS: PROCESSING AND INTERPRETATION OF DATA

4.1 Introduction

In this chapter the learners' responses to data handling and spatial tasks are analysed and interpreted. This study was set up to examine the role of visualization qualitatively through an understanding of the thought processes that occur during visualization when learners engage in data handling and spatial tasks. Therefore, the objective of the analysis of data was to provide evidence of the learner's use of visualization and to be able to describe those processes that occurred during visualization. This was achieved by identifying, through the learner's thought processes, the processes of visualization using Kosslyn' s categories, so as to be able to derive conclusions about the role of visualization. The tasks used for the study are divided into spatial tasks found in Worksheet 1 to Worksheet 3 and data handling tasks found in Worksheets 4 to Worksheet 6. Data from both written and interview responses were analysed through adapting to Kosslyn' s (1994) categories, which are image generation, image inspection, image transformation and image use. These were explained in Chapter 2.

The research design utilized followed the discussion on methodological issues as explained in Chapter three. The use of bar charts and tables to represent data was adopted in response to the nature of the objectives of the analysis and to explain by comparison the extent that visualization was used by the learner. In order to provide evidence that learners had used visualization, tables and graphs of percentage visuality were set up. Visuality is how often a learner uses visualization during problem solving. To understand more about the role of visualization, comparisons were done regarding visuality between boys and girls and among above average, average and below average ability in mathematics. By using percentages it was possible to determine the order of learners in terms of percentage visuality and it was possible to determine the average percentage visuality for the chosen comparative groups. With the aid of the results in the section 4.2 and the description of the mental processes involved during visualization given in the section 4.3, it is possible to deduce the role of visualization. Since the 12 learners chosen for the study had been at the same school for more than a year, the experiences they had undergone as regards mathematics were the same.

4.2 CATEGORIZING OF RESULTS

4.2.1 Results showing evidence of learners' use of visualization

Table 4.1 shows the distribution of learners used in this study and their percentage visuality.

Table 4.1 Visuality: The frequency with which the learners used visualization

LEARNER	Visuality from written responses	% visuality from written responses	Visuality from the interview responses	% visuality from the interview responses	Average % visuality
Learner A(F)	71	70	49	98	84.00
Learner B(M)	76	75	38	76	75.50
Learner C(M)	82	80	35	70	75.00
Learner D(F)	69	68	45	90	79.00
Learner E(F)	59	58	41	82	70.00
Learner F(M)	78	76	35	70	73.00
Learner G(F)	65	63	32	64	63.50
Learner H(F)	72	71	50	99	85.00
Learner I(F)	52	51	30	60	55.50
Learner J(M)	12	11	20	40	25.50
Learner K(F)	15	15	15	30	22.50
Learner L(M)	62	61	30	60	60.50
Total possible	102	100	50	100	100

Learners A-D---------above average
Learners E-H--------- average
Learners I-L ----------above average

(F)---------Female
(M)--------Male

By looking at Table 4.1 the average percentage visuality for the learners ranged from 22.5% to 84%. It is therefore evident that all learners used visualization during their problem solving activities. While engaged in a mathematical activity whether of a numerical or geometric nature, learners always construct images (Wheatley 1998:8). When a learner makes a diagram then he/she is operating from a constructed image. If a learner fails to initially construct an image in his/her mind then there is nothing to re-present, inspect, transform or use. The high percentage visuality of ten out of the twelve learners is an indication of tasks that required a lot of visualization. The visuality per learner per task depended mainly on the nature of the task and on the ability of the learner to understand the task. It is interesting to note that Learner H of average ability had a higher average visuality score of 85% as compared to the Learners A, B, C, and D who ranged from 79% to 84% but were rated as of above average ability. Also Learner L had a score of 60% visuality though he had been classified as of below average ability. The conclusion drawn was that some learners, even though not rated highly in mathematics have a high percentage of visuality. In the case where they are given problems that require a lot of visualization, they will perform exceptionally well. This can surprise a teacher who normally provides his/her class with tasks that do not require a lot of visualization. It must, however, not be forgotten that Learner H could have been erroneously placed in a wrong ability group.

Table 4.2 shows the distribution of learners used in this study according to gender and ability categories.

Table 4.2: Distribution of learners used for comparative analysis

	Frequency	**%**
GENDER		
Boys	5	41.67
Girls	7	58.33

ABILITY		
Above average	4	33.33
Average	4	33.33
Below average	4	33.33

Table 4.2 reveals that at least 41.67% of the learners were boys as compared to the 58, 33% girls. The learners were equally distributed according to the three ability groups.

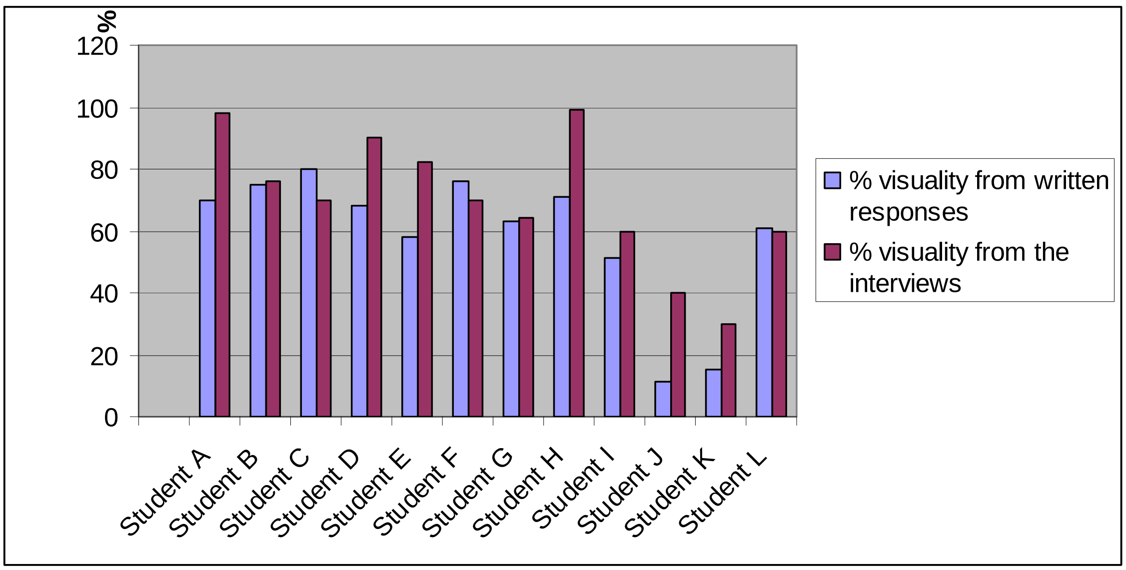
Table 4.3 shows the percentage visuality according to the type of data collected. Data was collected from written responses as well as from interviews and in each case the percentage visuality for each learner was measured.

Table 4.3 Distribution of learners' percentage visuality according to type of data collected

LEARNER	% visuality from written responses	% visuality from the interviews	Average % visuality
Learner A	70	98	84.00
Learner B	75	76	75.50
Learner C	80	70	75.00
Learner D	68	90	79.00
Learner E	58	82	70.00
Learner F	76	70	73.00
Learner G	63	64	63.50
Learner H	71	99	85.00
Learner I	51	60	55.50

Learner J	11	40	25.50
Learner K	15	30	22.50
Learner L	61	60	60.50
Average percentage for group	58.25	69.92	64.08
Total possible	100	100	100

Figure 4.1: Distribution of learners' percentage visuality according to type of data collected



In Table 4.3, 58.25% visuality was recorded from data collected from written responses as compared to 65.92% visuality collected from interview data. This is an indication that one can obtain better evidence of use of visualization from interviewing or from talking to the learner than from written responses. Supporting this is Figure 4.1 which reflects that overall high percentage visuality per learner was recorded during interviews than it was during written responses. For example, Learner E had 58% visuality recorded from the written responses as compared to the very high 82% visuality recorded from the interview. Similarly Learner J had 11% visuality recorded from data from the written responses as compared to the 40% visuality recorded from data from the interview. This pattern appears to be consistent with most of the learners. It therefore appears that learners provide a lot more information about visualization when they communicate with another person than they do when they are writing the information down on paper. When learners put responses on paper, they do not put down all their mental experiences. For example, Learner J had only 11% visuality according to his written response. Learner J engaged in the process of visualization during problem solving but he failed to put clear procedures on paper. After the researcher had an interview with Learner J, it was discovered that he had been involved in many of the processes of visualization, and so got 40% visuality. Most of the responses that are put down on paper by the learner are a result of the process of visualization which have taken place in the learner's mind but remain hidden as long as they do not talk about it or as long as they wish not to talk about it. The above results from Table 4.3 and Figure 4.1 highlight the fact that the interview stands out as an important instrument in understanding the process of visualization and therefore enables one to understand the role that visualization has in many aspects of mathematics. However, written responses still remain the original basis from which one checks these processes of visualization.

Table 4.4 and Figure 4.2 show the percentage visuality according to gender. The two groups of seven girls and five boys had their percentage visuality compared.

Table 4.4 Distribution of learners' visuality according to gender

	% visuality from written responses	% visuality from the interviews	Average % visuality
Average percentage for girls	57.00	75.00	66.00
Average percentage for boys	61	63	62.00
Total possible	100	100	100.00

In Table 4.4 it can be observed that an average of 57% visuality was recorded for girls as compared to an average of 61% visuality for boys in written responses. However, the data from the interviews indicate higher average percentage visuality of 75% for girls as compared to the boys' 63% visuality. Overall girls' and boys' percentage visuality differ by a small percentage of 4%. This implies that when working through data handling and spatial tasks girls and boys go through visualization at almost the same level of achievement.

