

Relationship between Visual Perceptual Skill and Mathematic Ability

by

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Abstract

Poor mathematics performance in South African schools is of national concern. An attempt to gain insight into the problem prompted a study into the possibility of a relationship between visual perceptual skill and mathematic ability. A theoretical review revealed that inherent limitations of traditional psychological theories hinder an adequate explanation for the possible existence of such a relationship. The theory of situated cognition seems to be better suited as an explanatory model, and simultaneously clarifies the nature of both visual perception and mathematics. A small exploratory study, with a sample of 70 Grade 6 learners, provided empirical evidence towards the plausibility of the relationship. Specifically, it proved the hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement. The results of the study, interpreted within the situated cognitive framework, suggest that a conceptual emphasis in mathematics education – as opposed to a factual emphasis – might improve mathematic ability, which may credibly reflect in scholastic performance.

Keywords: visual perceptual skill, mathematics, mathematic ability, mathematical cognition, situated cognition, embodiment, societal embedment, extended mind, enaction, cognitive emergence, conceptualisation, metaphor

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Relationship between Visual Perceptual Skill and Mathematic Ability

CHAPTER 1: Introduction

In 2008, approximately 60% of South Africa's matric students failed mathematics (South African Press Association [SAPA], 2009). In 2009, less than a quarter managed to achieve more than a 40% pass mark in mathematics. Experts were quoted as warning that these disastrous results would continue, unless the basics were properly taught in primary school (Govender, 2010). A study by Stellenbosch University in 2010, commissioned by the Western Cape Education Department, subsequently revealed that only 35% of Grade 3 learners, in 45 Western Cape schools were sufficiently skilled in mathematics (Mokone & Davids, 2010).

Mathematics education has been a priority of both the South African government and the private sector since 1994 (Bernstein, 2005). Yet, more than fifteen years later, less than half the students who wrote the matric final exam managed to pass. This dismal statistic is despite a noticeable increase in the number of children attending extramural maths classes. One well-known maths class franchise (Own a Master Maths education franchise, [s.a.]) boasted a 38% growth in their enrolment between 1995 and 2000, with an additional 8 000 students enrolling over the next four years. The Centre for Development and Enterprise (CDE) also recently reported (cited in Bernstein, McCarthy & Oliphant, 2013) finding significant increases in private extra maths class attendance.

Mathematics is becoming increasingly important in our increasingly technological world. The government recognises this trend, as evidenced by the fact that mathematics literacy is now compulsory in the South African education system. However, our children are failing to cope with the demand in school. Education is a critical multi-disciplinary field, where psychological theory has much to contribute to furthering education research, as well as to ensuring a well-educated future generation.

1.1 Context

Practice in mathematics is essential in order to consolidate and strengthen certain critical skills and techniques. However, practice is of no benefit where a concept is not fully grasped. Furthermore, mathematics teaching is hierarchical, where new concepts build on those previously taught. It is therefore essential for learners to grasp each concept consecutively in order to avoid mathematical stagnation; an insight first endorsed by Piaget (1970). Bernstein et al. (2013) have also claimed that early mathematics learning deficits hinder subsequent mathematics learning.

Difficulties in mathematics may therefore stem from levels that are more elementary, with the most basic level involving the cognitive tools required for mathematics. It is widely accepted that visual perceptual skill forms the foundation of all learning, and for mathematics and reading in particular (Halliwell & Solan, 1972; Leonard, 1986; Piaget, 1970; Rosner, 1982; Wurzburg, 1994; Hegarty & Waller, 2005), on the basis that they both involve decoding symbolic information.

A set of five essential visual perceptual component skills were categorised by Chalfant and Scheffelin (cited in Martin, 2006) in 1969, thus:

1. **Visual discrimination** is the ability to discern similarities and differences of visually dominant features.
2. **Figure-ground** is the ability to distinguish distinct shapes from an irrelevant background.
3. **Visual closure** is the ability to identify a figure when visually fragmented or disorganised.
4. **Spatial relations** are the ability to perceive the position of objects in space both in relation to each other and to oneself, incorporating directionality and laterality.
5. **Visual memory** is the ability to recall previously seen visual information.

Together, these skills enable the complex visual perceptual facility. The collective contribution of the component skills thus determines composite visual perceptual skill. Rosner (1982) suggested that the perceptual skills appear to be partly innate and partly learned, and are consequently trainable. Moore (1997) has similarly expressed that “visual skills and visual integrative abilities are inherent [...] However, the degree to which its

structures are efficient and integrated with each other becomes a developmental question” (p. 27).

Well-developed visual perceptual skill may be a particularly beneficial cognitive tool when it comes to the practice of mathematics.

- Visual perceptual skill enables identification of the representation of numerals, which enhances the memory of number facts.
- One proposal, supported by Lakoff and Núñez (2000), is that perception is the basis of conception. Concepts metaphorically map to abstract thought, a definitive aspect of mathematics.

These claims support Rosner’s (1982) assertion that, irrespective of whether mathematics is taught by number facts or by number concepts, visual perceptual skill will aid learning.

Taking cognisance of this, mathematics lessons ought perhaps to comprise a combination of mathematics teaching as well as a visual perceptual skill training program, thereby addressing the two most critical components of learning: knowledge acquisition and analytic ability. Rosner (1982) has encapsulated this by saying, “the better one’s perceptual skills, the more one learns from concrete experiences; and the more one learns from concrete experiences, the better his perceptual skills” (p. 302). However, before implementation, or even researching the beneficence of such a scheme, clarification is required as to whether visual perceptual skill does indeed relate to mathematic ability in children.

Although the importance of visual perceptual skill for mathematics learning seems to be generally accepted, there is little agreement on an explanation for why this should be the case. Controversy is therefore evident as to the precise nature of the relationship between visual perceptual skill and mathematic ability, and to the specific aspects or component perceptual skills that are most relevant when it comes to mathematics.

1.2 Goals and Hypotheses

The goal then, is to explore the apparent link between visual perceptual skill and mathematic ability in children. A discussion pursuant to this goal suggests that classical cognitive theories are somewhat inadequate in their conception of both mathematics and visual perception, which may contribute to the uncertainty surrounding the possible existence

of a relationship between them. The relatively recent situated cognitive theory (Robbins & Aydede, 2009) is introduced as a basis for the development of a more coherent explanation for the suspected relationship, which simultaneously clarifies the nature of both mathematics and visual perception. Subsequently, a small exploratory study is used to test the ensuing hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement, and the relative contribution of a set of component skills is thereby determined.

1.3 Design and Methodology

A natural correlational design was considered most appropriate at the outset, due to the focus of the hypothesis on *scholastic* mathematics achievement. The research sample consisted of 70 Grade 6 learners from diverse socio-economic and cultural backgrounds. The third edition of the Test of Visual Perceptual Skills (TVPS-3) was used to assess participants' visual perceptual strengths and weaknesses. Performance on a grade-appropriate mathematics test served as a measure of scholastic mathematics achievement (the dependent variable). Pearson's product-moment correlation determined the significance of the relationship between visual perceptual skill and scholastic mathematics achievement. A regression analysis, utilising the TVPS-3 subtests' scores as the independent variables, and a correlation matrix, determined the relative contribution of individual component skills to this relationship.

1.4 Delineation

Is there a relationship between visual perceptual skill and mathematic ability? The simplicity of this question belies the complexity of its component constructs. Both visual perception and mathematics are multi-disciplinary topics, involving – amongst contributions from other fields – insight from psychology, philosophy, neurophysiology, linguistics, artificial intelligence, education, and mathematic theory, where each discipline may harbour several pertinent theories. Consequently, there are a number of potentially conflicting paradigms.

The growing sphere of cognitive science involves the attempt, in all aspects of cognition, to resolve conflicts among various extant paradigms and disciplines, with the

subsequent creation of a unified inter-disciplinary field. Insight into the plausibility of a relationship between visual perceptual skill and mathematic ability therefore entails a striving towards a broad cognitive science of mathematical cognition and visual perception. Such an endeavour demands analysis and synthesis, not only of empirical evidence, but also of various select theoretical contributions. Constructing this study as a researcher with an educational background in classical cognitive psychology, investigation necessarily begins with the wisdom of conventional theories largely considered as legitimate within the field of psychology. Conventional psychological theory intimates a relationship between visual perceptual skill and mathematic ability. However, it has proven difficult to provide a feasible description of this relationship, or an explanation for its existence, within this conventional framework. Inherent shortcomings of traditional theories have steered the direction of further investigation towards the cognitive science theory of situated cognition. The present study follows the course of this investigative path.

The complexity of mathematics eludes simple definition. In addition to the precepts of various theories through which it is analysed and understood, there is a pervasive and resolute folk belief in mathematics as being universal, infallible, and transcendent of human experience. Chapter 2 therefore presents various philosophical and cognitive perspectives on mathematics in an attempt to gain clarification on the subject.

The nature of mathematics is generally viewed as an essentially philosophical consideration; hence, an assessment of the foremost contemporary philosophies of mathematics is pursued in this study, to include rational realism, formalism, and constructivism. Although none of these can adequately account for all aspects of mathematics, the assessment made here reveals the intuitionist augmentation of constructivism as the most acceptable when taken from a cognitive psychological perspective. Accordingly, mathematics is presumed to be a fundamentally mental construction, built upon a base of particular, innately intuitive concepts, and developed through perceptual and cultural experience.

The cognitive and conceptual nature of mathematics suggests the necessity for a deeper probe in to how we actually *do* mathematics, in order to enhance our understanding of mathematics. Thus, a brief review of mathematical cognition presents a selection of pertinent conclusions from decades of research. The review begins by looking at the cognitive concepts that support arithmetic, which include the number sense, counting principles, symbolic

representation, and the mental number line. The development of these concepts is examined, which exposes the explicit involvement of various visual perceptual component skills, which carries through to basic arithmetic. When it comes to mathematics beyond arithmetic, however, uncertainty prevails. Research in this aspect being scarce, deliberation on associated influences such as language, education, intelligence, working memory, and visuo-spatial abilities provides some insight into higher mathematical cognition. However, it is apparent that the view of mathematics as essentially a formally analytic process – as opposed to an experientially conceptual one – is still in evidence within the scientific community. It is suspected that conventional psychological theories of cognition may have contributed to the apparently discrepant beliefs in this regard.

Notwithstanding confusion as to what mathematics entails, it is only within the framework of a coherent and viable theory of visual perception and mathematical cognition that the possibility of a relationship between visual perceptual skill and mathematic ability can be determined. Traditional cognitive psychological theories of the twentieth century prove insufficient in accounting for either, which obfuscates an account of any relationship. In substantiation of this claim, Chapter 3 highlights select theories as representative of the tradition, while exposing their respective limitations.

A summary of traditional perceptual theories suggests that the nature of visual perception is poorly understood, yet these theories nonetheless intimate a connection between perception and cognition. Piaget's prominent theory of cognitive development is evaluated in terms of the support it provides for this connection. His hypotheses evidence an appreciation of the fundamental interaction between perception and the construction of logico-mathematical cognition. However, the recent criticism of certain aspects of Piaget's theory (see Section 3.2) mitigates its support of a relation between perception and mathematic ability. In the latter half of the 1900s, disillusionment with behavioural psychology instigated the 'cognitive revolution', which brought the information-processing paradigm to the fore. A brief introduction given here to various information-processing theories pertaining to perception and arithmetic exposes the fundamental inadequacies of the information-processing paradigm.

Classical cognitive theory thus appears inadequate in supporting a valid relation between visual perception and mathematical cognition. Despite this theoretical deficiency, there exists compelling empirical evidence (presented in Section 3.4.1) of a relationship

between visual perceptual skill and mathematic ability. There are, however, conflicting conclusions among the evidence. Furthermore, closer analysis reveals a generally superficial interpretation of the evidence in support of such a relationship. This misinterpretation may be attributed to the limitations of prevailing interpretative frameworks. These generally involve a conventional, formalist mathematical philosophy and classical cognitive theory, which is predicated on a belief in a pre-determined objective world. Cognition within this context demands inner representations, which prove problematic.

Inner symbolic representation, common to many perceptual theories, is particularly evident in information-processing theory and constitutes a problem that is inadequately addressed. The predicament involved in cognitive interpretation of perceptual representations is proposed here as constituting one of the primary obstacles to a perceptual and cognitive theoretical synthesis. Examination of three potential solutions follows. These include connectionism, ecological theory, and embodied cognition. The root of the problem, however, appears to rest in the underlying assumption – evident in classical cognitive psychology – of an objective external world with predetermined properties. This indicates the need for a radical paradigm shift, to ensure further progress.

It is apparent that an innovative comprehensive cognitive science of mathematical cognition and visual perception is indispensable to the elucidation of a possible relationship between visual perceptual skill and mathematic ability. Chapter 4 introduces a current trend in cognitive science, namely *situated cognition*, which shows promise in meeting this challenge. Situated cognition successfully circumvents the dilemma of interpreting representations by eradicating the notion of mind-body, or inner and outer, duality of the world. Specifically, situated cognition occurs *with* the world, as opposed to *within* the world, thereby generating a holistic mind-body-world system. Embodied sensory-motor activity *with the enacted world* thus grounds the regress of meaning. Categorisation and perceptual metaphor promote conceptualisation, which supports abstract cognition. In addition, the human world includes embodied individuals who exist as a society of language-users. Linguistic metaphor thus extends embodied meaning into the social context, supporting higher levels of abstract cognition. The cognitive science theory of situated cognition helps to clarify the natures of visual perception, mathematical cognition, and the connection between them. In doing so, it exposes the viability of an elucidated relationship between visual perceptual skill and mathematic ability.