Figure 4.2 Distribution of learners' visuality according to gender

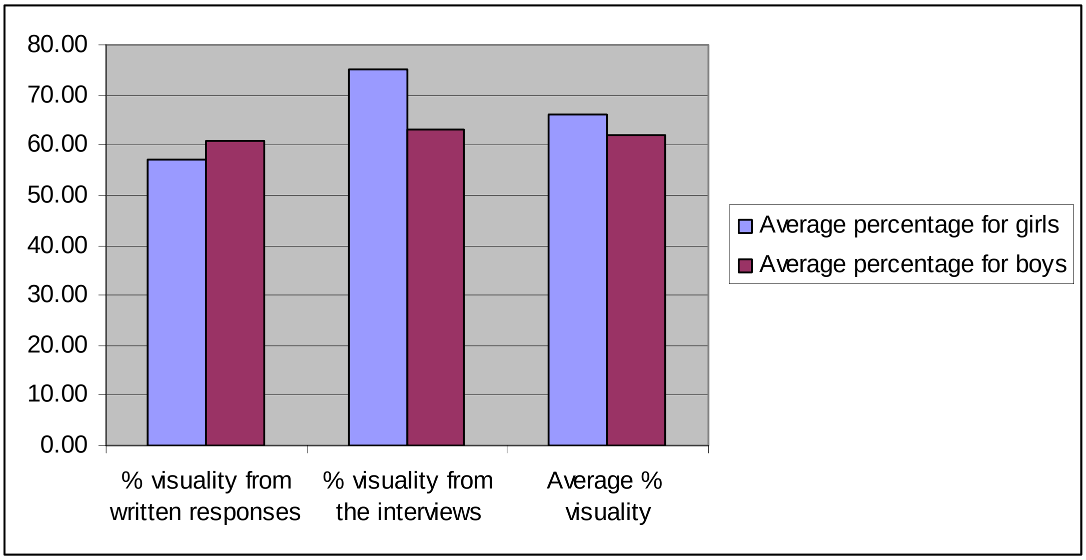


Figure 4.2 demonstrates clearly that boys put down on paper a clearer indication of their processes of visualization than the girls. On the other hand, girls tend to talk more about their processes of visualization than the boys. These results conclude that though learners have almost the same ability to involve visualization, some can clarify it more during written responses while others clarify it better during conversations. It can be concluded that girls talk more about what they are thinking about than what the boys do. Therefore boys have a higher percentage of visuality during written responses while the girls have a higher percentage of visuality from the interview responses.

Table 4.3 supported by Figure 4.3 was presented in an attempt to summarize the findings in this study by categorizing the learners into their ability groups, namely above average, average and below average.

Table 4.5 Distribution of learners' percentage visuality according to ability

	% visuality from written responses	% visuality from the interview	Average % visuality

		responses	
Average percentage for above average learners	73.00	84.00	78.50
Average percentage for average learners	67	73	70.00
Average percentage for below average learners	35	48	41.50

Figure 4.3 Distribution of learners' percentage visuality according to ability

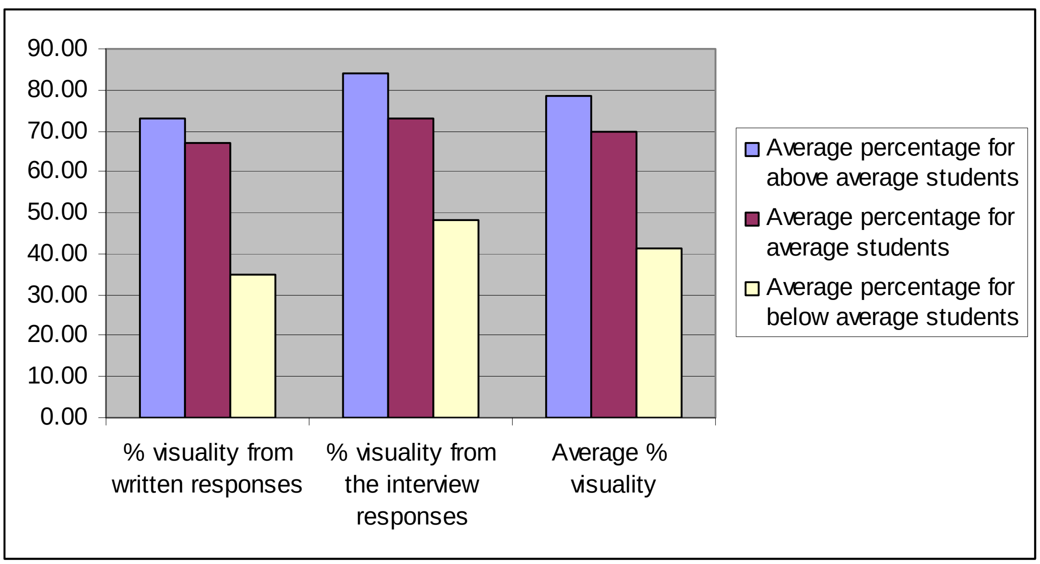
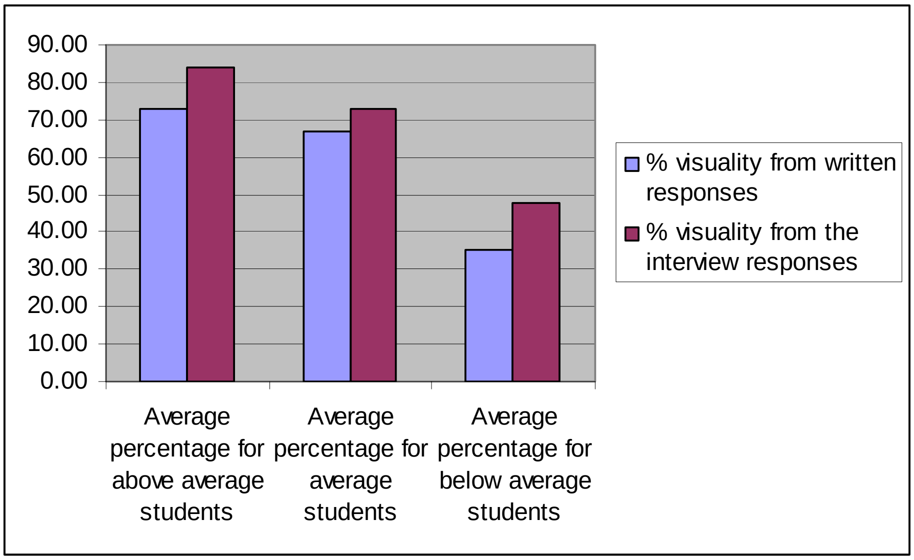


Table 4.3 shows that the visuality of learners vary proportionally with ability as all the sections of data are positively skewed. This implies that the learners who are of above average ability had a higher visuality than those learners who were rated average and the learners who were rated average also had a higher visuality than those who were rated below average. It can therefore be concluded that learners generally do well in mathematics if they make greater

use of visualization. Visualization therefore, improves mathematical performance. Table 4.4 further justifies the fact that more information about a learner's visualization is retrieved during interviews than is done during written responses. In all the three cases the percentage visuality from written responses was lower than the percentage visuality from the interview responses.

Figure 4.4 Distribution of learners' percentage visuality according to ability and type of response



4.3 Description of the mental processes during visualization

4.3.1 Image generation

A number of learners kept referring to the fact that they were able to 'see' or that they were thinking of something in their minds while they were working through the tasks. For example, the researcher wanted to understand why Learner C kept playing around with his fingers. Learner C had this to say:

INTERVIEWER: What are you doing with your hands? What are you doing with your fingers?

LEARNER C: I am trying to make a shape and I am thinking of a shape in my head. I am thinking of the dice.

In the above case, Learner C was seeing a three dimensional figure, a cube, whilst what was on the net shape was a two dimensional figure. The learners may have heard descriptions of three dimensional figures prior to seeing the pictures, and their imagination may have created images for them to attach to those descriptions. Visualization offers a method of seeing the unseen (McCormick in Arcavi 2003:216). Moreover, when the researcher tried to probe those learners who failed to make a three dimensional figure out of the two dimensional net shape, their answers always referred to the fact that they could not 'see' it or that it was not in their minds. The learners were always aware of what they had in their minds and what they did not have. In this case the three dimensional figure was simply not generated in their minds. The self-monitoring process on the part of the learners played a role in the successful solution of their tasks. Visualization therefore allows self-monitoring during the problem solving processes. Visualization clearly influences learners' responsiveness and plays a key role in determining how learners selectively attended during problem solving.

Another very interesting observation was the constant occurrence of learners who produced only final correct answers on written responses and nothing to indicate the problem solving processes. During interviews these learners always indicated that they had already seen the whole situation in their minds and did not regard any need for a written detailed description. Learner E had this to say:

INTERVIEWER: Ok. Which years did you choose that exceeded 8 after 1991? Did you write them down? I don't see them.

LEARNER E: Yes. I chose 1995, 1999. It looks like I did not work them out properly because I was working it out in my mind. But I wrote some of it here.

Another Learner K had this to say when she was asked why she had nothing on paper to show her calculations.

LEARNER K: I didn't have to do the hard calculations because and that is why I did it in my mind. It adds up to a hundred, and yet in a pie chart you work with hundred. So it wasn't hard.

The above is evidence that some learners mentally operate through all the categories identified by Kosslyn and finally get the answer without having to go through the writing of the problem solving process. This goes hand in hand with MacFarlane Smith's (1964:132) comment:

> It is sometimes said that the most able mathematician can think entirely in abstractions and have no need of diagrammatic props. But this may be because these gifted individuals have their own internal blackboards and can visualize complicated structures without being aware that they are doing so.

The researcher made an attempt to find out why Learner I had drawn a cube that was not labelled for the task on Worksheet 3 even though the question clearly stated that it needed the geometric shape to be labelled with the given numbers 1 to 6. The following conversation ensued:

> **INTERVIEWER:** Your cube is coming from this shape isn't it?
> **LEARNER I:** Yes it is. I didn't draw it like this but I drew it like a 3 D.
> **INTERVIEWER:** What about putting the numbers?
> **LEARNER I:** Yaa, but then the numbers will only show on three faces.

Learner I was able to generate a three dimensional figure in her mind and was able to see that she could label it with all the numbers. She also realized that if the shape was drawn on paper then it was not going to be possible for her to label all the numbers onto the correct sides. All this clarification about the labelling was being done mentally. It can therefore be concluded that visualization enables one to see in three dimensions regardless of whether the shape is presented on a flat paper plane or not. The three dimensional objects are always manipulated mentally.