Analysis of the theory of situated cognition in Chapter 4 begins with a summary of key psychological contributions. Being a relatively new paradigm, clarification of the associated concepts and terminology follows. A review of various interpretations expands on the concepts of autopoiesis, embodiment, societal embedment, the amalgamated mind, and environmental extension. Thereafter, reflection on various germane implications of a situated cognition includes the issues of intelligence, knowledge, concepts, and rationality. The relevant constructs are then revisited from a situated cognitive perspective.

Situated visual perception, being a difficult concept to grasp, is elucidated through the concrete aspect of colour perception. Next, the important roles of context and action in visual perception are highlighted, leading to an appreciation of the particular significance of visuo-spatial perception, which is then more fully discussed. This notion is extended in a consideration of visual perception and temporality. According to situated theory, the primary purpose of visual perception is guided action. In the execution of this function, visual perception gives rise to abstract conceptualisation.

The exploration of the nature of situated mathematics reveals it to be contrary to all prevailing contemporary mathematical philosophies. Mathematics, an abstraction, is a metaphorical conceptualisation by human situated cognition. Situated mathematical cognition, by virtue of the nature of situated mathematics, operates primarily through the emergent cognitive capacity for conceptual metaphor. This is abstract conceptualisation, which arises by means of visual perception. Lakoff and Núñez's (2000) innovative comprehensive theory of mathematical cognition has clarified the crucial sensory-motor grounding of mathematical cognition.

The implications of situated theory for mathematics education are then considered, particularly in terms of situated mathematics and mathematical cognition. Examples are presented showing the potential practical applications of Lakoff and Núñez's (2000) theoretical insights. The implication is that a visual and conceptual focus in mathematics education – as opposed to a factual algorithmic focus – may prove beneficial.

A summation of the diverse theoretical contributions presented provides a firm argument for the validation of the situated cognitive paradigm. Accordingly, situated visual perception and mathematical cognition are co-constituted emergent cognitive capacities, through embodied sensory-motor activity embedded in, and extended into, the enacted

physical and social environment. This fosters the reasonable presumption of a relationship between visual perceptual skill and mathematic ability.

Such theoretical conclusions prompted the design of a small exploratory study to investigate the presumed relationship between visual perceptual skill and mathematic ability. Positive empirical evidence would further validate the situated cognitive paradigm. Specifically, the study tested the hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement, and the relative contribution of a set of visual perceptual component skills was determined. Chapter 5 outlines the research methodology of this study.

Chapter 6 presents a description of the data collected, and the results of the statistical analyses. The conclusion reached is that visual perceptual skill positively correlates with scholastic mathematics achievement. This is in validation of the situated cognitive theoretical prediction of a relationship between visual perceptual skill and mathematic ability. However, contrary to this theory, the visual memory component skill was found to be of greater significance than spatial relations were.

The theoretical significance of the results of this study is discussed in Chapter 7, and the consequences for the didactics of mathematics are considered. In addition, potential shortcomings of the study are reflected upon. Two major issues concerning the manner of visual perceptual skill testing, and the appropriateness of the methodology are discussed. Finally, recommendations for further research are proposed. The chapter concludes with a brief discussion on the merits of the pursuit towards a comprehensive cognitive science, from which the theory of situated cognition has evolved. Situated theory has already contributed significantly to our understanding of cognition in general, and elucidated the natures of visual perception and mathematics. In so doing, it presents a viable framework for addressing the presumed relationship between visual perceptual skill and mathematic ability. Furthermore, the knowledge gained from the efforts of cognitive science has the potential for a positive influence on mathematics education. Continued multi-disciplinary efforts are proposed as a worthwhile endeavour.

CHAPTER 2: Perspectives on Mathematics

The common view is that mathematics is calculation with numbers. However, this is more precisely taken as the definition for arithmetic. Arithmetic may have historically been synonymous with mathematics, but today it is only one important branch. The mathematical domain has significantly expanded throughout history, and its expansion is expected to continue. *The Concise Oxford Dictionary* (2008) defines mathematics as “the branch of science concerned with number, quantity, and space, as abstract concepts” (p. 881). Yet even this definition seems to fall short of the boundaries of the field, with its omission of motion and change.

Devlin (2000), in concordance with most modern mathematicians, has defined mathematics as “the science of patterns” (p. 7). Mathematical understanding of the term *pattern* includes: order, structure, and logical relationships. The more than sixty current branches of mathematics each study a different kind of pattern. Because these patterns are generally abstract, an abstract symbolic notation is required to describe them. These patterns originate from the world or the mind, and so are as varied as to include those that are real or imaginary, perceptual or cognitive, static or dynamic, numerical or qualitative.

The human brain is the ultimate pattern perceiver; in particular, visual perception is the interpretation of patterns of visual sensations. If mathematics is the science of patterns, it is quite conceivable then, that mathematical cognition may recruit the highly evolved human visual system to aid in the analysis of mathematical patterns. This is irrespective of whether they originate from patterns of light falling on the retina, or are alternatively more abstract, since the recruitment of basic neural abilities in the development of more complex facilities is the natural evolutionary process.

The mathematical definition therefore seems satisfactory, although it subtly implies some indecision as to whether mathematics exists in the world, awaiting discovery, or whether it is a human invention. The nature of mathematics, a description of its objects (numbers) and relationships, is an indispensable factor in the progress towards a comprehensive cognitive science of mathematics. Historically, the nature of mathematics was the preserve of philosophy, as was visual perception. Philosophical attitudes and paradigms

have therefore undeniably shaped, and will continue to affect, prevailing psychological and scientific thinking.

2.1 Contemporary Philosophies of Mathematics

Contemporary philosophy of mathematics is divided into three broad belief systems: rational realism, formalism, and constructivism (Ernest, 1991; Davies, 1992; Dehaene, 1997; Lakoff & Núñez, 2000; Sousa, 2008). Each has influentially contributed to the *apparent* eternal, absolute, infallible, and transcendental nature of mathematics.

2.1.1 Rational Realism

Realists believe, as a matter of faith, in the real physical existence of the external world, which is independent of the mind and of perception. This philosophy is epitomised by both Platonism and Cartesian dualism.

Platonism dictates the existence of abstract objects and concepts as real, independent, timeless, objective entities transcendental to human existence. Mathematical objects and the number concept therefore exist on an abstract plane, external to the mind. The function of mathematics is to discover and to observe these entities. Many mathematicians ascribe to this view since it resonates with their subjective experience of doing mathematics. Specifically, it lends credence to the apparent objectivity and autonomy of mathematics. According to this view, the fact that mathematics describes the necessary structure of reality presumably explains its scientific effectiveness. This may certainly justify the static structural aspects of mathematics, such as set theory. However, Ernest (1991) has argued that Platonism inadequately accounts for the important constructive computational side of mathematics, such as iterations and recursive functions.

Descartes established the mathematical principles of optics, and his hypotheses laid the foundations for modern perceptual theory. He regarded mind and matter as separate, divinely linked substances; with reason favoured over the senses as the source of truth and knowledge acquisition. In her critique of Descartes' theory, Wolf-Devine (1993) has contended that it presents an initial clarity and logical argument, yet the causal connections between the rational mind and the mechanical body remain elusive. For nearly 400 years,

philosophers have struggled to reconcile this Cartesian legacy of the mind-body problem. Despite these problems, perhaps due to its intuitive appeal, various forms of dualistic rationalism have dominated psychological theory, with the notable exception of the materialist philosophy of behavioural psychology.

From a contemporary psychological perspective, with its accompanying neurophysiologic evidence, the rational realist position is indefensible. Cartesian dualism is now deemed unacceptable, in terms of the lack of a viable description of the interaction between reason and the physical body. Similarly, it is also unclear as to how the physical Platonic mathematician interacts with the abstract realm of real, but immaterial, mathematical objects (Dehaene, 1997). It is possible that the reality of mathematics to mathematicians is but an illusion, and simply reflects an exceptional capacity for visualisation; an aptitude considered to promote mathematical thinking (discussed further in Section 2.3.5).

2.1.2 Formalism

The formalist attitude towards knowledge acquisition emphasises organisation, consistency, and formal principles. Few psychological theories are formalist, although computational theory and connectionism intimate a degree of formalism.

As a philosophy, formalism is dedicated to mathematics. Euclid (cited in Lakoff & Núñez, 2000), around 300 BC, organised the entire body of geometric knowledge of that time to show that all then-known geometric facts could be logically deduced from five fundamental self-evident facts. This gave rise to the notion, as mathematics expanded, that each sub-discipline was similarly deductive, and rested on a foundation of limited postulates, or axioms, taken as truths. The development of Cartesian analytic geometry in the 1600s further established a link between geometry and algebra (Dehaene, 1997). However, acceptance of the legitimacy of non-Euclidean geometry in the early nineteenth century challenged the grounding of algebraic mathematics in geometry. In the following decades, attention turned to the exploration of the connection between logic and mathematics, culminating in the invention of formal symbolic Boolean logic, which formed the basis of the modern digital computer. Boole (cited in Dehaene, 1997) believed that logic, besides providing the grounding of mathematics, also grounded the 'laws of thought'. Since all

formal systems are equivalent, it was therefore hypothesised that if the Turing machine¹ could successfully process all forms of mathematics, then it may also simulate, and thus expose, all logical thought (symbolic processing is discussed further in Chapter 3). Consequently, formalism rose to prominence, its progress following the rise of the digital computer and its logical foundations.

The attempt to formalise all mathematics and reduce it to logic thus dominated early twentieth century mathematics. Mathematics, as such, is a small compilation of axioms conveyed in symbolically written formal logic, from which all mathematical truths are logically derived. The challenge for formalist mathematicians is to disprove Gödel's incompleteness theorem (cited in Davies, 1992), presented in the 1930s. Gödel showed that such an axiomatic set could never be both complete *and* consistent. In other words, logical proofs are inadequate to the demonstration of *all* mathematical truths. This is a consequence of a paradox associated with self-referential systems in general. The following example illustrates this paradox.

Consider this statement:

“This statement is a lie.”

If the statement were true, then it would be false. Yet, if it were false, then it would be true.

Therefore, mathematical theorems may be logically deduced from the axiomatic set, however, the axioms themselves cannot be mathematically proved true. The only conclusion, therefore, is that mathematics must rest on a base of assumptions. This has the undesirable effect of introducing the fallibility of mathematics.

Formalist mathematics is thus a hypothetico-deductive system, which is able to describe arithmetic by its rules, but for which there is no basis. In stark contrast to realism, formalist mathematical objects bear no relation to reality whatsoever. They are simply a set of entirely abstract symbols that satisfy certain axioms and geometric theorems. Such symbols are intrinsically meaningless; meaning within the system derives from their rules of use. Mathematical statements therefore consist of un-interpreted strings of symbols, which display syntax, but are semantically void. Wittgenstein (cited in Dehaene, 1997) captured the crux of this philosophical standpoint in his pithy comment: “all mathematical propositions

¹ A Turing machine is a hypothetical computational device that manipulates symbols according to formal rules. It mathematically models a machine that can be adapted to simulate the logic of any computer algorithm.

mean the same thing, namely nothing” (p. 243). Mathematics is thus a purely human-invented ‘meaningless’ activity, involving the manipulation of symbols according to precise formal rules. As such, any connection between visual perceptual skill and mathematic ability would be taken as specious, at best. The only possible role of visual perceptual skill in this type of mathematics would be the decoding of the arbitrary symbol strings; a task related more to linguistic ability than to mathematic ability per se.

To date however, the formalist program has not only failed in its own goals, but has also failed to satisfy a number of other crucial concerns. One of these, raised by Dehaene (1997), is as to why the formal rules of arithmetic should be judged as being more fundamental than the rules of other formal systems, such as the game of chess. According to the formalist philosophy, there is no substantial difference between these two formal systems. Yet the common human perception is that the laws of nature operate in accordance with only one of these systems. Essentially, nothing in the formalist philosophy justifies the effectiveness of mathematics in modelling the real world. Neither does it account for innate mental concepts such as number (see Section 2.2.1 for discussion on the number sense).

In addition to the inherent failings of the formalist view of mathematics, it is also deficient in terms of a broader philosophical foundation and in a supporting framework of available psychological theories. In spite of this, it unfortunately seems to have had the most influential impact on the common perception of mathematics, namely as simple formal symbol manipulation divorced from reality. This influence manifests even in the classroom, despite the availability of alternative teaching methodologies based on tested educational psychology theories (cited in Davis & Maher, 1997; Alexander, White & Daugherty, 1997).

2.1.3 Constructivism

Constructivism advocates that there is no such thing as a *knowable* objective reality, in that all knowledge derives from mental constructions. This is not a rejection of physical reality, as in the idealism of Berkeley (2004), which reduces the external to the mental, leading to the contentious mental creation of external reality. Rather, there is acknowledgement of the reality of the external, but this reality is unknown to us, due to our perceptual and conceptual distortions.

Accordingly, as in formalism, mathematics does not exist in the external world. The physical existence of any abstract structure being impossible, since it is only through mental conception that abstraction comes into being. Although, in contrast to the formalists' meaningless inventions, constructivists take mathematical objects to be mental constructions directed by the internal structure and function of the brain, and constrained by external reality.

Intuitionism evolved from constructivism to include particular innate, intuitive concepts independent of experience, such as that of number, as a base for further constructions. Mathematical objects are thus primitive, a priori mental categories. Although Kant, a founding intuitionist, believed these to be synthetic a priori categories. This is because of his radical conception of innateness, whereby knowledge is the product of our conceptualisation of intuitive information from the senses. This is not knowledge construction, as per social constructivism. Rather, truths are a priori in that they are independent of experience; yet synthetic, as they cannot be reduced to logical truths independent of the synthesising activity of the mind. Kant (cited in Dehaene, 1997) thus nevertheless rationalises that, “the ultimate truth of mathematics lies in the possibility that its concepts can be constructed by the human mind” (p. 244). In other words, the effectiveness of mathematics in nature is the result of our own mental projection. Davies (1992) expressed this clearly when saying that “we read mathematical order into nature rather than read it out of nature” (p. 150).

Social constructivism and intuitionism thus found mathematics upon social-linguistic rules and conventions, and innate mental categories respectively. Yet, this replacement of the foundational assumptions of formalist mathematics is not entirely successful in redressing the fallibility of mathematics. However, it may be argued that *together*, cultural and neural determination and constraint is sufficient to render mathematics *contextually* infallible. In essence, as Ernest (1991) explained, this negates the apparent absolute, universal truth of mathematics, restricting it instead to human mind-based and social relativity.

From a classical psychology perspective, amongst the broad philosophies of mathematics discussed, intuitionism is the most acceptable in terms of how that philosophy's claims relate to what is known of the brain. Empirical psychological evidence (cited in Dehaene, 1997) affirms the innate basic number sense, located in specialised neural circuits of the inferior parietal lobe. Hence number, like colour, is a fundamental dimension,

perceived by virtue of brain structure having evolved to reflect, or to represent, the properties of the external world. Brain structure and function, through perception and conception, thus determines our view of objective reality, with any other objectivity being fundamentally *unknowable*. Constructivism, and intuitionism, essentially conveys active construction of all conceptual knowledge, based on perceptual experiences and previous individual and cultural knowledge. Yet a definitive definition of mathematics remains elusive.

Núñez (2008) has characterised mathematics as precise, objective, rigorous, generalisable, stable, and effective in the real world; and none of the three broad philosophies of mathematics can adequately account for all of these characteristics. It is argued that the recent introduction of an alternative philosophy of the nature of mathematics and mathematical cognition, expounded in Section 4.6, provides a more adequate account. It endorses the relevance of social construction and limited innate functional capabilities, while the incorporation of situated cognitive emergence essentially endows the enacted environment with a more pivotal contribution in conceptualisation. This perspective emphasises the creation of mathematics – a cognitive creation together *with* the environment, however – rather than exclusively a social construction *within* the context of the environment. Mathematics is thus a product of specific neural capabilities, positioned within the human body, located within a specific environment, together with the particular biological and social evolution of these factors (Varela, Thompson & Rosch, 1991; Sfard, 1997; Lakoff & Núñez, 2000). Despite a disparity in the intuitionist and situated processes of conceptualisation, there is nevertheless agreement over the conceptual nature of mathematics.

2.2 Mathematical Cognition

In line with contemporary philosophy, cognitive science conceives of mathematics as being concerned with concepts. These concepts, which are largely founded in sensory-motor experience, manifest in mathematical formal proofs with symbolic notation. With this emphasis on the conceptual nature of mathematics, clarity on the social evolution of mathematics and the realisation of mathematic ability demands insight into the underlying cognitive processes. A discussion of contemporary views on mathematical cognition offers deeper insight, by bringing attention to the particular neural structures involved, and elucidating contributory influences.

Although mathematics is not solely calculation, arithmetic is yet an important part. A sense of number is, in turn, an important pre-requisite for arithmetical cognition (Dehaene, 1997; Devlin, 2000; Lakoff & Núñez, 2000; Bobis, 2008; Sousa, 2008).

2.2.1 The Number Sense

A sense of number is in its essence the conception of numerosity, namely an intuition that endows numbers with meaning. Dehaene (1997) has credited Tobias Danzig with coining the term *number sense* in 1954. Danzig defined it as the ability to perceive the change in a small collection after the addition or removal of an item. Devlin (2000) expanded this definition, suggesting two important sub-components:

1. The ability to compare collections presented simultaneously.
2. The ability to remember and compare collections presented successively.

This definition suggests that the number sense is a direct consequence of the sensory processing system. Van Loosbroek and Smitsman (1990), Dehaene (1997), and Giaquinto (2007) have all conceded that the numerosity of a collection is a concrete, invariant property that is extracted and perceived as other properties, such as colour and form.

Piaget's (1963, 1969) influential constructivist ideas asserted that successive maturation of the infant nervous system, together with information gathered through observation of the environment, enabled the construction of understanding, and the conceptualisation of the world, including that of number. According to Piaget's theory (discussed further in Section 3.2), the number sense manifests between 2 to 7 years of age. However, contemporary research (cited in Dehaene, 1997; Devlin, 2000; Lakoff & Núñez, 2000; Cordes & Gelman, 2005; Sousa, 2008; Bobis, 2008) questions the validity of Piaget's developmental timeline. Empirical studies demonstrate a rudimentary number sense in infants as young as 6 months old. The evidence further suggests that it is probably a neonatal attribute. Unfortunately, support for this claim is elusive due to practical constraints. Evolutionary survival depends on the number sense abilities, in order to assess predator threats and food source opportunities. It is thus considered a genetic acquisition within most animal species, and cannot be learned. Devlin (2000) has cited cases of patients with specific brain abnormalities as providing neurophysiologic confirmation of this. He found that these patients could learn number facts; however, the facts have no meaning for them. Wheatley (1997) provides a concise discussion of what such a deficiency in numerical meaning entails.

Debate still exists as to the exact mechanism of number processing. Limited evidence of discrete number-detecting neurons lends little support for Dehaene's (1997) speculative accumulator model of analogical number processing. Yet, regardless of the specific neurophysiology, the evidence (cited in Dehaene, 1997) supports an imprecise or 'fuzzy' representation of number. The innate number sense is consequently severely limited. For larger collections, it merely allows for comparisons, approximations, and estimations. Instant and accurate numerosity perception, known as perceptual subitizing, is apparent only for four or less objects (Atkinson, Campbell & Francis, 1976; Dehaene, 1997; Devlin, 2000; Lakoff & Núñez, 2000; Sousa, 2008). Positron emission tomography (PET) scans, during subitizing, reveal activation solely in the visual cortex, confirming the role of the visual processing system (Sousa, 2008). Specifically, Dehaene (1997) considered subitizing to depend on the visual pathways dedicated to object localisation and tracking. The capacity to subitize, inherent even in non-human animals, supports a nominal degree of arithmetic ability. It enables rapid enumeration of small collections, as well as the addition, subtraction, and comparisons of their numerosities.

The importance of the basic number sense to mathematical development is well-recognised in the literature (Dehaene, 1997; Lakoff & Núñez, 2000; Giaquinto, 2007; Bobis, 2008; Sousa, 2008). Dehaene (1997) suggested that cognitive development and education function to connect this foundational sense to other cognitive structures. An additional cognitive capability, not present in other animals, markedly promotes human number sense expansion. This capability, believed to be a defining human distinction, is the creation and manipulation of symbol systems. Yet, more fundamentally important than symbolic representation, is the underlying mental facility of abstraction.

Abstraction (discussed further in Sections 2.2.6 and 4.3.2) facilitates the generalization of the language-independent number sense across different perceptual modalities, encouraging the conceptual development of number. A study by Sfard (1997), in collaboration with Linchevsky, offers a sound portrayal of the consequences of the incomplete conception of number, which they attribute to inadequate abstraction. Their sample of dyslexic children showed a significant discrepancy between their superior aptitudes for calculation with the relatively concrete concept of money, as opposed to purely abstract number.

2.2.2 Counting

Developmentally, there is initially an apparent disconnection between number concept and counting, and uncertainty still exists as to whether counting is innate or learned. Devlin (2000) reported that anthropologists estimate that our ancestors acquired the capacity for counting between 100 thousand and 200 thousand years ago. However, certain isolated cultures, such as the Australian Aborigines, do not exhibit counting beyond the basic number sense of *one, two, and many*. This would seem to provide evidence against the theory of an innate sense of counting, although on the other hand, may simply reflect a lack of necessity to enumerate accurately within these cultures. Consensus on the matter has yet to be reached.

The anthropologist Sheets-Johnstone (1990) resolved several ambiguities concerning counting development with a lucid account of its tactile-kinaesthetic grounding and cognitive evolution within humans. She maintains that a similar process occurs during infant development, with the added benefit of an already extant cultural number system. It may be due to this benefit that confusion regarding the development of counting originates. Counting is customarily described as a complex process, incorporating two important principles (Sousa, 2008):

1. The one-to-one principle necessitates sequentially and systematically assigning the correct number word to each item counted.
2. The cardinal principle involves the recognition that the last number in the sequence is the total number, an important concept for addition and subtraction.

This description demands an existing verbal number system. However, the account provided by Sheets-Johnstone (1990) reveals that neither numbers, as abstract entities, nor their linguistic counterparts, are essential to primitive counting. It is the basic number *sense* that necessarily supports counting. Extending beyond approximate comparative perceptual subitizing, by one-to-one correspondence of the elements themselves it is possible to ascertain whether two categories have exactly the same quantity of elements, without individual enumeration. This represents a non-standard qualitative concept of number, based on perceptual matching. Sheets-Johnstone has claimed that various non-human animals display comparable pre-literate counting abilities. It is the increasing abstraction of the number concept, and the human symbolic facility, that eventually enables the individual enumeration of elements counted. Individuals, as well as most cultures, appear to learn to count spontaneously, through finger enumeration as the original symbol system. Although

some cultures increased their counting repertoire by including other body parts, modern brain imaging studies (cited in Devlin, 2000) show substantial activity in the left parietal lobe when performing basic arithmetic. This is the neural region responsible for finger motor control.

After mastering concrete object, and symbolic finger counting, further abstraction takes place. The inclusion of the abstract number sense expands the intuitive number sense, permitting mental counting and basic arithmetic. The level of abstraction now endorses a number as an abstract entity, rather than as a simple qualitative description. It thus becomes an object unto itself (Devlin, 2000). Symbolic number representation reflects recognition of this fact.

2.2.3 Symbolic Representation

Archaeologists have discovered notched bones from about 35 thousand years ago, and have interpreted them as number tallies. Tallies, like symbolic finger counting, indicate a basic conception of numerosity. The purely abstract concept of number, essential to mathematics, appears to have developed much later. Around 8 000 BC the Sumerians conveniently replaced their commercial clay envelopes containing tokens with clay tablets engraved with abstract markings to denote the number of tokens. Although there was still no culturally universal number system, the use of these symbolic markings evidenced a step towards a higher level of numerical abstraction (Devlin, 2000).

Symbolic representation is intrinsically important to mathematics (Davis & Maher, 1997). Dehaene (1997) credited this seminal human development to dispelling the fuzzy quantitative representation of numbers. The Arabic number notational system has since proved the most efficient and successful. This creative invention has substantially simplified the practice of mathematics. Arabic numerals, and symbolic algebraic notation developed in Alexandria in 250 AD, enable the expression of mathematical formulae, facilitating the dissemination of mathematical ideas and thus substantially contributing to the expansion of mathematics. Written symbolic notation also surmounts the constraints of mental arithmetic imposed by a limited working memory (Tversky, 2005a; Rowlands, 2010; Clark, 2011).

In spite of this, mathematics is not simply a form of symbol manipulation; it further involves abstract patterns and numerical concepts. Even though it is possible to ‘do’ arithmetic purely through the rules governing the manipulation of symbols, the process will

only make sense if the meaning of the symbols and rules are understood (Presmeg, 1997a; Wheatley, 1997; Alexander et al., 1997; Devlin, 2000; Lakoff & Núñez, 2000; Gallistel & Gelman, 2005). This understanding is achieved precisely through the innate number sense and the proper abstraction of number.

2.2.4 The Mental Number Line

Providentially, unlike language words in general, digits are more than symbols that hold the *possibility* of an attachment of meaning. Mental conception of the symbol occurs in conjunction with the meaning, namely its magnitude, as a unit (Brysbaert, 2005). This unconscious and automatic process results in direct cognitive comprehension of digits as quantities (Dehaene, 1997). Whereas language processing, including number words, is located in Broca's area of the left frontal lobe, the left parietal lobe processes Arabic digits of the number system. Devlin (2000) suggested that this is a result of the abstraction of number from finger enumeration.

Quantity is not the only attribution of numbers. They also seem to elicit a spontaneous spatial association (Fias & Fischer, 2005). On these grounds, cognitive scientists construe the principal conceptual structure for whole, positive integers to be a *mental number line*. Several experiments (cited in Dehaene, 1997; Gallistel & Gelman, 2005) over the past 50 years reveal important effects of number comparisons, which elucidate this conceptual mental representation of number.

- Number comparisons take longer for numbers with similar magnitudes, namely 'the size effect', due to their closeness on the mental number line, which creates confusion.
- Number comparison speed is also slower for larger numbers, known as 'the distance effect', which is thought to stem from an increasing compression along the mental number line.
- Dehaene (1997) reported a spontaneous left-to-right association for small-to-large numbers that he termed the Spatial-Numerical Association of Response Codes (SNARC) effect. Although the SNARC effect serves to support a spatial component in number conception, Dehaene admits that Western cultural factors, related to the direction of reading, largely influence the left-right orientation. Subsequent studies confirm that cultures who read from right to left, display the appropriately opposite

spatial number association. A similar down-up vertical association for small to large numbers was also found to exist (Devlin, 2000).