Learners brought forth different images from more abstract representations. When learners were asked to draw a rectangle or a trapezium or a geometric body, the different types of diagrams produced were clear evidence that learners have different mental images for the same mentioned shape. For example, Learner K drew a square, Learner F drew a rectangle, Learner C drew a rhombus and all were their ideas of what a parallelogram was. A cube was named in many ways as a box, a dice, a ring box, a toy box, a jack in the box, and even as a closet. Visualization allows the generation of different images for the same described shape. It can be deduced that what the teacher is thinking about, is not always what the learners are also thinking about. It is through conversation and discussions that one is able identify the similarities and differences in the thinking.

The researcher made an effort to find out how different learners managed to generate the images that they had for the different problems during problem solving. The following were results extracted from the interviews with three learners on Worksheet 3:

LEARNER I: A geometric body has air in it so I made a box.

LEARNER A: A geometric body has got sides all equal like it can form something eee…..

LEARNER C: I discovered that a geometric body has a….should have space inside….

From the problems on Worksheet 4 to 5 the following are results extracted from the interviews:

LEARNER D: I don't see a pattern.

LEARNER A: I could see the pattern and I therefore made a formulae.

LEARNER H: The pattern was queer and so I couldn't use it.

Judging from the extracts above, learners found key words to be useful for image generation. On the geometric problem on Worksheet 3 the term "geometric body" got the learners working with several initial images and for the problems in Worksheet 4 to 5 the word 'pattern' initiated the generation. It would be relevant to conclude from the above results that visualization always seeks the aid of key words. It is therefore true that visualization highlights the importance of key words in the discipline of mathematics.

The researcher checked the method used by Learner A while going through the problem on Worksheet 4. Upon reading the problem on Worksheet 4, Learner A quickly worked out the problem routinely by checking the results of each consecutive stage in the pattern without trying to develop a deeper understanding of the situation. After the pattern went on and on for the respective stages, she got stuck and the following conversation was recorded:

INTERVIEWER: So how did you get to 210 dots? Because I don't seem to see how you get to 210 dots.

LEARNER A: Yaa. I was trying to work it out on another paper, but I didn't reach the 210 dots.

INTERVIEWER: Where is it? (The learner gets the paper quickly) So you have got another paper. So you continued on to the 13^{th} Figure 14^{th} 15^{th} 16^{th}.

LEARNER A: I didn't reach the 100th figure. (Showing the researcher)

INTERVIEWER: So you failed to reach the 100th figure?

LEARNER A: Ya.

Learner A was forced to construct a relationship that would guide her with the solution towards the 100th figure. Learner A was used to routine problems and was happy to follow the known procedures in solving this problem but this problem was good in that it involved an activity in genuinely problematic situations. Learner A therefore worked out the number of dots using the calculation of numbers one after another but the 100th figure proved to be too far away, and that prompted her to generate a generalization that could be used to find the 100th figure. Visualization encourages the choice of non-routine and realistic problems by the educators and therefore plays a key role in determining how teachers select tasks to be used in the classroom. Failure to get a routine problem forced the learner to inspect constructive useful mental images and therefore generate richer images. If the above problem had not asked for the 100th figure, the learner would have easily solved it by numerical calculation of numbers one stage after another, and would not have needed to think of generalizing the situation.

The memory images of formula were so vivid in some learners that they could sometimes work out a whole part of the problem in their heads. For example, in one problem about vehicles that passed a school gate on Worksheet 6, Learner D did not calculate on paper the number of degrees which are supposed to be used for drawing a pie chart, but managed to mentally calculate and draw segments for the pie chart. The following was recorded:

INTERVIEWER: What did you use to demarcate your pie chart?

LEARNER D: I used a compass. (She meant a pair of compasses)

INTERVIEWER: So it means you calculated something?

LEARNER D: Yes.

INTERVIEWER: What was that?

LEARNER D: I didn't have to do the hard calculations because if you take the bicycle, car, motor-bike, lorry, taxi and bus and all these numbers. (points to the list of numbers given) It adds up to a hundred, and yet in a pie chart you work with hundred. So it wasn't hard.

INTERVIEWER: So for this one (pointing to one of the sectors of the pie chart) How did you get the value for this sector?

LEARNER D: I used a compass

INTERVIEWER: How did you get the degrees is my question, because I do not see any calculations.

LEARNER D: I didn't use degrees. First I split my circle into half then into one quarter and then into one eighth.

INTERVIEWER: So that is how you got to demarcate this section? Say it clearly again.

LEARNER D: I took my circle.I split it into half. I drew a halfway line and then a one quarter way line and then into (**PAUSE)** I think it was an eighth. I am not quite sure. I think it was eighth and then I calculated it in my head.

The learner was recalling and calculating the fraction for each sector in her mind which she explained as 'easy' since the values of the number of vehicles added up to a 100. Demarcations were done through mentally dividing a circle from whole to half and so forth. Visualization provided a quick means of recall and calculation of formulas. Richardson (in Presmeg 1986:4) agrees that memory images of formulae, and pattern images, are two types of imagery which provide a quick means of recall of abstract general principles and procedures, the former in a concrete image which encapsulates a procedure, the latter in a more schematic image which stressed regularities.

The following conversation was recorded between the researcher and Learner E. Learner E was explaining the process that got him through realization of a pattern. Later Learner E was able to make a prediction of the number of accidents that occurred in 1991.

LEARNER E: (Showing the researcher her written response). These ones are added, these ones are subtracted.

INTERVIEWER: Say it in years.

LEARNER E: 1981was added.2 was added so that you can get the 1982.One was added to get the 1982 answer. And here 1982 one was added to get 1983 and in 1983 four was added to get 1984.And here in 1984 six was subtracted to get the 1985 answer. In 1985 three was added to get a 1986 answer instead of 2 that was added in 1981. They started again in 1985 and they added three to get 1986.

INTERVIEWER: In other words, are you saying that you have started again in 1985?

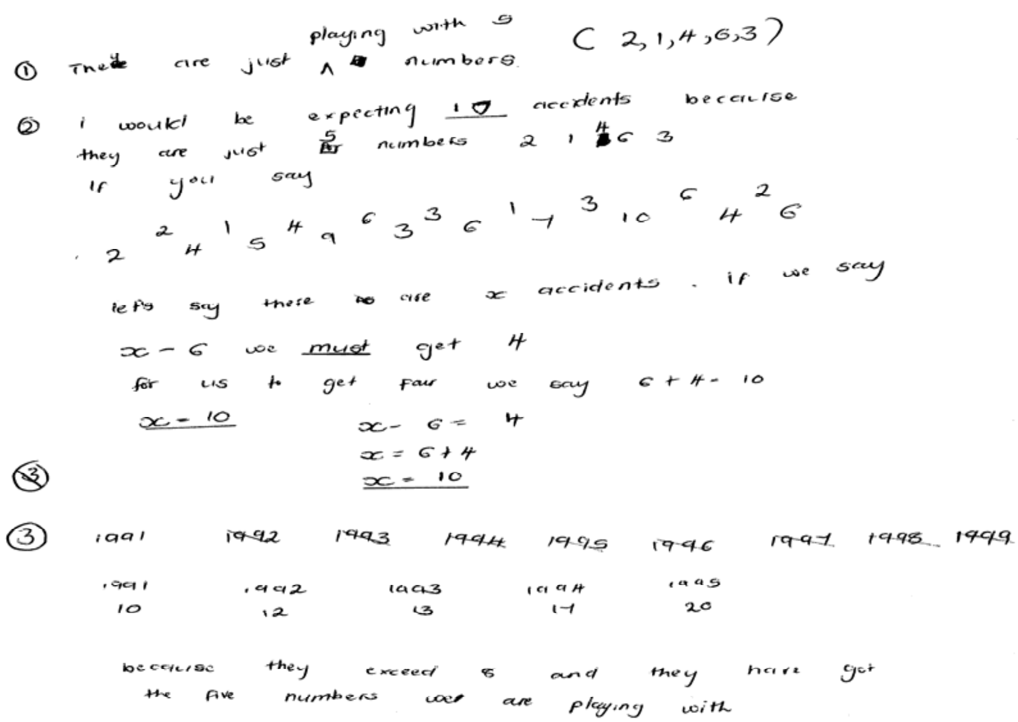
LEARNER E: Yes they started from the pattern but this time they added the number.

INTERVIEWER: And what did they add? From three they added what?

LEARNER E: They added three to get the 1986 answer. Then the pattern is still the same here in 1986. One was added to get 1987‘s answer. And also a one was added here in 1982.Then in 1987 three was added to get 1988’s answer.

Supporting the above is Learner E‘s written response shown in Figure 4.4 below.