Based on these effects, the conceived mental number line is taken as a row of points, with a culturally specific orientation in space, along which numbers are sequentially located, following unequal graduations approximating a logarithmic scale. The number concept, through the organisation of the mental number line, thus incorporates a strong spatial component. The spatial sense rests upon spatial relations, a crucial visual perceptual skill enabling the perception of objects in space both in relation to each other and to oneself, incorporating directionality and laterality.

2.2.5 Arithmetic

The number sense, cognitive counting principles, the facility for abstraction and symbolic representation, and the conceptual mental number line, constitute the elementary cognitive tools that support basic arithmetic. All are either innate, or spontaneously develop through the functional architecture of the brain, and are subsequently honed through cultural influence (Ernest, 1991; English, 1997a; Bisanz, Sherman, Rasmussen & Ho, 2005; Giaquinto, 2007).

Already at this level, an intimate relationship between visual perceptual skill and mathematic ability is beginning to unfold. Visual perceptual skill is directly implicated in the number sense, particularly the component skills of visual discrimination, figure-ground and visual memory. Finger counting also relies heavily on visual discrimination, together with visual-motor integration. Abstraction, the facility underlying symbolic representation, is a direct consequence of categorisation (Tversky, 2001). Categorisation relies on the perception of similarities and differences, a complex ability that necessitates visual discrimination in particular, and to a lesser extent figure-ground, visual closure and visual memory. The mental number line conception depends upon the spatial relations visual perceptual component skill. Possessing these basic cognitive tools enables children, with little instruction, to deduce correct solution strategies for simple addition and subtraction (Alexander et al., 1997; Dehaene, 1997).

More advanced arithmetic involving multi-digit calculation, multiplication, and division is often the stage at which problems with mathematics originate. Dehaene (1997) has

rationalised this by the supposition that number processing rests on the foundation of the analogical approximate number sense. Although this drawback may certainly prevent the kind of speed and efficiency of calculation skills of a digital computer, it should not preclude a natural extension to advanced arithmetic. Gurganus (2004), Bobis (2008), and Sousa (2008) have all asserted that number sense development is attainable through appropriate stimulation.

Conventional schooling in the Western world, however, does not seem to encourage this development. On the contrary, there appears to be active suppression of the perceptually based intuitive number sense, through over-emphasis on rote learning. Unfortunately, this occurs regardless of the plausibility of the fact that the innate non-verbal number sense imbues linguistic numerical representation with meaning, and further mediates verbal numerical reasoning (Gallistel & Gelman, 2005). Moreover, rote learning of arithmetic tables necessitates the recruitment of the verbal memory system. Dehaene (1997) presumed that the surfeit of deceptive verbal patterns in the tables confuses the natural pattern-perceiving ability of the brain, an ability that should, by definition, aid mathematical cognition. He deftly illustrates this informally, with a challenge to memorise an address book, such as what follows here:

- Charlie David lives on George Avenue.
- Charlie George lives on Albert Zoe Avenue.
- George Ernie lives on Albert Bruno Avenue.

and a second example, for professional addresses:

- Charlie David works on Albert Bruno Avenue.
- Charlie George works on Bruno Albert Avenue.
- George Ernie works on Charlie Ernie Avenue. (p. 127)

Every child faces this monumental task if taught arithmetic tables by rote learning, as these addresses are in fact merely disguised arithmetic tables. The home and work addresses represent addition and multiplication respectively, with names rather than number words. Their mathematical equivalences, which could alternatively be easily computed (for example through finger counting), are:

- $3 + 4 = 7$
- $3 + 7 = 10$
- $7 + 5 = 12$

- $3 \times 4 = 12$
- $3 \times 7 = 21$

- $7 \times 5 = 35$

The customary school ban on finger counting thus degrades the addition and multiplication tables to mere mathematical facts. As a result, intuitive mathematical *understanding*, or conception, is lost (English, 1997a; Wheatley, 1997; Mayer, 2005). This impedes learning, as understanding enhances associative memory. In addition, a child's interest in mathematics may begin to wane through its disconnection from the real world (Sousa, 2008). These considerations possibly also contribute to the negative perceptions towards mathematics, which further frustrate mathematical learning.

2.2.6 Higher Mathematics

Mathematics beyond arithmetic has very little to do with numerical calculation. Yet, perhaps surprisingly, this level in particular demands a well-developed conception of numbers and their spatial arrangement. Arithmetic rote learning offers no assistance, notwithstanding even greater redundancy since the advent of the electronic calculator. Novel problem solving through analysis, involving concept formation, analogy, metaphor, and inductive and deductive reasoning, now receives emphasis. This is achievable only through a comprehensive awareness of the mathematical relationships and patterns involved. The proper abstraction of number is essential for the perception of these relationships and patterns.

Devlin (2000) has identified four levels of abstraction, or symbolic thought:

1. The lowest level, shared by most animal species, is in essence the absence of abstraction, where objects of thought are restricted to real objects within the perceptual environment.
2. In the second level, the range of thought extends to include real, familiar objects presently beyond the immediate perceptual environment. The apes function comfortably at this level of abstraction.
3. The third level, attainable only by human beings, is a prerequisite for language, where symbolic objects of thought may also include unfamiliar or imaginary versions of real objects.
4. Finally, in the highest level, the range of thought includes even that which has no *direct* correspondence to the real world of objects.

It may be argued that Devlin's first level is an underestimation of non-human animals' capabilities. Alternatively, it simply indicates diverse interpretations of abstraction. All animals must at least be capable of differentiation, for example, to differentiate between that which is *food* and *not food*. Such pre-lingual categorisation, even if restricted to real objects within the perceptual environment, may be considered a demonstration of minimal abstraction (see Section 4.3.2 for a revision of Devlin's levels of abstraction). Nevertheless, it is the fourth level mentioned above that is relevant to higher mathematics. This level is the epitome of mathematical cognition, since mathematical objects are entirely abstract. Núñez (2008) described such abstract concepts as "rich and precise entities that lack any concrete instantiation in the real world" (p. 333).

Through the process of abstraction, abstract entities become objects in their own right (Devlin, 2000). As an instance of an extreme degree of abstraction, Dehaene (1997) has reported that maths geniuses describe the seemingly real existence of numbers as objects populating a virtual, symbolic mathematic landscape. Albeit an extreme example, this nonetheless lends credence to the speculation that generally, with sufficient abstraction, the natural faculties of the brain that assist in navigating the real world may similarly be utilised in the abstracted mathematical world. Visual perceptual skill constitutes one such resource, acknowledged as such since Gibson's theory of direct perception.

Considerable research (cited in Sections 2.2.1 – 2.2.5), particularly neuropsychological, has led to a modest appreciation of foundational mathematical cognition, exposing more fully its relationship to visual perception. Until recently, cognitive science contributed very little beyond this, possibly due to the misconception that mathematics is merely calculation, involving simple formal manipulation of symbols. To date, there is no all-encompassing science of mathematical cognition, nor yet consensus on interdependent mental influences.

2.3 Influences Associated with Mathematical Cognition

Further insight into mathematical cognition calls for an assessment of the contribution of several cognitive capacities and related influences. Evidenced by the content of current research, the influence of language, education, intelligence, working memory, and visuo-spatial abilities are of particular interest to the practice of mathematics.

2.3.1 Language

An inevitable consequence of the view of mathematics as simply a form of arithmetic symbol manipulation, is the impression that mathematics involves merely an alternative use of, or expansion upon, linguistic ability. According to Devlin (2000), symbolic thought enables the development of language from which, together with the basic number sense, mathematic ability arises. Gallistel and Gelman (1992) have similarly hypothesised that, building on innate number sense, the generative structure of number is inferred from linguistic and grammatical properties of verbal counting and quantifying. Gleitman and Papafragou (2005), on the other hand, express their scepticism of the plausibility of such linguistic mediation of number ability. In brief, they highlight the non-recursive nature of English language counting words – particularly up to eleven – along with most natural language quantifiers.

The impression given by views such as those of Devlin (2000), and Gallistel and Gelman (1992), is that numerical ability is considered subordinate to language. Sheets-Johnstone (1990) condemned this view for contributing to the enigma of numerical origins. There can be no doubt, though, that language supports the progressive development of mathematics. It undeniably promotes the consolidation of mathematical ideas, and facilitates their propagation between individuals, across cultures, and over generations. However, this does not necessarily implicate a linguistic cognitive foundation for basic mathematic ability.

An interesting incident in 1913 (cited in Dehaene, 1997) relates to the present discussion, showing that even mathematical discourse is not always an absolute necessity to mathematical cognition and innovation. Professor Hardy, a Royal Society contemporary of Whitehead and Russell, received a letter from Srinavasa Ramanujan Iyengar, a poorly educated Brahmin in India. The letter contained novel mathematical formulae, in addition to accepted mathematical theorems, presented as if they were Ramanujan's own. Many were derived indirectly from obscure mathematics, with which Hardy was nevertheless familiar, since he had personally contributed to them. Later collaboration between the two men revealed that Ramanujan possessed a sophisticated sense of number that enabled him to conceive of innovative arithmetical relations. In his youth, he had effectively reinvented past mathematics, independently. Dehaene (1997) described Ramanujan as a mathematical genius, due to the fact that "he had seen further than any other mathematician without sitting on anybody's shoulders" (p. 145). Language and mathematical formulae may communicate

mathematical ideas, and assist in their evolvment, but rarely reflect the essence of the mathematical cognitive process.

Even so, previously believed to be limited to mere expression of thought, language is increasingly seen as an indispensable component of cognition (Vygotsky, 1962; Ernest, 1991; English, 1997a; Sfard, 1997; Clark, 2011). Gleitman and Papafragou (2005) have cited Wittgenstein to illustrate this position where he states that “the limits of my language are the limits of my world” (p. 633). Ernest’s (1991) construal was that all logical truth, including that of mathematics, is “dependent on the socially accepted linguistic rules of grammar” (p. 31). However, Gleitman and Papafragou (2005) again remained unconvinced of such extremism, due to the relative impoverishment of language, in their view, which necessitates non-linguistic inference in order to achieve a richness comparable to that of thought. Sheets-Johnstone (1990) also argued against cognitive language-dependence, although she does acknowledge a degree of linguistic contribution to higher-level cognition. Her argument is based on evidence of non-verbal modes of cognition, plus primitive pre-verbal cognition. Language development and its cognitive value are contentious issues that have been widely researched. Much of the contention is possibly a result of different interpretations of the definition and constitution of language. Modern theories tend towards the more moderate, with a common critical thread as reflected in Ortony’s (cited in Sfard, 1997) statement that “language, perception, and knowledge are inextricably intertwined” (p. 344).

An increasingly popular view is that mathematics is mostly inductive, comprising imagistic, analogical, and metaphorical reasoning, as opposed to purely logical deductive reasoning (English, 1997a; Wheatley, 1997; Sfard, 1997; Lakoff & Núñez, 2000). This superficially appears to further distance the role of language from the practice of mathematics. Yet, on the contrary, it exposes more clearly the linguistic contribution. Presmeg (1997b) has stressed the role of mental images, constructed from concrete perceptual experiences, during the process of conceptualisation in metaphorical, or abstract, cognition. Taking a broader view of language, she has then gone on to explain that metaphorical conceptualisation is also symbolically mediated, arguing that language is the *only* medium in which it can occur. Although a controversially exclusive argument, linguistic metaphorical conceptualisation, in general, cannot be easily refuted. In a cogent treatise of relevance to this argument, Kruger (2003) articulated that “language is not the communication of pre-existing thought, but is itself the form in which thought becomes incarnate” (p. 44). To elaborate, Núñez (2008) regarded metaphor primarily as a

subconscious human cognitive mechanism, which subsequently manifests via its pervasion into language. Sfard (1997) has insightfully admitted a consequence of this, which simultaneously reinforces the explanation: “Construction of new knowledge”, through metaphorical conceptualisation, is actually the “creation of a new discourse” (p. 358). Metaphorical thinking thus entails a harmony of both images and linguistic sketches, or descriptions. In mathematics, perceptual experiences lead to the formation of mental images, which are indispensable to the creation of new concepts and the concurrent convergence of mathematical discourse.

Both linguistic and mathematical cognition thus present as commonly a result of the primitive brain functions of categorisation and abstraction. Certainly, labelling and verbally defining categories promotes categorisation refinement, augmenting abstraction and leading to higher levels of symbolic thought (Clark, 2011); thereby fortuitously benefiting mathematical cognition. However, on the other hand, pre-lingual categorisation is an evolutionary-derived perceptual skill. Effective perceptual categorisation manifests in symbolic thought, through the facility for abstraction. Symbolic language words and symbolic numerical digits are thereafter processed distinctively, both neurally and cognitively (refer to Section 2.2.4; Barnes, Smith-Chant & Landry, 2005), which is suggestive of them as two distinct capabilities. This view is supported by neurophysiologic evidence of literacy and numeracy double dissociations (e.g. Cipolotti, Butterworth, & Denes, 1991 and Cappelletti, Butterworth, & Kopelman, 2001 (cited in Butterworth, 2005)). Several instances have been observed in brain-damaged patients. Additional cases have been artificially invoked by electrical brain stimulation, causing temporary dysphasia and calculation impairments. A remarkable double dissociation between the use of verbal numbers and Arabic numerals is reported by Anderson, Damasio, and Damasio (1990), Cipolotti (1995), and Mc Closky (1992), (cited in Fayol & Seron, 2005). There have also been reports of double dissociations between linguistic arithmetic factual knowledge and numerical conceptual knowledge (Dehaene, Piazza, Pinel & Cohen, 2005). That dysphasic children show superior performance when working with Arabic digits (Butterworth, 2005) lends further evidence to the conclusion that linguistic and mathematical cognition are indeed distinct capabilities.