Figure 4.4 Learner E‘s written response of Worksheet 4

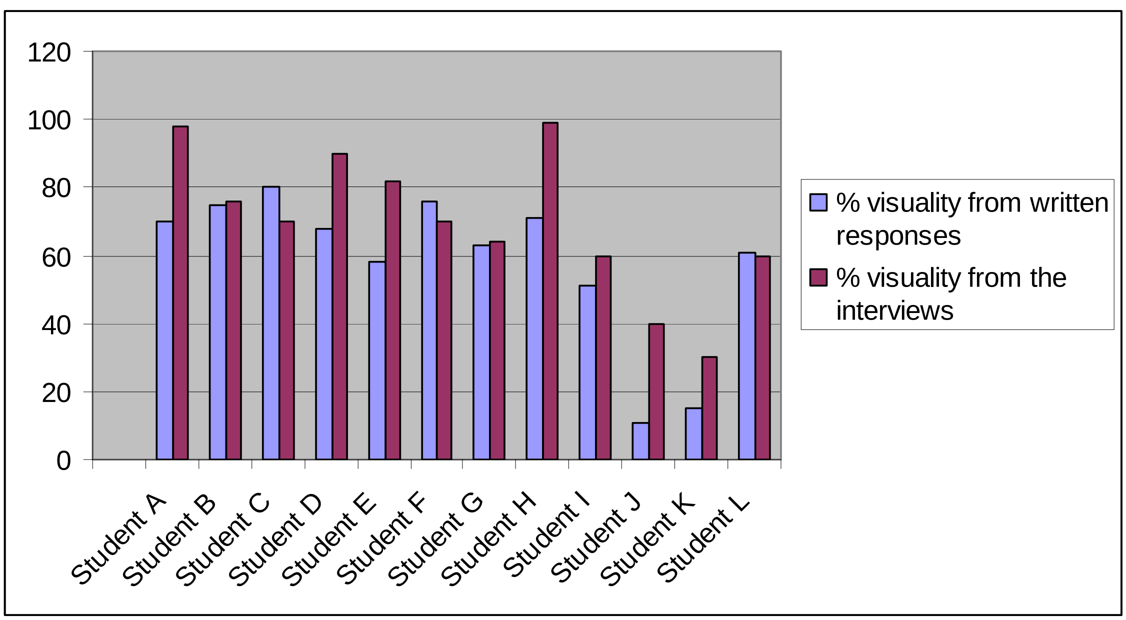

① There are just playing with 5 numbers (2, 1, 4, 6, 3)
② i would be expecting 10 accidents because they are just 5 numbers 2 1 4 6 3
if you say
lets say there are x accidents . if we say
x − 6 we must get 4
for us to get four we say 6 + 4 = 10
x = 10
x − 6 = 4
x = 6 + 4
x = 10
③ 1991 1992 1993 1994 1995 1996 1997 1998 1999
1991 1992 1993 1994 1995
10 12 13 14 20
because they exceed 5 and they have got the five numbers we are playing with

Learner E developed her understanding of the problem and demonstrated increasingly abstract solution activity so that she could anticipate its results and was able to use the results to make future predictions. Always after image generation it could be seen that visualization generated guided tactics, manipulations and decisions during problem solving. Visualization, at the service of problem solving, plays a central role in inspiring a whole solution beyond merely procedural activity towards a more organized structure. For example, when learner E required generalizing from a pattern of numbers above, the learner through visualization identified the stages as substructures of the whole in which a clear pattern can be established. Arcavi (2003:216-233) commented that visualization serves to adjust our wrong intuitions and harmonize them with the opaque and correctness of symbolic argument. This means that visualization serves as a support and illustration of essentially symbolic results by in effect providing a proof in its own right. So it is used as a possible way of resolving conflict between correct symbolic solutions and incorrect intuitions and it is also used in a way to help re-engage with and recover conceptual underpinnings which may be easily bypassed by formal solutions. Visualization also influenced responses and involved the perception of the different types of shapes to enable the unique description of a particular type of a shape.

4.3.2 Image inspection

Figure 4.4 provides evidence that learners clarified their ideas better when they were talking to the teacher than when they were writing those ideas on paper. All the learners A to L had higher percentage visuality during interviews as compared to their percentage visuality recorded from written responses.

Figure 4.4 Distribution of the learners' percentage visuality according to type of data collected



The results from Figure 4.4 show that the type of mental images used were made clearer when learners were able to discuss the problem freely with the researcher than when they had to provide them as written responses. In this case clarifying an image means the learner is internally inspecting his/her image so as to later transform and use it. When the learners realized that the researcher was appreciating all their ideas, they even asked if they could write down more about what they had done during problem solving to reach a final answer. Talking to learners and discussing problems with the learners is therefore encouraged to improve image inspection and therefore to improve visualization. Middle school teachers should strongly encourage, if not demand, that their learners verbalize their thoughts, and should utilize a number of concrete manipulative objects in the classroom (Robichaux 2000:17). Visualization as a process is not intended to exclude verbalization, it actually complements it.

Learner C worked through the problem on Worksheet 3 for about thirty minutes. He had one paper with a rectangle drawn on it and another paper with a cube drawn on it. The following extract indicates what the learner said when asked about the two papers on his desk. It should be noted that the learner had actually been trying to hide this paper away from the researcher.

LEARNER C: Yaa. I managed to work out my problem but at first I did it the wrong way.

INTERVIEWER: Why do you think it was the wrong way?

LEARNER C: Because it says 'On the plane piece of paper provided draw the image obtained by unfolding a geometric body (actually perceived or mentally represented) by the Figure above. And this Figure has what? Six faces. So I just said let me say 1, 2, 3 they will be on this side 4, 5, 6, they will be on the other side, and that gave me a what? a rectangle, and then later I discovered that a geometric body has a….should have space inside, and then I wrote on another paper.

Learner C is indicated above as to how he went into a process of self monitoring and self-regulation by re-reading the problem again. He discovered that what he had done was not what he thought was required. He was convinced that the original image of the net shape that he had on one of his papers was not correct and he was able to give the reason why he thought this way. Visualization during inspection is responsible for the selecting and finalizing of a quality type of visual image. In another way it can be concluded that the learner was guided in the right direction during inspection of the image. In this case visualization aided as a tool to extricate oneself from situations in which one may be uncertain about how to proceed (Davis 1984:35). During the solving of this problem Learner C moved from thinking that the geometric shape was a rectangle to deducing that it was must have been a shape with space. There is need to strive for the quality of mental images during visualization or else without self-monitoring and self-regulation, an unhelpful image emerges to the disadvantage of the problem solving process. In support of this Hoz (in Presmeg 1985:295) comments that some qualities of visualizations often appear in a negative frame relating more to the obstacles which they can create, one of which is called "geometric rigidity".

During interviews the learners continually explained why they had to follow a certain way with the solution and not another. Learner D had this to say on the problem on Worksheet 6.

LEARNER D: I did not draw a bar chart because the smallest amount is zero for buses and there is two for taxis and the biggest amount is 48 for bicycles and then that would be just too much of a big gap and it would mean a big big chart so and then I thought a pie chart would be better.

Learner C responded this way to the problem on Worksheet 3.

LEARNER C: Because a cuboid, it's long and this (pointing to the shapes). We don't have a rectangle. There are just squares, and squares just make a cube.

Learner D was able to give a reason why, although she had thought of a drawing a bar chart, she had resorted to a pie chart. Learner D explained about big gaps between the discreet data of buses, bicycles etc, that would not provide an effective bar chart. The statistical reason given by the learner showed her competence in the handling of data. Visualization involves identification and selection of suitable materials to use for a particular problem which were evident in the case of the three learners above. Meta-cognitive strategies were evidently used throughout the problem solving process. Reflective thinking enabled image inspection and therefore resulted in image transformations. Reflective thinking is a process whereby one keeps on checking on one's problem solving process. In other words, reflection does not only take place after one has resolved a problem, but it also takes place during the process of solving a problem. For example, in the problem on Worksheet 6, Learner D was able to explain why he had to choose a particular type of graph (in this case a pie chart) and not the other ones, for example a bar chart with reasons during his problem solving process. All the other graphs that were not relevant to the learner were eliminated mentally and the learner was able to explain why they did not warrant attention. In substantiating this idea it is said persons who reflect on their activities or solutions have control over their thinking and they are able to pursue several paths to the solution rather than being limited (Wheatley in Zimmerman & Cumming 1991:535). On the other hand, the net shape provided was in Learner C's mind, any of the shapes that included a cuboid but he finally deduced mentally that it was a cube instead. Learner C's mental image was that of small pieces of squares. Through inspection he could see that he could build the small pieces mentally into other shapes that included a cuboid. This is similar to a child who can build boats, houses, churches, and many other things if you provide them with loose shapes. If the shapes are not loose, as in Worksheet 3, visualization then plays an important part. All the stages in visualization were done mentally. The learner was also able to give a reason why the shape that would result from the net shape had to be a cube and not a cuboid. Barbeau (in Arcavi 2003:225) comments that visualization makes use of visual representation of the problem statement. This means that a learner can imagine a visual story and impose it on the problem and derive from it the solution. Learner C saw the net shape in terms of square pieces broken into small pieces and put together to form a cube.

In Worksheet 2 given in Appendix, Learner G was asked to draw a shape that Learner D described without naming the shape. Learner D had drawn a trapezium and written down the description of the trapezium that was supposed to be enough to make another learner draw the same shape, without being told what it was. She was then required to read out her written response to the learner who happened to be Learner G. Learner G was supposed to listen, draw and deduce the said shape without seeing the other learner's diagram. Therefore, Learner D had to give instructions about how Learner G was to draw a trapezium without telling her that it was a trapezium.

LEARNER G: Can I have it again?
LEARNER D: Ok. Ok. I will start again. Draw a 5cm line horizontally.
LEARNER G: Uh. I have finished.

LEARNER D:	And then from were you started draw another line vertically going down about 5cm.
LEARNER G:	I have finished.
LEARNER D:	And then from where you last stopped draw a horizontal line to the right about 6.5cm
LEARNER G:	To the right?
LEARNER D:	Ok lastly now draw a line from the end of the top line going to the end of the horizontal line.

The conversation between the two learners shows Learner D trying to set up very clear and complicated mental images in a bid to describe what a trapezium was to Learner G. It was clear that Learner D was not able to put down on paper all that was required and necessary to be able to draw a trapezium. Learner D used measurements and directions in order to help herself come up with the picture of a proper trapezium. Therefore, a lot of the descriptive work came from the learner‘s mind since there was not much written on paper. Learner D had to mentally inspect the trapezium, transform it and use the information to describe a trapezium. The learner who did not use measurements or directions to give instructions about how to draw a trapezium had problems trying to come up with the shape that he/she had in his/her mind. Explaining her method to the other served the important purpose of inducing Learner D to reflect upon her methods, upon what she had done to solve her problem. In this case Learner D was reflecting about the best way to define and describe a trapezium. It was interesting to note that while learners were going through the activity in Worksheet 1 and Worksheet 2, they suddenly realised that the information they had about a shape was not enough to indicate the shape to someone else. Visualization, apart from consolidating of knowledge and correcting of mistakes, induces important refinements in methods. Gagatsis (in Carpenter & Franke 1996:33) in accordance comments that if one is aware of the processes involved in solving a problem one is able to describe and explain them to others.