Therefore, the totality of mathematics, as a social discipline, may certainly be contingent upon certain social practices, such as language. However, contrary to a hierarchy with mathematical cognitive ability built upon a linguistic foundation, parallel co-development within the individual – originating in the capacity for conceptual metaphor –

seems more probable (Sfard, 1997). Mathematical cognition, like linguistic ability, is thus likely to be founded directly upon perceptual skill (Lakoff & Núñez, 1997; Sfard, 1997; Holyoak, 2005).

2.3.2 Education

Teaching mathematics necessitates more than simply relaying mathematical ideas and formulae. To learn mathematics effectively, the mathematical cognitive process requires appropriate stimulation. This insight has been widely acknowledged at least since the 1950s.

Previously, mathematics teaching in the Western world was in accordance with Humanist and Utilitarian principles. Mathematics education thus involved the transmission of mathematical facts, emphasising rote learning (Ernest, 1991). Evaluation was strictly according to the final answers on assessment tests, with no consideration for cognitive processes. Such a curriculum limits intellectual growth in its neglect of abstract concepts, which are the building blocks of higher-level mathematics (Davis & Maher, 1997).

Influenced by the developmental theories of Piaget and the social constructivism of Vygotsky, Western educators later adopted a more progressive approach. The psychology based on these theories suggests, in simple terms, that mathematic ability develops through environmental experience and social interaction. Consequently, the focus of mathematics education shifted towards the provision of appropriate environmental and social experiences, suitable for promoting active and independent mathematical enquiry by the learner. The teacher's role is primarily to facilitate the learners' investigations, encouraging them to devise and record their own solutions (Ernest, 1991). Assessment is informal and criteria-based, in an attempt to reveal the interplay of knowledge, motivation, and context (Alexander et al., 1997).

Such a progressive educational strategy is, however, in contradiction with the pervasive absolutist view of mathematics. It may be due to the resultant tension that, in practice, mathematics education still tends toward the previous content-driven methodology (see Sections 4.7 and 7.2; Núñez, 2007). This is a disastrous consequence when considering the premise of the progressive philosophy, which predicts the mode and quality of education, and prevailing classroom mathematical discourse, as having a severe impact on mathematic ability.

In addition, the misperception of mathematics as symbol manipulation and as a form of formal deductive logic has eclipsed its problem-solving aspect. The associated apprehension and construction of mathematical patterns and relationships cannot be assessed in terms of memorised facts and algorithms. Problem-solving situations yield a form of assessment that is ultimately more accurate (English, 1997b; Wheatley, 1997). In support of this view, Alexander et al. (1997) have linked the pattern perception and association of basic mathematical reasoning to the performance components of Sternberg's componential theory of analogical reasoning, an aspect of his triarchic theory of intelligence. The crux of Sternberg's (2005) theory is that intelligence is not the accumulation of factual knowledge, but is, rather, related to successful environmental coping, through problem solving.

2.3.3 Intelligence

Pesenti (2005) has suggested that because not all intelligent people are good at mathematics, a dissociation exists, requiring the separate consideration of these two constructs. Yet, there is evidence (Siegler & Booth, 2005) that children's mathematic ability both positively and significantly correlates with IQ measures.

This disparity may be a result of the broad, and manifold, traditional notions of intelligence. Spearman (cited in Sternberg, 2005; Van Eeden & De Beer, 2005) was the first proponent of a single intelligence factor g , but he simultaneously identified diverse s factors, specific to particular activities. Moreover, Cattell and Horn (cited in Sternberg, 2005) later split factor g into fluid and crystallized intelligence, each reflected in myriad mental abilities. Thurstone (cited in Sternberg, 2005; Van Eeden & De Beer, 2005) went on to propose a seven-factor theory of intelligence. Gardner's theory of multiple intelligences (cited in Sternberg, 2005; Van Eeden & De Beer, 2005) originally included seven separate intelligences, and has since expanded to include even more.

Notably, though, mathematic ability features in each of these theories of intelligence, and their accompanying tests, providing psychometric evidence of a connection between mathematics and intelligence, as measured by IQ tests. In fact, mathematic ability, as problem solving, seems to be so entwined with the Western notion of intelligence that it becomes difficult to isolate either without detracting from the original constructs. In addition, many tests of intelligence also contain a visual perceptual component, often referred to as 'nonverbal' IQ. This would seem to lend credibility to the view that IQ is not necessarily a

confounding effect in studies specifically investigating the relationship between mathematic ability and visual perceptual skill. In fact, the recognition that several cognitive resources contribute to measures of intelligence – including that of visuo-spatial abilities (Kyttälä & Lehto, 2008) – necessarily puts into question the validity of controlling for IQ.

The neurobiological hypothesis (cited in Di Blasi, Elia, Buono, Ramakers & Di Nuovo, 2007) also closely links perceptual and intellectual abilities. Visuo-spatial information-processing is accordingly not only a visual perceptual construct, but it also underlies the Performance IQ score. Loikith (1997) consequently stressed the impossibility of separating perception from intelligence. Kulp et al. (2004) similarly reasoned that poor visual discrimination will lead not only to perceptual inefficiency, but will also affect information-processing, thereby influencing visual memory, problem solving, and IQ. Vernon (1970) has concurred that the speed of information-processing, which is partially dependent upon visual perception, forms an integral part of general intelligence.

There is thus a connection between visual perception and intelligence, in addition to that to be found between mathematics and intelligence. The coincidence with intelligence, of both mathematic ability and visual perceptual skill, indicates a potential relationship between the two constructs.

2.3.4 Working Memory

Working memory is central to active cognition. A broad literature survey undertaken by LeFevre, DeStefano, Coleman, and Shanahan (2005) reveals much acclaim for a close connection between mathematics and working memory. However, empirical research is sparse. Mathematics involves a wide range of cognitive abilities, which makes it difficult to assess the extent of influence of working memory. Of the few recent studies available, most (including those of Wilson & Swanson, 2001; Reuhkala, 2001; Holmes, Adams & Hamilton, 2008; Kyttälä & Lehto, 2008) acknowledged the importance of all three components of Baddeley's model of working memory: the phonological loop, the visual-spatial sketchpad, and the central executive. Correlational research, though, emphasises the role of the visual-spatial sketchpad, in its support of mental rotation (LeFevre et al., 2005).

Mental rotation is a crucial ability for the processes of mathematics. It is also an aspect of the visual perceptual component skills of form constancy, figure-ground, and visual

memory. This commonality further supports an association between mathematic ability and visual perceptual skill.

2.3.5 Visuo-Spatial Abilities

Visual perceptual skill underpins the lower-level visuo-spatial abilities, such as object recognition and location. Visual perceptual skill also exerts a supplementary influence on higher-level visuo-spatial cognition, which involves visual or spatial representations, either perceived or imagined, together with the mental operations performed on them (Tversky, 2005a). This is in accordance with the theory (Kosslyn, Thompson & Alpert, 1997) that vision and visual imagery form part of the same system. The difference between the two is that vision involves predominantly bottom-up processing, whereas imagery occurs as a result of input to associative memory, initiating top-down processing. Mental rotation and other image transformations exemplify visuo-spatial cognitive abilities.

The value of internal visualisation or external diagrams in mathematics, and problem solving in general, is common knowledge to most people through their personal experience. It is also widely upheld within the scientific community (English, 1997a; Presmeg, 1997b; Reisberg & Heuer, 2005; Newcombe & Learmonth, 2005). However, contemporary theories reveal that visuo-spatial abilities are more than a mere superfluous aid to mathematical cognition; rather, they are of deeper epistemic importance (Presmeg, 1997a; Sfard, 1997; Wheatley, 1997; Lakoff & Núñez, 2000; Hegarty & Waller, 2005; Giaquinto, 2007). The growing recognition of this fact is a natural concomitance with the realisation that mathematics is not entirely a deductive discipline, the emphasis accordingly shifting towards imagistic, analogical, and metaphorical reasoning. Presmeg's (1997b) definition of a visual image as "a mental construct depicting visual or spatial information" (p. 303) is generic enough to include all abstract patterns, which are the pertinent domain of mathematics.

In his text entitled *Visual Thinking in Mathematics*, Giaquinto (2007) has posited that abstract general truths can be inductively discerned through specific images, and therefore visualisation is a pertinent means of discovery. By way of explanation, he has suggested that the representations in associative memory activated by visualisation are more than simple pictorial images; they are perceptual concepts. The phenomenal, or experienced, image is therefore loaded with extracted and schematized information relating to the concept, thus allowing inference, generalisation, and analogy, which are the bases of abstraction.

According to Giaquinto's (2007) position, the role of visuo-spatial cognition in geometry is obvious, due to its spatial nature. Visual perceptual experience creates category specifications that characterise the perceptual concepts of shapes. Geometric concepts of basic shapes are simply perfect exemplars of these perceptual concepts. Visual imagination, which involves the generalisation of experience in memory, then extends geometric knowledge. This notion of geometry does not necessitate definitions or logical deduction, and is therefore non-analytic, thus it is the antithesis of a conventional view of mathematics.

Visuo-spatial cognition also plays a significant and diverse role in arithmetic. The simple appreciation of a multi-digit numeral relies on several visuo-spatial abilities. These include perception of shape and orientation for recognition of the constituent digits, their horizontal alignment, and left-right ordering. Besides this accepted peripheral function in comprehending numerals, Dehaene (1997) and Giaquinto (2007) have further proposed that arithmetical thinking is essentially visual. In the standard addition algorithm, each step except one, involves visuo-spatial ability. Likewise, the rule-governed rearrangements, substitutions, additions, and deletions of symbols typical of algebra are spatial manipulations, which further depend on the spatial arrangement of the input symbol array. In addition, although numbers, unlike geometric figures, are not overtly spatial, the mental number line indicates a natural association with the visual imagery system and with spatial sense. Neuro-imaging corroborates the use of visual imagery in multi-digit calculations, in bilateral inferior temporal activation of the ventral visual pathway. Visuo-spatial abilities are therefore crucial in number perception, counting, multi-digit number representation, finger addition, algorithms for calculation, as well as for digital and analogue arithmetic problem solving techniques.

Furthermore, an entire branch of higher mathematics is, in part, a consequence of visualisation. Calculus arose through the inability of geometry or arithmetic to solve certain physical quantitative problems, many of which were nonetheless easy to visualise. The challenge in calculus, according to Giaquinto (2007), lies in the weak intrinsic connection between basic analytical and perceptual concepts, as compared to that found in geometry. Assumptions are therefore required to strengthen the link. Giaquinto concludes that analytic concepts thereby not only provide formal solutions for visualisable problems, but also extend the domain beyond these problems, into the realm of the un-visualisable. This is evidenced by differential functions whose curves cannot be visualised.

Visualisation can therefore be seen to have instigated the conception of calculus, which subsequently surpasses visuo-spatial cognition in its reach and methodology. Giaquinto (2007) regarded this complex and tentative connection between the analytical and perceptual as paradoxical. However, the supposition of a paradox might reflect a general continuation of the misconception of mathematics and mathematical cognition as formal symbol manipulation. The apparent paradox arises from the difficulty Giaquinto faces in explaining the link between analytical and perceptual concepts. He resorts to relying on assumptions to strengthen the link. The result is inadvertently a reversion to formalist mathematics (refer to Section 2.1.2). Yet, discussion on foundational mathematical concepts explores their development from an experiential visual perceptual base (refer to Section 2.2), and the constructivist and situated philosophies extend this perceptual foundation to higher mathematics, implicating the conceptual nature of mathematics (refer to Section 2.1.3). It is clear from the above discussion, however, that higher mathematics is still commonly considered as algorithmic manipulation of symbols founded on formalist assumptions.

2.4 Conclusion

When assessing the discourse on this subject, there appears to be conflict between the constructivist belief in the conceptual nature of mathematics and a lingering view of mathematics as a logical analytic discipline. This is possibly partly a consequence of the influence of traditional theories of cognition in general, as well as those of visual perception in particular. A dominant view is that cognitive processing of perceptual input produces an output. In terms of mathematical cognition, this view cultivates a formalist impression of mathematics. Perception of the visual symbol array provides the input for algorithmic cognitive processing, which produces the output symbol array. This model, however, undervalues the functions of perception and visualisation, as has been discussed. It is said (reference may be made to Section 2.2) that cognitive science conceives of mathematics as concerning concepts, which arise through sensory-motor experience. If this is the case, then a detailed and precise theory of sensory perception is required in order to comprehend the subsequent cognitive conceptual process. Unfortunately, this manner of theory seems to be lacking within classical cognitive psychology.

The discourses of contemporary mathematical cognition have therefore not as yet been entirely successful in clarifying what lies behind mathematics and mathematic ability.

Mathematical cognition is not an isolated capacity, but simply an instance of cognition in general. In any cognitive model, sensory perception is an indispensable component. It is thus apparent that the question of a relationship between visual perceptual skill and mathematic ability can only be addressed within the framework of a coherent theory, encompassing both visual perception and cognition. The following chapter presents an argument that traditional cognitive psychology theories provide an insufficient perceptual-cognitive framework for a comprehensive theory of mathematical cognition. However, their inadequacies direct the path towards such a theory. To this end, the traditional view of perception and its relation to cognition is discussed in Chapter 3, while Chapter 4 explores an alternative view, with the aim of clarifying the apparent link between visual perception and mathematical cognition.

CHAPTER 3: Perception and Cognition

The constructivist philosophy of mathematics, in conjunction with neuropsychological evidence on foundational numerical cognition, provides the prospect of a link between visual perception and mathematics. Yet controversy continues, perhaps due not only to previous misconceptions of mathematics, but also of visual perception itself, and of cognition in general.

The discussion of various perspectives on mathematics conducted in Chapter 2 reveals that there is lack of agreement regarding the fundamental constitution of mathematics. Likewise, a summary of perceptual theories in Section 3.1 reveals that visual perception was not well comprehended by the discourses of the twentieth century, and it is still not fully understood. The outcome of the discussion in Chapter 2 is that a coherent psychological theory of perception and cognition is paramount to the confirmation of a relationship between visual perceptual skill and mathematic ability. However, only relatively recently has a sound holistic theory (to be discussed in Chapter 4) found introduction. Previous cognitive theories thus provide inadequate explanation for the possible existence of a relationship. In support of this claim, the contributions of two dominant cognitive theories of the twentieth century are evaluated within this chapter:

- *Piaget's theory of cognitive development.* Piaget (1963, 1969) advanced the broader relation between perception and logico-mathematical cognition. This is congruent with a proposed relationship between visual perceptual skill and mathematic ability. However, certain crucial failings within the theory have hindered further exploration of the proposed perceptual-cognitive relations.
- *Information-processing theory.* Although support for information-processing theory is waning, it remains prominent. The paradigm offers details of the separate theories of perception and cognition. However, it provides insufficiently sound explanation of perceptual-cognitive relations, and presents with inherent inadequacies.

Existing empirical evidence (presented in Section 3.4) compounds the problem of determining a definitive relationship between visual perceptual skill and mathematic ability. The body of evidence yields conflicting results, and its interpretation is often superficial. The inherent failings of the prevailing perceptual and cognitive theories have possibly obscured a

more reasonable account of the relations between visual perception, cognition, and mathematic ability. The particular weakness of perceptual representation is deemed especially obstructive to a holistic theory.

3.1 Visual Perception

There have been many psychological theories of visual perception, each of which has made important contributions to the development of our understanding. Still, no theory has been able to fully capture the complexity of the process of visual perception. Consequently, a definitive operational definition of the skills involved is not yet possible.

The Gestalt theory of the early twentieth century has proven to be one of the most compelling and enduring models by means of which we may explain visual perception. Martin (2006) went so far as to have asserted that it provides the foundation for all current tests of visual perceptual skill. The major premise of the theory (cited in Matlin, 2002) is that “the whole is greater than the sum of its parts” (p. 6). The mind organises visual patterns according to laws of perceptual organisation, to generate holistic perception. The laws of *pragnanz*, similarity, good continuation, proximity, common fate, and familiarity, originate from knowledge gained through experience in the environment (Bruce, Green & Georgeson, 2003). These valuable insights into perceptual organisation have not diminished, despite the lack of support for the theory in terms of what is known of the physiology of the brain.

Behavioural psychology, prevalent in the early 1900s, is a notable exception that contributed little to a theory of visual perception. Behaviourism’s primary concern is objectively observable stimulus-response reactions, which mostly involve associative learning. All mental concepts, such as sensation and perception, are therefore ignored.

Subsequent disillusionment with behaviourism facilitated demise of the view of perception as simple stimulus-response relationships. Constructivism has since played a central role in the psychology of perception. With the exception of Gibson’s theory of direct perception, other diverse theories advocate mental construction of perceptual experience. Construction typically employs incomplete data from the senses, together with memories, which are themselves constructions of the past. This constitutes indirect perception, where cognition or cognitive algorithms construct meaning from an imperfect retinal image.

Neuroscience, aided by developments in imaging technology, has considerably clarified many of the basic processes involved in perception (Bruce et al., 2003; Enns, 2004). However, it is becoming obvious that physiology and cognitive architecture alone are insufficient as a full explanation of visual perception, since a sensation-based approach is unable to explain a sense of its ability to capture spatial and temporal features. Even at the lowest perceptual level, for a distorted two-dimensional retinal image to reveal a meaningful three-dimensional world, necessitates additional information.

It has also become apparent that there is more to perception than just visual awareness. Object recognition is indeed a complex perceptual problem. It has received much attention in theoretical research, as well as in practical applications, such as robotics. This focus, however, has perhaps been a factor in the neglect of more crucial aspects of perception. Examination of non-primate animal species reflects an important alternative function of perception. Bruce et al. (2003) have provided a detailed exposition of an insect's visual system, the primary purpose of which is to provide swift and accurate navigation, to the total exclusion of object recognition. It is possible that object awareness may be only a peripheral benefit of a highly developed visual perceptual system, with guided action as the core function. Guided action is, on balance, an obvious evolutionary advantage. Perceptually guided action, in concert with perceptual awareness, may possibly promote knowledge acquisition to support higher-level cognition.

The majority of the diverse array of perceptual theories concur that visual perception entails selecting, interpreting, and organising visual sensations into meaningful experiences. Moreover, in general, it is currently accepted across them that cognition and knowledge of the world obtained through experience, influences, and are influenced by, visual perception (Matlin, 2002). It is therefore possible to postulate that there are deep connections between vision and cognition. Perceptual experience of the world, particularly through vision, guides our thoughts, ideas, and concepts (Piaget, 1969; Sfard, 1997; Lakoff & Núñez, 2000). Conversely, cognition often entails visual imagery and imagination (English, 1997a; Presmeg, 1997b; Lakoff & Núñez, 2000; Reisberg & Heuer, 2005; Newcombe & Learmonth, 2005). In fact, cognition and visual perception utilise many of the same neural centres and pathways, albeit in a different manner (Enns, 2004).

3.2 Piaget's Theory of Cognitive Development

Piaget (1963, 1969) in particular, has written of the fundamental interaction between perception and cognition. He viewed intelligence as effectively logico-mathematical thinking, and firmly advocated its grounding in sensory-motor activity. His constructivist developmental model presents learning as the result of a dynamic interaction between the child and the environment. Existing knowledge modifies perceptual input for *assimilation*, which thereby modifies existing knowledge in order to *accommodate* the perceptual input. To initiate the process, Piaget acceded only to phylogenetic general regulatory mechanisms of processing environmental inputs, and autonomic reflexes. Perceptual and intellectual development thus progress in parallel, influenced by a combination of neural maturation and experience.

Piaget identified a universal sequence of four logical developmental stages, each associated with a specific average age range:

1. Sensori-motor period (0 – 2 years): The only world known is in terms of perceptions and actions. The major culminating cognitive development is object permanence.
2. Pre-operational period (2 – 7 years): A limited capacity for symbolic thought expands the known world beyond the immediate perceptual environment. Reasoning is from the particular to the particular, which Piaget describes as transductive.
3. Concrete operational period (7 – 11 years): Imagery and cognition are more developed but still constrained to concrete reality.
4. Formal operational period (adolescence): Abstract and logical thought is now possible through inductive, deductive, and inter-propositional reasoning.

In a detailed discussion of this sequence, Loikith (1997) highlighted how it clearly portrays the close alignment between cognitive and perceptual development. This further illustrates the impossibility of separating perception from intelligence (refer to Section 2.3.3). The grounding of logico-mathematical cognition in sensory-motor actions, leading ultimately to abstract mental operations, provides a basis for a sound argument for the relationship between visual perceptual skill and mathematic ability.

Piaget's cognitive developmental theory has been the most influential theory of development of the twentieth century. Its enduring prominence in psychology literature and its lingering legacy in education, in spite of current recognition that it is flawed, is testament to its final importance. Unresolved flaws, however, frustrate its overall claim to legitimacy,

precluding its efficacy here in reasoning for a relationship between perception and mathematical cognition.

Within the general censure, the main criticism directed at the theory is that the average age ranges of each stage vastly underestimate developing cognitive ability (Ernest, 1991; Meyer & Van Ede, 1998; Halford, 2005). For example, recent research suggests the likelihood that the basic number sense is a neonatal attribute (refer to Section 2.2.1), a concept that, according to cognitive developmental theory, only develops between the ages of 2 to 7 years. Criticism also extends to Piaget's notion of intelligence as logico-mathematical cognition. Sheets-Johnstone (1990) has maintained that this emphasis on deductive cognition is largely a result of Western cultural influences. She argues that Piaget's limited focus neglects the remarkably human feature of creative intelligence.

A flawed theory, however, does not necessarily imply that it is fundamentally incorrect. Many researchers postulate that the miscalculation of the age ranges of the developmental stages stems simply from a misunderstanding of the instructions given during experimental tasks. Piaget's core theoretical reasoning may have been sound, with his theory's main limitations arising from his traditional and largely Western notions of cognition as the rational inner processing of an independent external world. It is natural, however, for scientific and psychological theories to evolve. The incorporation of new evidence and insights serve to gradually modify, extend, and refine existing theories, as evidenced by Vygotsky's social theory of mind and Von Glasersfeld's (1995) radical constructivism, which demonstrate such an evolution, where both developed from a foundation of Piagetian epistemological assumptions and ideas, with the rejection of many of the problematic aspects.

Piaget's theory contains the seeds of embodied cognition (discussed in Chapter 4). To this, Von Glasersfeld and Vygotsky added the tentative ideas of cognition as embedded in the environment, and in a society. The natural course of further evolution seems likely to have involved the progression to a theory of fully situated cognition (explored in Chapter 4). However, the severe, and perhaps unduly harsh, criticism of Piaget's hypotheses retarded this process. His revolutionary psychological insights into the holistic functioning of a person as a unified whole within the context of the environment, were unfortunately overlooked. Over the subsequent half-century, the majority of research interest was thereby diverted, focusing attention on the actual processing of information between stimulus and response.

3.3 Information-Processing Theory

Reaction to the Behavioural school of thought, in concert with the advent of the digital computer, led to the so-called *cognitive revolution* in the 1960s. Computational theory then dominated cognitive psychology for the next fifty years, and remains prominent. Portrayal of the mind as an information-processor accentuates notions such as sensory coding, storage, and retrieval of information. Empirically based information-processing theories, using computer model simulation, have made important contributions to the knowledge of both cognition and perception.

3.3.1 Visual Perceptual Information-Processing

Within this paradigm, the author is of the opinion that the most fully developed and eminent perceptual theory is that of Marr (1982). His computational theory is constructivist, but only in terms of visual perception, with the omission of any reference to knowledge construction from sensory perception.

Marr's (1982) central concern was the transformation of retinal light patterns into an awareness of the visible world. He describes both image and awareness as representations of the world. An image, or sensory code, is the representation of the external spatial array of various light intensities and hues. Awareness is a symbolic representation of the positions, motions, and identities of objects, algorithmically constructed from the image representation. According to Marr, the serial modularity of visual processing explicates this construction. Each module begins with one representation and converts it into another. The primal sketch, representing the two-dimensional pattern of light intensities on the retina, is successively transformed through a two and a half-dimensional sketch, incorporating distances and orientation, to a final three-dimensional model representation. The three-dimensional representation is compared with representations held in memory for object identification and awareness.

The theory is primarily data-driven, aimed at expressing bottom-up algorithmic processing. This unidirectional flow of information negates any role for existing knowledge in interpreting the visual input. Yet Marr (1982) did not strictly oppose top-down processing. He acknowledged the potential contribution of a knowledge base, and accepted the role of

general properties, aspects of the natural world that constrain possible variants of the retinal image. These properties are not *logically* true, but are frequently experienced to be true. An example is that *object surfaces have regular textures*. The visual system embodies the knowledge of these processing clues, thus at least allowing for the role of unconscious inference in visual perception.

The majority of early computational theories emphasise data-driven bottom-up processing, by placing what I argue to be too much importance on the visual stimulus. Treisman's well-known feature integration theory of object recognition (cited in Goldstein, 2005) reflects an attempt to redress this imbalance. According to this theory, perception involves two stages. During the pre-attentive stage, an object is automatically analysed into its features. The focused attention stage then combines these features, and perception occurs as a result. Although this stage does not strictly entail top-down processing – since there is no need for knowledge – meaning and expectations can control attention in some situations.

In general, information-processing theories of perception involve the construction and manipulation of abstract symbolic representations. The nominal function of perceptual awareness is the natural accompaniment to these processes. This consequently marginalises guided action and knowledge acquisition as functions of perception.

3.3.2 Arithmetic Information-Processing

Modularity, similar to that presented in Marr's perceptual theory, is also presumed to exist in the information-processing cognitive architecture believed to support basic arithmetic encoding, calculation, and answer production. In their review of recent research, Campbell and Epp (2005) have detailed three important opposing models:

- *The abstract code model.* Proposed by McClosky (cited in Campbell & Epp, 2005) in collaboration with several colleagues, this model is modular in terms of numerical function. It identifies separate comprehension, calculation, and response production modules, and purports that these sub-systems connect by means of their use of a common abstract code. The comprehension module encodes numerical input, irrespective of whether these are digits or number words, into the abstract code. This code acts as input to the calculation module, which includes memory for number rules and number facts in the same abstract code. The output of the calculation module is input to the production module, which converts the

abstract code back into digits or number words, as required. The encoding and calculation processes are independent, and thus, additive.

- *The triple code model.* Dehaene (cited in Campbell & Epp, 2005) alternatively proposes a model that is modular with respect to representational code. The transformation of input into one of three numerical processing codes depends on the number-processing task. The analogue-magnitude code supports estimation, number comparisons, and possibly subitizing. The processing of digital operations occurs in visual, Arabic digit code. In contrast to the abstract code model, memory of number facts is language-based, utilising auditory-verbal code. Despite the variously encoded inputs however, the encoding and calculation processes are still independent and additive, as in the abstract code model.