More evidence of image inspection occurred during interview discussions. At one point Learner H drew a rectangle in the problem on Worksheet 3 that required the drawing of a geometric body. After the probing from the researcher during the interview, the learner was able to realize his mistake and resolved that the geometric body was therefore a cube not a rectangle. The way the researcher generated questions for the interview caused the learners to re-examine their problem solving process. Visualization is improved if there is discussion in the classroom. Self-monitoring and meta-cognitive behaviour which is evident during visualization occurs during these discussions. Learners talk about solutions, their problems, their doubts and many other things, if they are convinced that they will receive positive support in a conducive environment.

It was noted that learners tended to constantly relate their visual mental images to some physically visual object they knew. Learner I's response to the task on Worksheet 3 in Appendix was interesting. The task required the learner to draw on paper the shape that could be made from the net shape that was given. She made a cube out of paper, glue and cellotape so as to clarify for herself the shape that was made from the net shape. During the interview she had this to say:

INTERVIEWER: You made a box with the sides 2cm each?

LEARNER I: I thought maybe I was just supposed to identify what this thing could make.

INTERVIEWER: So you didn't want to draw anything on paper? It's like the problem says 'On the plane piece of paper provided draw the image obtained by unfolding a geometric body.'

LEARNER I: Yaa. Its like to me it was obvious. The problem was quite obvious to me. When I looked at the paper I could see that in this thing we could do something like this box which can be six sides which is 1, 2, 3, 4, 5, 6. So I thought maybe I could make this box (showing the researcher the box she had physically made from paper, glue and selo-tape)

Learner I could tell that the geometric body from Worksheet 3 was a cube. She physically constructed the cube, but completely failed to draw the cube on paper. She even managed to convince the researcher that, it was easier to make the cube and to number the sides if she made the cube. She totally failed to draw a cube and gave the reason that 'because it was a three dimensional object, it was therefore difficult to draw it'. All attempts to make her draw a cube, which is a three dimensional figure were unsuccessful. The movement from mental images to physical objects and then from physical objects to mental images to clarify understanding occurs during visualization. Visualization enables the use of concrete objects as mental visual aids during problem solving. The above result could also provide evidence that learners at times were not always able to draw or write what they imagine. In this case the use of action imagery was more evident. In substantiating this Wheatley (1998:10) maintains that care must be exercised in inferring imaging from learners' drawings, as learners cannot always draw what they can image. It is therefore true that the mental images of some learners have to work hand in hand with physical visual images if they are to be successful with their problem solving activities. It must also be noted that concrete materials play a significant part in the development of imagery, in focusing attention and in assisting links in concepts during visualization. Learner I, on the problem about a geometric body, was only able to make the geometric body physically but failed to draw the three dimensional body. This proves that it is sometimes necessary to involve concrete materials to allow for the development and understanding of rich mental images. It must however be noted that concrete materials are not used to transmit visual images or spatial concepts to learners but are vehicles to encourage problem solving, knowledge construction and the development and use of different kinds of imagery (Vinner in Owens & Clements 1998:215). Hershkowitz and Vinner (in Owens & Clements 1998:215) maintain that concrete and other physical representations can however limit concepts by restricting the development of flexible concept images.

Responding to Worksheet 3 from Appendix, Learner L had this to say

INTERVIEWER: What is it that I see here?
LEARNER L: I found out that the geometric body was a ring box.

Another Learner E said "I*t is a toy box or a jack in the box*" All the above responses show that the learners metaphorically associated the cube to a ring box, a toy box etc. Visualization in this case allows the use of figurative language as a means of visualizing (Robichaux 2000:11). Learner L made a geometric shape which she called a ring box and Learner E called her cube a toy box which she further described as a 'jack in the box'. This way they were referring to their own personal mental representations of their solution. The metaphors obviously resulted from previous cultural or social backgrounds. It can therefore be concluded that when most learners first read the question and understand it, they mentally search for similar or familiar clues to use so as to generate a meaningful image. This was fulfilled through learners searching for either previous knowledge or previous relationships when they were searching for the right name for the cube. One can safely conclude that visualization enables recall of information from cultural and social backgrounds.

Responding to the problem on Worksheet 3, Learner C drew a cube and named it a cube. He then called it a dice, and had this to say:

INTERVIEWER: You said Eh…. You can make a geometric shape with these six squares which will make a cube. It's some sort of a dice. What do you mean?

LEARNER C: I said that it's some sort of a dice because a dice like a cube also has 6 sides and it is numbered and it's sort of a cube but not that cube.

Looking at the geometric shape he had drawn, Learner C first called it a cube and then later called it a dice. He called it a dice since he found it to be similar to a cube as a result of concrete or pictorial imagery which was explained in Chapter two (Table 2.1) and the learner added that the cube was a dice that was numbered. When the researcher probed Learner A to explain why she had not called her cube a cuboid, the following conversation occurred:

INTERVIEWER: Ok. So are you aware of the difference between a cube and a cuboid? Could you have called it a cuboid?

LEARNER A: No. Because a cuboid, its long and this (pointing to the shape) we don't have a rectangle. They are just squares, and squares just make a cube.

Learner A explained why she could not have called the geometric shape, a cuboid. The reason to enable the researcher to see things the way she saw them and in this way the learner went through the process of animation. Animation is processes of letting one person see the way another person sees things. During visualization metaphors, similes and animation are used to guide the reasoning process (Presmeg 1992:599).

4.3.3 Image transformation

There was development of conceptual relationships assisted by the dynamic movement of images from one shape into another. Learners kept referring to motion and directional language when they had to describe shapes for another learner. Learner D had this to say when she was describing the parallelogram:

LEARNER D: From where you started the horizontal line please I want you to draw a 5cm line going down vertically. (repeats) Going down 5cm.

The statement above indicated the learner's reference to motion and direction in her attempt to make another learner draw a trapezium in Worksheet 2. This was evident through the learner's use of the words horizontal, vertical, down etc. On another note Learner C's fluid transformation of his image was evident while he was describing a parallelogram as a square that needed to be pushed around. To quote:

LEARNER C: I was thinking that a parallelogram is a shape. It is a shape like a square and its sides are equal but it is not straight. It is slant.

INTERVIEWER: Come again. You said its sides are..?

LEARNER C: The parallelogram sides are equal and it is slant, that is a parallelogram.

INTERVIEWER: What do you mean by it being slant?

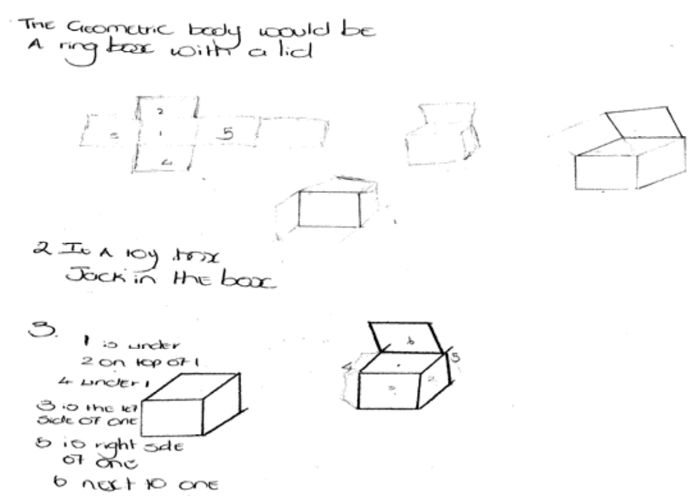
LEARNER C: It is not a square. A square is when you put the paper like this......up straight, it would also be up straight, but if it is a parallelogram now if you put a paper up straight it won't be up straight, it will be like sleeping down.

In the above example Learner C described a parallelogram as a shape that is first drawn as a square and then the square is pushed around. This is evident from the learner's constant reference to the need to make the square slant. He also mentions the idea that the square would be "like sleeping". From the above examples and explanations it can be concluded that visualization aids with the transformation of shapes in order to clarify them before they are understood drawn on paper.

Learners' use of hands to help their mental images was obvious during problem solving. While 'seeing the unseen' during visualization explained in section 4.2., Learner C kept using his hands while expressing that he could see 'it'. The 'it' was the mental image of a cube that he could see in his mind and the learner was inspecting and transforming his image with the help of body language. On the other hand Learner L drew a three dimensional shape but closed her eyes when she was explaining where the numbers had to be put on her geometric body. The use of parts of the body for example twitching of the eyes, movement of the whole body or parts of the body helped with the recalling of the mental images so as to inspect and transform them. It can be concluded that body movements help the learners with the generation, inspection, and transformation of mental images. It was interesting to note that, the changes or transformations that were done by the learners were rarely clearly shown in written responses, but were communicated to the researcher during interviews. Visualization is said to allow one to see in three dimensions. When learners can see in three dimensions, they will also be able to transform in three dimensions. It can therefore be deduced that visualization allows one to transform in three dimensions.

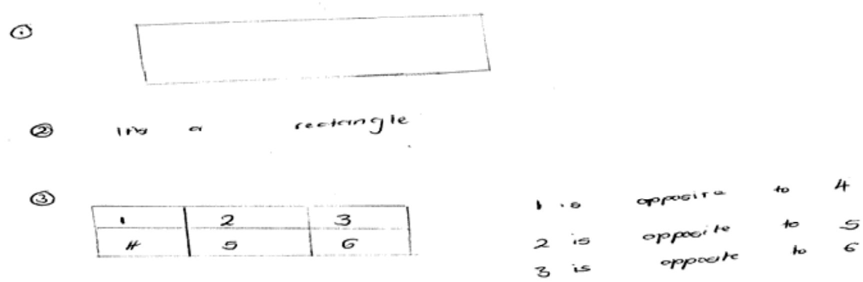
For the learners who put more detail into their written responses, one could observe the problem solving process showing evidence of visualization. Learner B's written work below shows evidence of an original image that was generated and went through a number of transformations achieved after inspection. The generated image was later used to describe some objects that can be derived from a cube. Cubes were in some cases named as toy boxes or as a jack in the box. Learner B‘s written response below shows evidence of image transformation on the geometric body as the learner progresses from one shape to another.

Figure 4.5 Learner B‘s written response of Worksheet 3



A written response for learner K is given in Figure 4.6 below. Learner K moved the shapes from the net shape to form a rectangle. The rectangle was later numbered 1 to 6, and an indication of which number opposite the other was written down. Learner K was able to generate some image but was not able to transform and clarify it so as to get a richer meaningful image. This is an example of something that hinders the successful solution of a problem. In accordance with this view, one aspect of learners' individual differences is each one of them was able to describe a shape clearly, and this is done through the controllability of the images which individuals generate. Richardson (in Lean & Clements 1981:273) suggests that, if imagery is to be useful in problem solving, it needs to be controllable. Tall (1991:31) in support of Richardson comments that in order to be successful in mathematics, it is desirable to have rich mental representations of concepts and this means generating images that one can control. A representation is rich if it contains many linked aspects of that concept, and a representation is poor if it has too few elements to allow for flexibility in problem solving.

Figure 4.6 Learner K's written response of Worksheet 3



Therefore, it can be concluded that Learner K was able to generate an image but was not able to transform it in any way. This problem during visualization hinders the successful solution of a problem. The knowledge of this particular kind of problem may help educators when they want to find ways of improving visualization.

4.3.4 Image use

The problem on Worksheet 6 required the learners to make a representation of some given data. Learner D made a pie chart which was also neatly labelled and coloured. Learner D described how she organized her solution, which included calculations and recall of formulas, in her mind in order to solve the problem on Worksheet 6. The following is part of the conversation between the learner and the researcher.

INTERVIEWER: What did you use to demarcate your pie-chart?

LEARNER D: I used a compass.

INTERVIEWER: So it means you calculated something?

LEARNER D: Yes.

INTERVIEWER: What was that?

LEARNER D: I didn't have to do the hard calculations because if you take the bicycle, car, motor-bike, lorry, taxi and bus and all these numbers, (points to the list of numbers given) they add up to a hundred, and yet in a pie chart you work with hundred. So it wasn't hard.

INTERVIEWER: So for this one (pointing to one of the sectors of the pie chart) How did you get to divide this sector?

LEARNER D: I took my circle.I split it into half. I drew a halfway line and then a one quarter way line and then into..(**PAUSE).** I think it was an eighth. I am not quite sure. I think it was eighth and then I calculated it in my head.

The learner worked out her problem in an organized way, and used colouring to clarify the sectors of her pie-chart. She was able to substantiate her steps and to explain her choice of this representation as opposed to the other representations. Visualization enabled the successful development of this solution. In support of this idea, visualization can accompany a symbolic development of a solution, since a visual image by virtue of its concreteness, can be an essential factor for creating the feeling of self-evidence and immediacy (Fischbein 1987: 101).

Learner A is described in section 4.3.1 as having first worked through the problem on Worksheet 5 without the use of generalization. When Learner A had the following conversation with the researcher, another role of visualization was evident.

INTERVIEWER: So how did you get to 210 dots? Because I don't seem to see how you get to 210 dots.

LEARNER A: Ya. I was trying to work it out on another paper, but I didn't reach the 210 dots.

INTERVIEWER: Where is it? (The learner gets the paper quickly)So you have got another paper. So you started on the 13^{th} figure 14^{th} 15^{th} 16^{th}.

LEARNER A: I didn't reach the figure. (Showing the researcher)

INTERVIEWER: So you failed to reach the figure?

Learner A soon realized that he could not solve this problem for up to and including the 100^{th} term. He then resorted, as was observed from her written response, to generalizing the stages. Visualization serves here as a cognitive technology which was described in Chapter 2. When numbers or patterns are too small or too large to see or visualize, visualization comes into play by acting as a cognitive technology. Visualization aided with the construction of generalization of the pattern, allowed learners to get answers for very large numbers. It can therefore be concluded that visualization helps simplify the limitation of the mind in thinking, learning and problem solving activities. It provides support for the successful development of the solution and therefore also provides the proof of the correctness of the generalization. This provides learners with mathematical power. Mathematical power involves the capacity to make connections, both between mathematical objects and concepts and between mathematics and the physical world. Visualization, whether in the form of concrete images, pattern images or dynamic images, has a key role to play in the development of learners' mathematical power.

Learner H, Learner D and Learner L worked through Worksheet 3, Worksheet 4 and Worksheet 6 respectively. The following conversations were recorded. Learner H had this to say on the problem from Worksheet 3:

INTERVIEWER: Where is your geometric shape?

LEARNER H: It is here (Showing the researcher the following shape).

Closet

1	**2**
3	**4**
5	**6**

INTERVIEWER: Is this your geometric shape? Can't you make any other shape if for instance you cut out these squares (Showing the learner the net shape).

LEARNER H: Yes, I can cut the shapes out, but still this is the shape that I will get..

INTERVIEWER: Can't you possibly make a box if you put some of the squares up and some, side by side?

LEARNER H: No. No. It is impossible. These shapes will fall off……

Learner D had this to say on the problem on Worksheet 5:

INTERVIEWER: Did you predict the number of accidents for 1991?

LEARNER D: No. Look what I went through, but I did not get any pattern.

INTERVIEWER: Can't you possibly check the pattern that comes after every four numbers?

LEARNER D: I tried it. It doesn't work.

Learner L had this to say on the same problem from Worksheet 4:

LEARNER L: I don't see any pattern. These numbers are disordered. There is no pattern here. They are just numbers.

LEARNER L: Those differences between numbers are not the same so what pattern are you talking about? A pattern must give exactly the same things, like a pattern on dresses.

After the researcher had discussed the problem with Learner D, the latter realised that a pattern actually existed and was then able to make a prediction successfully. On the other hand, Learner H managed to generate a mental image of the geometric body. His geometric body was a rectangle divided into six parts and this was not useful to the given problem. He could only see a rectangular shape being formed from the net shape. Learner H was unable to see the diagram in a different way, only in just one simple way. This way did not help him solve his mathematical problem. Learner H could not be moved from the idea that the net shape was not a rectangle to it being a cube even after probes and clues were given. This was an uncontrollable image. In contrast Learner D never managed to generate any meaningful image. He could not get a pattern that was meaningful and would help him succeed with problem solving. All the above three learners' experiences resulted in the generation of images that were later found to be inappropriate. Visualization can cause such problems for the learner and can happen with particular tasks. While learners do not have much difficulty in generating images, they do have great problems in producing rich useful images which they are able to control. Controlling an image means to be able to inspect it, transform it in several ways or even to get rid of it, if found to be useless so as to be able to get a useful image. Successful problem solving is possible if it is aided by guided mental images that can be controlled. Learner H generated an image but was not able to control it and so he could not use it to solve his problem. Another learner insisted even through probes and clues that he could not see the pattern in the problem on Worksheet 4. Learner L had a mental image of a pattern related to the patterns on the dresses only and therefore his definition of a pattern produced a mental image of patterns on dresses, and he could not relate it to the number context. Unfortunately this kind of an image was not useful to the learner's solution. The above discussion makes it clear that visualization is an aid to successful problem solving.

The last part of the Armidale problem on Worksheet 4 allowed the learners to use their results of part two to make future predictions. Approximating the number of accidents after 1991 was done through the study of relationships that were identified in the first part of the question. This was part of the conversation that was recorded during the interview:

RESEARCHER: How did you predict the number of accidents that occurred in 1992?

LEARNER F I could see it by looking at these numbers. There was a pattern that I recognized and if this pattern is to continue, 1991 will get about 10 accidents….

Learner F recognized the need to use the word 'about' to give his answer about the number of accidents that occurred in 1991. Statistically this shows the learner's success in dealing with the organization, analysis, and prediction of data while heavily involving his mental images. Estimation appears to have encouraged and improved the learners' ability to generate mental images. Visualization offers a method of estimation and therefore enables prediction of results in data handling. When learners study patterns in data and use the patterns to make future or previous predictions generating images for this purpose becomes a must. Mental images during visualization are responsible for organizing and reorganizing the data to derive meaning out of it. For example, Learner F above went through a process of studying the pattern and her answer was an estimate. No single correct answer was expected and evidently learners evoked very rich mental images.

4.4 Main findings based on the analysis of the quantitative data

This study was set up to examine the role of visualization qualitatively through an understanding of the thought processes that occur during visualization when learners engage in data handling and spatial tasks. Therefore, the objective of the analysis of data was to provide evidence of the learners' uses of visualization and to be able to describe the processes that occur during visualization so as to deduce the role of visualization. Some major findings which support the qualitative results of this study have been summarised and illustrated in short in Table 4.6.

Table 4.6 Summary of the major findings

LEARNER	A	B	C	D	E	F	G	H	I	J	K	L	Average % visuality
% visuality from written responses	70	75	80	68	58	76	63	71	51	11	15	61	58.25
% visuality from the interviews	98	76	70	90	82	70	64	99	60	40	30	60	69.92

% visuality	84	75.5	75	79	70	73	63	85	55.5	25.5	22.5	60.5	64.08

Distribution of learners used for comparative analysis					
	GENDER		**ABILITY(Maths subject performance)**		
	GIRLS	**BOYS**	**Above average**	**Average**	**Below average**
Frequency	5	7	4	4	4
% visuality from written responses	57	61	73	67	35
% visuality from interviews	75	63	84	73	48
Average % visuality	66	62	78.5	70	41.5
Total possible	100	100	100	100	100

Table 4.6 highlights high percentages of visuality ranging from 55% to 84% for the majority of the learners. It was therefore evident that all the learners in this study used visualization during their data handling problem solving activities. While engaged in a mathematical activity whether of a numerical or geometric nature, the learners in this study managed to construct images. The visuality per learner per task depended mainly on the nature of the task and on the way the learner used visualization in solving the task. According to Table 4.6 the difference between the average percentage visuality of boys and that of girls is very small. The percentage visuality from written responses always tended to be lower than the percentage visuality from interviews. Though written responses as well as interview responses are both important for getting a proper analysis of the individual learner’s visuality, there is, however, an indication that one can get better evidence of the use of visualization from interviewing or from talking to the learner than from written responses. Learners in the sample group who had been chosen as above average in mathematics during sampling appeared to be getting a higher visuality than the learners who had been chosen as below average during sampling. It can be concluded from this study that learners are likely to

improve their performance in mathematics if they are encouraged to use visualization. Encouraging learners to use visualization can be successful if the educators have a clear understanding of the role of visualization.