- *The encoding-complex hypothesis.* Campbell and Clark (cited in Campbell & Epp, 2005) reject adopting such simple additive cognitive architectures for the explanation of number processing. They assert that numerals automatically activate an extensive network of associations, including both task-relevant and irrelevant information. This is a result of the wide variety of numerical functions, such as number transcoding, comparison, estimation, arithmetic fact recall, and so forth. Similar to the triple code model, input activates one or more visual, visuo-spatial, verbal, or motoric representational codes. The complexity of simultaneous activation of numerous associations and representations dictates interactive – rather than additive – communication between the modular systems. Number processing skill develops as the interactive module connections become integrated and consolidated.

From their study into numerical cognition in Chinese-English bilinguals, together with a literature review, Campbell and Epp (2004) concluded that the evidence favours an explanation that presumes some role for linguistic code in arithmetic fact memory. In addition, because presentation format affects calculation, the encoding and calculation processes are likely to be interactive, rather than independent and additive. Therefore, the encoding-complex hypothesis, possibly in combination with Dehaene's triple code model, presents the most probable scenario.

True to the information-processing paradigm, all three of these models focus on coding, storage, and retrieval of information from a memory store. In doing so, there is a concurrent disregard for arithmetic meaning or understanding. Dehaene's singular focus on Arabic digits to the exclusion of other numerical symbol systems, compounds the problem. The inference is that arithmetical cognition simply involves a finite set of rote-learned facts

and rules. Yet this is not the case. A crucial aspect is the understanding of meaning denoted by the symbols, facts, and rules (refer to Sections 2.2.1 and 2.2.3). A degree of numerical meaning through the number sense in fact supports some basic arithmetic prior to instruction or formal learning, and therefore in the absence of any stored facts or rules. Mathematical cognition, including that of arithmetic, remains unexplained within this framework.

3.3.3 A Critique of the Information-Processing Paradigm

Information-processing theories seem to entail an inadequate account of visual perception, and ultimately fail to explain why mathematics is meaningful. It may be due to these failings that a primary relationship between visual perceptual skill and mathematic ability is obscured, within the information-processing paradigm.

Among the most common criticisms of this paradigm is the limited focus of information-processing theory (Meyer & Van Ede, 1998). Primary interest is given by this theory to the way in which the cognitive system processes symbolic information derived from a pre-given environment, with no concern for influencing factors. The paradigm acknowledges, by virtue of brain structure, the basic, innate capacities required to process information. The role of the objective environment is simply the provision of an input to the cognitive system. What is lacking is an explanation of the affective role of environmental, genetic, personal, and socio-cultural factors in the development and application of cognition. In such a case, every human would think the same. The solid computer analogy thus creates the impression of passivity. This however contradicts the alleged constructivist philosophy of the information-processing paradigm. In the absence of active initiation and construction of cognitive procedures and strategies, the possibility of a genuine relationship between visual perceptual skill and mathematic ability is groundless, and would be unwarranted. Yet, despite being unwarranted, the possibility of a relationship is frequently raised.

It is argued here that theories within this paradigm prove themselves to be too reductionist. Even in the early years of computational research, Minsky (1997) voiced a similar opinion. The cognitive system is analysed as an isolated system, within which specific cognitive tasks are isolated, progressive modularisation of which promotes further reductionism. Fodor's (1983) entire thesis of cognition actually rests on the principle of modularity. Ironically, though, even he later admitted (Fodor, 2000) that "we do not know how the cognitive mind works; all we know anything much about is modules" (p. 78). A

motor vehicle provides an illuminating analogy. The basis of this transport system is that fuel input enables travel from location A to B. The processes between this input and output are concealed to an observer. Alone, detailed knowledge of the gear mechanism, or complete understanding of the functioning of the pistons, proves insufficient in explaining the resultant output. In reality, even a mechanic's knowledge of the operation of the entire engine still fails to validate the output; since the vehicle will not travel anywhere without a cognitive consciousness navigating the environment, and dynamically co-ordinating the inner workings along the chosen path. Davies (1992) astutely recognised that scientific progress requires both the reductionist and holistic approaches to an explanation, in parallel, however, rather than in opposition. He suggests that, ideally, holism should subordinate reductionism, while it in turn complements the holism. In other words, any scientific model should begin with consideration of the system as a whole. Only then, can its parts be described and analysed, *within* the context of the operation of the system in its entirety.

Information-processing theories may aid in grasping likely processes occurring between information input and output, beyond those which physiology might be able to explain. However, as a comprehensive theory of perception or cognition, they dismally fail. Results in the field of artificial intelligence reflect this position. Although computer models successfully simulate particular limited cognitive aspects, there appears to be no clear direction in enabling simulations to capture the full range of complex, flexible, intuitive, and creative human cognition. Minsky (1997) maintained that all attempts, since Aristotelian times, to represent common-sense reasoning by a logistic formal system, fail not by fault of the formal system itself, but by the characteristics of logic, which are not flexible enough. Bruce et al. (2003) have argued that Marr recognised this inadequacy when he proffered his computational theory only as the first level in perception, and explained the necessity for physiological and psychophysical input for the actual implementation of the proposed algorithms; thereby unifying psychological and physiological explanations of visual perception.

Jackendoff (1987) bluntly commented that "it is completely unclear to me how computations, no matter how complex or abstract, can add up to an experience" (p. 18). However, his intention was not to renounce computational theory. Rather, building from the foundation of Marr's (1982) theory, he attempted to establish a link between the three-dimensional model and conceptual categorisation. His aim was to extend the scope from simple objects to the full spatial landscape of the perceived world that is essential for

navigation. This three-dimensional representation abounds in non-visual elements, such as the relations among objects, the co-ordinate axis system, and motion trajectories. The problem then is how a theory of perception can tolerate abstract *visual* information that cannot be *seen*. Part of Jackendoff's solution is the recognition of the perceptual or experiential *phenomenological* mind, in addition to the *computational* reasoning mind. However, Roy, Petitot, Pachoud, and Varela (1999) argued that this appends a mind-mind problem to the existing mind-brain problem. In other words, it exacerbates the Cartesian mind-body problem, and ultimately further dissociates visual perception and mathematical cognition.

Criticism may even extend deeper than mere inadequacy. Due, possibly, to the tenacious reliance on the computer analogy, information-processing theories are embedded with inferences fundamentally incompatible with contemporary cognitive psychology. Computers are formal systems, which introduces the formalist assumption that symbol manipulation suffices to explain all cognitive processes. However, to presume the simple manipulation of symbolic neural representations is to neglect the *content* of the perceptual and cognitive information processed by the brain. The crucial omission is the association of meaning to the symbolic representations. For instance, whereas computers manipulate meaningless numerical tokens, research previously discussed (reference can be made to Section 2.2.4) shows that humans cognitively manipulate *numerals*, digits concatenated with their meaning. The information-processing paradigm thus endorses only a superficial relation between visual perceptual skill and mathematic ability, involving digit recognition and decoding, leading to a rule-based mathematics. Furthermore, Haugeland (1997) has provided the reminder that the notion of formal systems emerged from mathematics (refer to Section 2.1.2), inspired by arithmetic and algebra. It seems incongruous then to describe mathematical cognition in terms of mathematically-based formal systems and algorithms.

The information-processing paradigm thus displays a number of inadequacies. In brief, however, the supposed role of the perceptual process is the transformation of data from the senses into inner representations. Cognition involves the algorithmic processing of these inner representations. Yet, this conception fails to explain the semantic content of either perception or cognition. The purported role of the environment as simply a form of input to the senses, ultimately disregards meaningful engagement with the world.

3.4 Visual Perceptual Skill and Mathematic Ability

Incongruously, despite the lack of consensus regarding the intricacies of the pertinent constructs, and all cognitive processes in general, it is commonly maintained that visual perceptual skill forms the foundation for all learning, and for mathematics and reading in particular (Piaget, 1970; Halliwell & Solan, 1972; Rosner, 1982; Leonard, 1986; Wurzburg, 1994, English, 1997a).

Rosner (1982) has upheld that vision actually accounts for 80% of all learning. This may, however, be construed as a reflection of bias within contemporary text-oriented societies. In contrast, Sheets-Johnstone (1990) has advocated the primacy of tactile-kinaesthetic perception, based on experimental data showing the progression from tactility to vision in the course of human sensory development. This is in accord with Berkeley's (2004) assertion that tactile knowledge informs vision. Price (cited in Sheets-Johnstone, 1990) however, without denying the evolutionary and developmental value of tactility, has accredited the importance of visual learning to the rise of modern science. He believes that without vision there could be no science. Swartwout (1974) and Enns (2004) speculated that the relative importance of the visual sense can be explained, possibly, by the fact that it processes the most remote stimuli of all the senses, expanding the range of our perceptual domain. Furthermore, mathematics, like reading, involves decoding visual symbolic information.

Speculative views such as these, however, are inconsequential in the absence of a viable perceptual-cognitive interpretative framework. Debate as to the definitive relationship between visual perceptual skill and mathematic ability therefore continues, despite empirical evidence in support of its existence.

3.4.1 Empirical Evidence of a Relationship

Kavale (1982) meanwhile maintained that controversy continues due firstly to the difficulties in defining visual perception and its component abilities, as well as to a scarcity of longitudinal studies and methodological inconsistencies in current research. Many researchers (Larsen & Hammill, 1975; Kavale, 1982; Kulp et al., 2004) have concurred that these inconsistencies include controlling for the effects of age, cognitive ability, verbal intelligence,

and IQ. The confounding effect of IQ has, in particular, been vociferously argued. This is perhaps due to debate concerning the notion of IQ itself, and the practical value of IQ tests. The common contention is that children who are more intelligent do better on tests in general, resulting in a perceived significant relationship between visual perceptual skill and mathematics achievement. However, in a number of studies which controlled for IQ, the relationship was maintained (Kavale, 1982; Kulp, 1999; Kulp et al., 2004). Moreover, Section 2.3.3 discloses a theoretical basis for denying the confounding effect of IQ in the relationship between visual perceptual skill and mathematic ability.

A plethora of research attests to some degree of relationship. A number of studies that support a positive correlation have focused on a particular aspect of visual perceptual skill. Guay and McDaniel (1977) found that high mathematics achievers scored significantly higher than low mathematics achievers on four tests of visual spatial abilities. Kulp (1999) concluded that visual-motor integration skill significantly relates to academic performance in 7 to 9 year olds. In a later study, Sortor and Kulp (2003) confirmed that the Beery Developmental Test of Visual-Motor Integration (VMI) standard score significantly correlated with the Stanford total math score, as did the visual perception and motor coordination standard scores. They further concluded that children with poor mathematics or reading achievement ought to have their visual perceptual ability tested.

Some researchers assert the singular importance of visual memory to academic achievement. Kulp, Edwards, and Mitchell (2002) administered the visual memory subtest of the Test of Visual Perceptual Skills (TVPS) and the Stanford Achievement Test, to children across Grades 2 to 4. The results showed poor visual memory ability, significantly related to below-average reading decoding, mathematics, and overall academic under-achievement. Another study, by Kulp et al. (2004), with children across Grades 2 to 6, corroborated these results. In this study, the authors used a new test of visual memory, in addition to the VMI Supplemental Developmental Test of Visual Perception, to assess visual perceptual skill. Initial analysis of the results revealed that poor visual perceptual skill is significantly related to poor mathematic ability. However, logistic regressions, that included both visual discrimination and visual memory, yielded a significant relation for only the new test of visual memory.

Absolute denial of a correlation between visual perceptual skill and academic achievement is rare. According to Kulp et al. (2004), where authors are sceptical of a direct

relation to mathematics achievement, they at least accept a relation to reading ability, which would indirectly affect mathematics performance. An exception is a research review conducted by the authors Larsen and Hammill (1975). They reviewed 60 studies that explored the relationship of visual discrimination, spatial relations, memory, and auditory-visual integration to school learning in primary grades. Although the majority of these had found significant correlations, Larsen and Hammill claimed that the *combined* results suggested that “measured visual-perceptual skills are not sufficiently related to academic achievement to be particularly useful” (p. 287). They admitted, however, that this was not in agreement with other investigations that had tested the significance of difference on various measures of visual perceptual ability between that of achieving and underachieving children.

The evidence thus suggests the possibility of a relationship between visual perceptual skill and mathematic ability, although its precise nature remains contentious. Besides conflicting results among empirical studies, the prevailing interpretative framework may further confound clarification of this relationship.

3.4.2 Interpretation of the Relationship

Typically, the suspected importance of visual perceptual skill for mathematics is limited to little more than symbolic decoding of written mathematical problems. This is possibly a consequence of a prevailing formalist mathematical philosophy and information-processing psychology theory. Evidence of this interpretation is to be found within the popular media (Lambert, 2007) as well as within the scientific community, as detailed by Kavale (1982):

- **Visual discrimination** enables classification of geometric shapes, differentiation between acute and obtuse angles, and discrimination between mathematical symbols and numerals (e.g. 6 & 9; < & >; + & x; - & /).
- **Figure-ground** prevents skipping lines ensuring correct copying and reading of number work. It also contributes to understanding detailed diagrams, such as geometry problems.
- **Visual closure** aids in understanding fractions, and reading mathematics problems in full or decomposing them into manageable components.

- **Spatial relations** contribute to correct sequencing of numerals (e.g. 204 & 240), and the aligning of vertical sums. It also aids in graph-related activities, and in copying numerals.
- **Visual memory** enables the reproduction of numerals, as well as efficient error-free copying. It is also essential for following a sequence of instructions, or activities, in solving a mathematics problem.