4.5 The Problem-Centred Context's contribution to a successful investigation

The organization of the administering of tasks and interviews was successful since an effort was made to work in a problem-centred context. Sessions for the empirical investigation always started with a clear explanation from the researcher about what was expected from the learner. Moreover, the classroom atmosphere contributed to an easy working environment. If a task was misunderstood due to the way it was worded or through any other language problem, the researcher offered clarification. The emphasis was for the learner to develop meaning from the tasks as opposed to getting correct answers. In this setting learner were aware that it was their responsibility to think through tasks presented and to make sense of them. In turn they knew that the researcher was genuinely interested in their thinking, respected their efforts and that the researcher enjoyed listening to any explanation of everything that was taking place in their minds. The classroom was therefore characterized by trust between the teacher and the learner. In this environment learner felt comfortable to respond freely and to express their ideas. In addition to this prevailing sense of trust, the classroom was characterized by a problem solving atmosphere in which learner were routinely expected to verbalize their attempts at solutions and reflect upon these solutions. Lastly the open-ended nature of the activities contributed greatly to the generally favourable atmosphere that characterizes a successful problem-centred setting.

4.6 Conclusion

The two research instruments, the written responses and the interview, made it possible to accomplish the aim and objectives of this study. Data collected from the written responses and the interviews were presented and analysed. The focus was on finding evidence that the learner had used visualization and then to describe the mental processes involved during visualization while learners worked through data handling and spatial tasks. The investigation provided evidence of the use of visualization during the learner' problem solving endeavours with data handling and spatial tasks. It was revealed that all learners used visualization while involved in problem solving. The mental processes involved during visualization are better collected during interviews than from written responses. This is revealed through a higher percentage visuality recorded during interviews than recorded in written responses.

Comparisons were made of the different percentages visuality among above average learner, average learner and below average learner. It was revealed that the above average group had the highest percentage visuality, followed by the average and then the below average group. When boys and girls were compared it was found out that all learners go through visualization at almost the same rate. On the whole, a number of significant implications relevant to mathematics emerged from this investigation which will be discussed in the next Chapter.

In this study the domain-specific instructional theory was the cognitively guided model (Fennema, Carpenter & Peterson 1991) that uses research based on learners' thinking to inform instruction. Although the group involved in this study was not large, nevertheless some valuable information was retrieved from the study as a whole. As a result conclusions were drawn and recommendations were made. One thing is certain, that enormous diversity exists not only in individual abilities to form images, but also in the utilization of visual imagery when different individuals do mathematics (Presmeg 1985). In this instance, the learners used their visualization strengths to help them better understand and solve the data handling and geometric tasks given to them. Wheatley (1998) also observed this interplay between visualization and mathematics in a series of interviews he conducted with elementary school learners. (Wheatley (1998:14) concluded that "In learning mathematics, each of us builds a set of personal images which may assist us in solving problems or making new mathematical constructions".

CHAPTER FIVE

SUMMATION, RECOMMENDATIONS AND CONCLUSION OF THE STUDY

5.1 Introduction

The purpose of this dissertation was to investigate the role of visualization in a problem-centred context through the analysis of the mental processes during visualization that Grade 9 learner went through while solving data handling and spatial tasks. This will in turn provide educators and learners with a broad picture of the role of visualization, so as to help them make informed learning or instructional decisions. The process of visualization does not just happen by accident and is a complex one which should therefore be understood clearly. The focus was on the individual nature of this process and it has also been acknowledged that studying individuals' process ideas is not an easy task (Presmeg 1986b). In this study the role of visualization was analyzed through carrying out an empirical investigation to understand visualization by determining and describing the thought processes that occur during visualization when Grade 9 learners engage in problem solving via learners' responses to tasks in data handling and the field of spatial development. Learners go through the generation and inspection of mental pictures, transforming and using them in so many ways, so as to reach a solution. This is evidenced in this study by what learners have demonstrated during interviews and written responses.

Throughout the study links were established with the literature on visualization and an attempt was made to prove some of the researched ideas about the role of visualization. For example, Presmeg (1986:44) comments on other positive qualities associated with visual imagery during visualization such as its integrative power, its exemplary use, its concretization of abstract ideas, and its sometimes sudden illuminative aspect. These ideas are confirmed in this study by the results of the empirical investigation. Kosslyn (1980:222) defines four classes of image processes: generating an image, inspecting an image to answer questions about it, transforming and operating on an image and using and maintaining an image in the service of some other mental operation which are used to aid identification of the visualization process. Tasks and interviews were given to individual learners, but it was discovered that certain images can be common to more than one learner. This means that different learners can generate, transform and use images in exactly the same way. This occurred when there was utilization of the same tasks and environments or backgrounds.

5.2 Summation

When the learner is placed in specific material and social environments and required to carry out certain tasks, it is found that children create and use very different kinds of imagery, and they operate with those visualizations in very different ways. Different tasks can stimulate different images. This is because the visualization process appears to involve the learner constructing certain kinds of images and using them appropriately. Images are internally perceived, that is, holistic representations of objects or scenes that are isomorphic to their referents (Bishop 1980). They are mentally changed by continuous transformations corresponding to physical transformation. How much of this activity was required in this study depended entirely on the interaction between the stimulus and the individual concerned. Each individual interpreted a task in his/her own way, for there was never a unique way to carry out a task. What then happened depended on the learners' preferences, on their memory of visualization, on their ability to recall or to generate appropriate visualizations, or to choose appropriate ones, and then on their ability to operate appropriately with the chosen visualizations. The implication of this analysis is clearly that visualization and the use of visual imagery should be regarded as a very personal matter. Therefore, individual learners need to be given time to develop their own images, and there seems little value in teachers expecting that identical images will result from such a personal process. As a teacher one must anticipate, and to some extent therefore encourage the possibility of diversity in visualization.

It was obvious from this empirical study that learners normally write just a small fraction of what is actively going on in their minds when they are solving problems. To get more information, and to be able to help learner with their mental images, individual or class discussions with the teacher must come into play. Learner have to be convinced that they are in control of their solutions, and are not going to be penalized for thinking otherwise. Discussions with the teacher help the learners to arrive at answers logically and enable them to provide good reasons for getting such an answer. There must also be room for the learners to argue with their peers, check their peers' approach in order for them to have what are called controlled images. Individual solutions of problems sometimes cause uncontrollable images. For example in the study, Learner I stuck to the idea that she could only form a square from the net shape she had been given until she was given the opportunity to physically play with the different shapes. Therefore, learners can learn visual skills by involving themselves in a rich combination of drawing, manipulating, imagining, paper-folding, building, and using computerized enhancements or simulations of any or all of these. The activities may be physical, but the essential skills are mental and must be complemented by reflection on what learners have done.

More specifically, the investigation revealed some major roles of visualization and verbalization actually complemented it. During image generation it was revealed that learners are able to see in three dimensions regardless of how the shape is presented. Different types of images were generated for the same described problem and the importance of key words as a guide to generating images could not be ignored. During image inspection, meta-cognition which is thinking about thinking, manages the mental process during visualization. It was deduced that during problem solving there is monitoring of the processes which occur during visualization and these include organization of the problem solving procedure, providing proof of the results, selecting and finalizing of quality type of images and selection of suitable materials to use for a particular problem. Metaphors, similes and animation are used to guide the reasoning process. The results also reveal that during image transformation the learner is able to transform in three dimensions successfully. Transformation results from reflection or meta-cognition. Body movements help the learner to transform his/her generated mental image. The finding further revealed that the learner recalls previously lent formal or cultural knowledge and uses it at appropriate times. Visualization simplifies the learner's limitation of the mind in thinking, learning and problem solving activities. This occurred in this study when the learner was working with problems that involved patterns that were too long to determine the nth term or if there was need for learners to make future or previous predictions. The learner's need to integrate and apply his/her mathematical knowledge is highlighted in this case with his/her involvement with estimation and prediction.

Further probing during the investigation revealed that problems can occur to some learners during visualization and that this provides a positive evaluation or assessment for the educator responsible. Finally the study overall identified that a learner's inability to monitor and regulate his/her thought process can be a main hindrance to effective problem solving. Therefore, visualization has a key role in determining how learners selectively attended to tasks during problem solving.

5.3 Limitations

There were a number of limitations experienced while carrying out this study. Although the facilitation of tasks was done in a problem-centred context, it should be noted that this study could not test or check the effects of a problem-centred context as the empirical study was done in a limited amount of time. Greater interest was in the understanding of the role of visualization in data handling by facilitating learners with the tasks in a problem-centred setting. The empirical study was done in a problem-centred context for as much as it was possible for the researcher. Data was collected from written responses and interviews. While the categories of Kosslyn (1994) were used to identify and describe the processes during visualization, the detailed description of the sub-categories was done only to the extent that the researcher could identify the relevant sub-categories. It was also found out that some rubrics and their performance indicators were used to mark and analyse the written responses. Again the creation of the performance indicators and the relating of the performance indicators to the Kosslyn categories were made possible at the researcher's discretion. Subjectivity on the part of the researcher could not be avoided totally since it was the researcher who was personally involved with the categorization of the different processes. The choice of the sub-categories and the performance indicators was an objective exercise and this meant that it was sometimes difficult to distinguish among sub-categories and performance indicators.

The understanding of the instructions written in English in the tasks posed many problems for the learner and therefore the responses that the learner gave depended on how they had understood the problem. Mathematics is currently taught in a second language for most learners in South Africa. The medium in which mathematics is taught and facilitated plays an important role in both the development of the skills as well as to indicate some form of competence. In this study the tasks that were given to the learner were facilitated in English, which is a second language to the learners at this school. It was noted that at the beginning of each exercise learners needed clear explanations of what the various English expressions or words meant. Without a fruitful discussion of these terms and expressions no progress could have been achieved.

The time that was taken to carry out the investigation with each learner was dependent on individual responses to the task and to the interview. For example, during one session the learner requested to redo his problem as soon as the researcher had started the interview. Some transformation had obviously occurred in his mind and he wanted to change the way he had approached his problem. The result was that the researcher spent more time than what was budgeted for the investigation per day.

5.4 Recommendations

Visualization must serve as an integral part of teaching. It should help teachers to make instructional decisions about how to teach. Also decisions must be made about the content and the nature of tasks. Through the interviews carried out it appeared that opportunities for learners to recognize the importance of questioning the reasonability of their problem-solving processes or solutions should be provided. Visualization is enriched during environments where discussions are encouraged and in environments where learners are not penalized or humiliated for the paths they follow during problem solving. For example, some learners arrived at wrong solutions but only realised their short-comings during the interview with the researcher. The researcher in this case acted as a guide and mentor and as a person who understood all that the learners had gone through during their problem solving. If a learner withholds what he/she has in his/her mind, it is possible for the educator to miss those important issues that make the learner successful or unsuccessful during problem solving. Educators must appreciate the role of visualization during the solution of problems. This implies that the teacher must appreciate the process involved in reaching a solution rather than giving more attention to the end product in a solution. From the researcher's own experience as a teacher, it was interesting to note that learners usually realise what the teacher prefers: the end product which is the answer or the process towards the solution. Educators should understand the need for learners to communicate and interact with each other during mathematics lessons. Enriched with this knowledge the educator is able to provide an atmosphere that encourages fruitful, helpful and meaningful mental processes which Arcavi (2003:234) calls the micro-cosmos. This in turn will provide a better understanding of data handling and can hopefully be extended to all other subjects in the school curriculum. When a classroom, according to Arcavi (2003:234), is considered as a micro-cosmos, as a community practice, learning is no longer viewed only as instruction and exercising, but also becomes a form of participation in a disciplinary practice. Visualization is a central theme that develops and stabilizes an interaction between people and things.

If the visualization skills of the learner are not appreciated, one is likely to encourage a lecture style when teaching. Successful solutions of problems during visualization allow the learner to make connections between the mathematics curriculum and many other areas of learners' experiences, including other subjects, other parts of the curriculum, mathematics learned in the past years and, above all, the real world. Creativity is encouraged and developed.

Educators should try and make the role of visualization clear to the learners if the learners are to take maximum advantage of it during problem solving. Engaging learners through the process of visualization will enable learners to look at their own problem-solving processes critically. Thornton (2000) advises that what is of crucial importance is to promote flexibility of thinking, and to encourage learners to look for the connections between alternative representatives of mathematical entities. Noss (in Thornton 2000:255) describes mathematical thought as being characterised by the capacity to move freely between the visual and the symbolic, the formal and the informal, the analytic and the perceptual and the rigorous and the intuitive. It has been noted that it is not the size or complexity of the data set that is necessarily significant in encouraging learners to handle data in more sophisticated ways, but rather the questions and discussions in which they are encouraged to engage.

The research findings of Kosslyn (1980; 1983; 1994) have very important implications for education. Visualization involves imagery. An understanding of the precise limitations of children's imagery could inform one how to use imagery in teaching. For example, if children's images are static, in that they cannot transform their images (see them melt into a new shape), they may still be able to make use of 'blink' transformations (that is erasing the first image and imaging the object anew in some altered way). If so, then it would make sense to shift from trying to teach the child the rules of transformation to teaching him/her the rules of formation. Learning how to break up an initial image and see it flow into a new shape is different from learning how to form a second image that is related to the first in some way. Similarly, if young learners are able to form images easily, while they have difficulty in maintaining them, it would be better to try to teach them by inducing a series of rather simple images and so forth. Learners should be encouraged to acknowledge the limitations of the available mental representation in order to identify which questions are answerable from the given data. Learners should exercise caution when reading and interpreting the data and always attempt to link all items involved in data which are representing, summarizing and prediction. Whilst it is tempting to believe that all visualizations must necessarily play a

useful role in mathematical activity, we clearly need to understand the difficulties that come with visualization in a given mathematical situation. Presmeg (in Aspinwall, Shaw & Presmeg (1997:304) describes cases where imagery was not helpful but a hindrence to solving mathematical problems. Uncontrollable images were identified. By identifying uncontrollable images it should be noted that one is not attempting to abandon evoking imagery, but we are saying educators must be aware (Bartlett in Aspinwall, Shaw, & Presmeg 1997:315) that possible difficulties with imagery in the learning of mathematics are the price of its peculiar excellences. With these contrasting viewpoints on the benefits of visualization in the literature, one thing remains certain that enormous diversity exists not only in the individual ability to form images, but also in the use made of visual imagery when different individuals do mathematics (Presmeg 1985:303). One aspect of these individual differences is the controllability of the images which individuals use. Richardson (in Aspinwall et al 1997:312) suggests that if imagery is to be useful in problem-solving, it needs to be controllable.
The ability to create a mental image of an object and then to manipulate it mentally has significant practical application in fields such as mathematics, physics, architecture, engineering and design (Rhoades 1981:247). The results produced through an investigation of the role of visualization in this study provide a need for more research in order to understand all aspects of this process of visualization. Understanding the role of visualization is important for the teaching and learning of mathematics. If one understands and appreciates the role of visualization there will be need for more research to be done on how to help learners improve their processes of visualization.

5.5 Conclusion derived about the role of visualization

It has been concluded in this study that visualization is an important aspect in the learning of mathematics. The study has managed to identify the role of visualization through the description of the thought processes that learners go through while solving data handling and spatial sense problems. The role of visualization was highlighted when particular highlevel tasks which require high level processing were chosen for this dissertation. These are tasks in which information has to be mentally processed in some way, via remembering, integrating with other information and involving judgements. High level processing typically requires comparisons, mental computations, or transformations of information held in human memory, deeper and more elaborate mental processing, and therefore information also tends to be remembered well. Visualization requires high-level processing and is therefore important when learners learn data handling. The goal of this research was set by means of data handling and spatial problems. Spatial and data handling problems were used in this investigation because these problems are used by most researchers in their studies on improving visualization. However, it should be noted that both the spatial and data handling problems were chosen on the basis of a number of general, explicitly articulated requirements that were necessitated by the research goals. It can therefore be expected with some confidence that at least a part of the findings can be generalized to other situations that satisfy these requirements.

Visualizing can lead to the development of tactics and the monitoring of progress in problem solving. It has been noted that it is not the size or complexity of the data set that is necessarily significant in encouraging learners to handle data in more sophisticated ways, but rather the questions and discussions in which they are encouraged to engage. The role of visualization is important in mathematics and in all the other subjects of the curriculum, and it must therefore be appreciated and taken advantage of. The highest minds are probably those in which it is not lost, but subordinated, and is ready for use on suitable occasions (Presmeg 1986:42).The important role of visualization that was identified within this study led the researcher to conclude that encouraging visualization during the learning and teaching of mathematics should be a significant part of the school mathematics curriculum on all grade levels.

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APPENDIX A

WORKSHEET 1

1. Draw a parallelogram on the given piece of paper.

2. Write down a description of this Figure without mentioning its name, as you would to a friend over the telephone, so that the friend will be able to tell what kind of a Figure this is and also be able to draw this Figure as well.

3. You will be asked to read out your description slowly and carefully (repeating if necessary) to the other group without showing them your diagram so that they are able to draw the shape you are talking about.

Adapted from Robichaux, R.R.2000. The Improvement of Spatial visualization: A case study (pp21).

WORKSHEET 2

1. Draw a trapezium on the given piece of paper.

2. Write down a description of this Figure without mentioning its name, as you would to a friend over the telephone, so that the friend will be able to tell what kind of a Figure this is and also be able to draw this Figure as well.

3. You will be asked to read out your description slowly and carefully (repeating if necessary) to one member of the other group without showing him your diagram so that he/she will be able to draw the shape you are talking about.

Adapted from Robichaux, R.R.2000. The Improvement of Spatial visualization: A case study pp24

WORKSHEET 3

1. Identify the geometric body or image which could be obtained by imagining the folding back of the multi-dimensional drawing shown above.

2. On the plane piece of paper provided draw the image obtained by unfolding or folding the geometric body (actually perceived or mentally represented) by the Figure above.

3. Clarify your diagram by indicating the number which will be opposite the other.

(Adapted from Efraim Fischbein)

Worksheet 4

Data Interpretation Question

A well-known intersection in Armidale has had a number of serious accidents. The number of serious accidents was recorded for the last ten years.

1981	1982	1983	1984	1985	1986	1987
2	4	5	9	3	6	7

1988	1989	1990
10	4	6

1. Describe any pattern that you can see in the data.

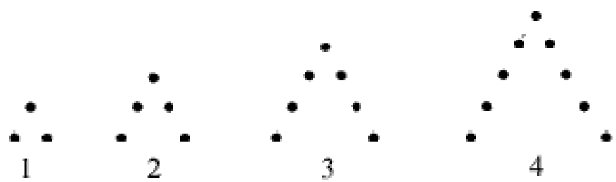
2. Approximately how many accidents would you expect in 1991? Why?

3. Suggest four other years in the future (after 1990) when you think the number of accidents might exceed 8. Why did you select those years?

(Adapted from Chris Reading, University of New England, Australia)

WORKSHEET 5

The following question refers to the following pattern of dot-Figures.



If this pattern of dot-Figures is continued, how many dots will be in the 100th Figure?

Give your answer and explain, writing down all procedures of how you found your answer to the question on the given plane paper. More paper can be provided.

WORKSHEET 6

The learners in your class were asked to take note of the first 100 vehicles that passed by the school's main gate road in Pretoria. The numbers of each type of vehicle were given as follows: lorry---7; bus-----0; car----28; taxi----2; motor-bike-----15; bicycle-----48

Represent the above information in a way you would like it displayed on a poster for the school notice-board for all the people to see. Use the plane paper provided to draw your representation. Show on the paper provided all the steps you will take in order to draw your representation. Explain why you prefer this representation and not the others.