Although these functions denote important practical mathematical skills, they are unrelated to the essence of mathematics or to its understanding, more accurately reflecting linguistic ability. Linguistic dependence, albeit a recognised adjunct to mathematic ability (refer to Section 2.3.1), is insufficient in accounting for mathematical cognition. In 2008, the annual South African national education assessments tellingly exposed an average literacy level of only 28%, despite average numeracy levels of 43% amongst Grade 3 learners in Gauteng Province (Grobelaar, 2011). Basic mathematic ability therefore appears to entail a fundamental natural core, which is more resilient to either the absence or presence of education.

The root of the misinterpretation as to the relevance of visual perceptual skill may lie in the common supposition that mathematics is a purely ‘mental’ pursuit. The senses are simply input devices. Cognitive processing of the input according to formalist rules then produces the output. This however disregards actual meaningful engagement with the physical and social world. Cognition in isolation from *within* a pre-given world demands an inner representation of that world as cognitive input.

3.5 Representation

It is possible to argue that the perennial problem of perceptual representation, exacerbated by the inherent inadequacies of the information-processing paradigm, has eclipsed the true nature of the relationship between visual perceptual skill and mathematic ability. Representation poses a primary obstacle to the synthesis of perceptual and cognitive theories, in general.

Representation refers to any symbolic descriptions of the world. Common to many perceptual theories is the reasonable – though problematic – notion that an internal

representation of the world is required for its perception. Since the seventeenth century, when Descartes established the principles of geometric optics and ‘discovered’ the retinal image, the dilemma arose as to how the brain interprets this representation, without resorting to an infinite regress of homunculi. Cognitive psychologists adopted a constructivist philosophy, partly in the attempt to solve this mind-body conundrum.

Yet computational cognitive theories are foundationally representational; since by definition, algorithms transform representations. Contemporary information-processing theories, especially in the areas of perception, conception, memory, and problem solving, portray intelligent behaviour as no more than symbol-manipulation. Haugeland (1997) has raised the valid concern that computational formalism purges symbols, or neural representations, of all meaning. A computer may manipulate such non-semantic symbols, whereas cognition undeniably implies a semantic understanding.

A number of potential solutions have since arisen to meet the challenges posed by symbolic representations of an objective external world. Three of these are of particular interest to the present discussion: connectionism, ecological theory, and embodied cognition.

3.5.1 Connectionism

Reluctance in renouncing the computer analogy has evolved traditional symbolic information-processing into theories involving connectionism and neural networks. Traditional information-processing theories, such as that of Marr (1982), describe perception as involving the construction and algorithmic manipulation of abstract symbolic representations. To eliminate the problematic interpretation of the abstract symbols, connectionist models and neural networks instead represent perceptual content by patterns of activity in units based on neuronal properties.

Although connectionist theories still functionally hint at formalism, with that philosophy’s inherent cognitive failings, they do nonetheless offer several advantages over symbol-processing models (Bruce et al., 2003). Connectionism appears more biologically plausible, emulating neuronal structure and function. Rule-based weighting adjustments in connectivity, and the resultant varying activation levels, suitably simulate perceptual learning. Parallel distributed processing, important for pattern and object classification, specifically eradicates a one-to-one correspondence between a neural unit and a percept or

concept. Different patterns of activity in the same unit contribute to different concepts, and a particular concept is characterised by the parallel activity of many distributed units.

However, connectionism appears ultimately unsuccessful in its failure to address the challenges of symbolic representation. The patterns of activation, although supposedly non-symbolic, nevertheless *represent* objects or properties of the world. Fodor and Pylyshyn (1988) have argued, as has Rowlands (2010), that this type of retinal-neural mapping still necessitates construction and manipulation of representations, albeit at a sub-symbolic level.

3.5.2 Ecological Theory

The complete elimination of representations proffers a sensible option, which explains the renewed interest in Gibson's sixty-year-old ecological theory of direct perception. Like Piaget (1963, 1969), Gibson (1950) rejected object awareness as the primary function of visual perception. This automatically circumvents the need for the construction of representations of the external world, despite a contradictory realist orientation to their theories. Whereas Piaget's constructivist egocentric approach focused on the intimate link between perception and cognition, Gibson founded his ecological theory of direct perception on the inseparability of perception and action. This contrasts with the concept of a *central executive* utilizing perceptual representation, primarily for awareness, in decision-making and controlling action.

Direct perception entails the total absence of representations and mediating cognitive processes. The physical properties of the environment determine the structure of light through reflection, refraction, and transmission. All necessary perceptual information is thus contained within this light structure, which has been termed the *optic array*. The image formed on the retina is not a symbolic representation of the external world, but is purely a reproduction of the spatial-temporal pattern of light from a segment of the optic array. *Invariants* are constant properties of the optic array, determined by specific environmental properties, by which the optic array unambiguously specifies the world. Unambiguous specification excludes the need for cognitive interpretation of visual input. Perception is direct, by means of resonance with, or by becoming attuned to, the invariants.

Gibson (1950) further maintained that the real function of perception is not the creation of a conscious percept, but the detection of *affordances*. Affordances, the values and

meanings of objects in the environment, are implicit in the manner of interaction with them. Norman (1998), in *The Design of Everyday Things*, shows how industrial designers capitalise on this natural phenomenon. By exploiting natural bodily and environmental constraints, good designs are rich in affordances that make explicit the manner of their human operation. The control of action forms an important extension to Gibson's theory. Visual perception is the means by which information is detected that is necessary to guide action, and active movement produces transformations in the optic array, providing essential information for visual perception. Perception and action each constrain the other; being so intimately bound as to be in fact inseparable.

Remarkably, despite Gibson's predominantly environmental theoretical orientation, the concomitant emphasis on active movement discloses a revealing insight. According to Gibson, a hierarchy of organs constitutes a perceptual system. The visual perceptual system includes photoreceptors that are stimulated, activating the eyes, which are part of a dual organ, set in a head that can swivel, and attached to a moveable body. Unfortunately, there is little elaboration on this holistic embodied view. In addition, so-called 'resonating with invariants' (Gibson, 1950) is an expression that imparts little about the neural organisation for detection of the invariants. The concern solely with the ecological level of perception neglects the physiological level.

Fodor & Pylyshyn (1981) have criticised Gibson's theory for the use of the terms 'invariants', 'affordances', and 'directly detected', which are in fact pivotal to the theory. Although they do acknowledge the possibility of direct perception of some simple properties of the world, they argue that the unconstrained definitions of these terms render them as vague, and therefore meaningless. In a similar vein, Cutting (cited in Bruce et al., 2003) has argued that the structurally rich optic array gives rise to gross over-specification of the environment. Consequently, an actively directed selective perception is called for.

Since the late 1990s, neurophysiologic evidence of a partially independent pathway linking the early stages of visual processing and control of action, provides some support for Gibson's views. Milner and Goodale (1998) theorised two interacting visual processing pathways, originating in the occipital primary visual cortex of the brain. The ventral pathway, which extends to the inferior temporal cortex, processes visual awareness information for object perception. The dorsal pathway, connected to the posterior parietal cortex, mediates visuo-motor control and enables spatial perception.

The ecological approach successfully offers an alternative function of perception. However, perception for action and perceptual awareness cannot be taken as mutually exclusive in humans, as is the case in insects (refer to Section 3.1). Disregarding perceptual awareness is problematic for explaining the acquisition of perceptual knowledge used to guide action later. In fact, it renders ecological theory as insufficient to cognition in general, including mathematical cognition.

3.5.3 Embodied Cognition

After only ten years of artificial intelligence research, it became apparent, as stated by Dreyfus (1997), that “intelligence and basic sensory-motor human life is inseparable” (p. 181). The natural consequence is embodied intelligence. The following chapter presents an in-depth discussion on the notion of embodiment. The implication, however, is a direct association between cognition and perception of the world. This, ironically, is comparable to the basic premise on which Piaget constructed his hypotheses. Fifty years worth of research has effectively led full circle back to a previously held psychological theory. Embodied cognition expressly negates the need for symbolic representation of an objective world inside an inner space called ‘mind’ as a requirement for explaining cognition.

3.6 Conclusion: The Need for a Paradigm Shift

It is clear that the frame of reference within traditional psychology theory is based on a metaphysics that inadequately accounts for both visual perception and mathematical cognition. A viable relationship between them is therefore unjustifiable, when working exclusively within these existing frameworks.

With image transformation beginning in retinal neural processes, it is difficult to differentiate sensation from perception. Similarly, James Gibson (cited in Loikith, 1997) noted that it is not possible to determine where perceiving stops and remembering begins. This emphasises the active neural construction of vision. Yet the common human experience of visual perception is a seemingly passive inflow of visual sensations. Phenomenologically, or experientially, this feels very different to conscious thought. It is possibly due to this phenomenal discrepancy that theories that successfully integrate perception and cognition have until now been scarce. A consequence of this dearth is the partial theoretical

disconnection of perception from cognition, resulting in the notion of perception as merely a store of environmental information relayed to higher-order cognitive centres. This notion is particularly evident within computational theory, which dominated cognitive psychology for approximately fifty years. However, with mounting disillusionment in the information-processing computer analogy, a further paradigm shift is currently taking place.

Previous assumptions entail the existence of a world with predetermined properties. Perception thus involved recovering these visual properties from an image, through symbolic or sub-symbolic representations. The only alternative seemed to be an apparent projected reality, as reflected by the internal structure and operation of the visual system, such as that proposed by the idealists. Cognitive science has recently introduced a viable compromise between these two extremes. Perception is accordingly considered a product of particular sensory-motor capabilities embedded in their encompassing biological, psychological and environmental context, which enables perceptually-guided action in a world enacted by these same processes.

Further progress seems to require a revolutionary paradigm shift towards an alternative metaphysics of the mind. One such current trend in cognitive science is the notion of *situated cognition*, in which the boundaries between mind, body, and environment are becoming increasingly blurred. A consequence of the blurring of these boundaries is an explanation that provides a direct grounding of cognition in sensory-motor experience. The new paradigm clarifies the nature of visual perception and mathematical cognition, thereby simultaneously providing a more coherent explanation for a relationship between visual perceptual skill and mathematic ability than that proffered by traditional views.

CHAPTER 4: Situated Cognition

Situated cognition represents a relatively new paradigm within cognitive science. However, its core ideas originated centuries ago. In a more contemporary age, the basic principle is found in the phenomenological philosophy of Husserl (cited in Merleau-Ponty, 1962), propounded in 1913. Among others, Dewey (cited in Gallagher, 2009), Heidegger (cited in Gallagher, 2009), Wittgenstein (cited in Gallagher, 2009), and Merleau-Ponty (1962) further developed Husserl's philosophy. Building on this foundation, the incorporation of current knowledge from a diverse range of disciplines has given rise to this new *cognitive science* of the mind. Philosophy, education, sociology, linguistics, physics, robotics, biology, anthropology, neurology, and cognitive psychology have all contributed to the theory of situated cognition.

Following in the path of prevailing philosophical and scientific views, traditional cognitive psychology is based on the assumption that information about the world, obtained via the senses, is used to construct knowledge by means of the application of reason. This necessitates an internal symbolic representation of the external world. Cognition is taken as the manipulation of these symbolic representations according to logical formal rules. Concomitantly, during this era, mathematics was commonly believed to similarly involve formal symbol manipulation according to the laws of logic (refer to Section 2.1.2). However, in neither case is there any indication of the derivation of meaning. Symbolic representation is not inherently meaningful. Symbols are meaningful only through interpretation, which poses a theoretical problem that proves difficult to resolve (refer to Section 3.5).

In the twentieth century, Gödel's (cited in Davies, 1992) incompleteness theorem eroded belief in the logical foundations of mathematics, which simultaneously cast doubt on the equivalent cognitive model. However, proposed cognitive models based on processes other than logic, still represented cognition *within* an objective environment. Circumventing the dilemma of representation necessitated the eradication of the notion of an internal-external duality of the world. Therefore, the concurrent resurgence of interest in phenomenology may have been an inevitable consequence. An alternative view developed as a result, whereby meaning arises through natural bodily involvement *with* the world. Such embodied cognition, however, does not exist in isolation. Humans, in particular, live as a society of language-users. Linguistic expression of common experience in a shared social

context extends embodied meaning into the generated socio-historical frame of reference. This is the basis of situated cognition, whereby situated sensory-motor interaction with the enacted physical and social world grounds the regress of meaning.

If the situated paradigm proves to be a viable framework for cognition, then it must extend to mathematical cognition, which is simply a specific instance in the application of cognition. Yet, it is difficult to reconcile the common view of mathematics as absolute, infallible and transcendental, with the view of mathematical cognition as relative embodied and social experience. Mathematics is not an empirical science, therefore it appears distanced from the realm of experience. The situated paradigm, however, clarifies the nature of mathematics as metaphorical. Situated sensory-motor activity with the enacted world thus grounds mathematical meaning, which develops through its extension into the social context by means of language. The suggestion that perceptual-motor activity grounds the regress of mathematical meaning allows for the reasonable expectation of a relationship between visual perceptual skill and mathematic ability.

4.1 Psychological Contributions

Situated cognitive theory provides a more coherent paradigm than that of traditional theories for understanding visual perception, mathematical cognition, and the nature of mathematics. Yet, many classical cognitive psychology theories contain the seeds of situated cognition.

- The principal tenet of Gestalt psychology presages situated cognition, where it holds that the whole is greater than the sum of its parts. Gestaltists adequately applied this important principle to particular phenomena, but Merleau-Ponty (1962) maintained that their contradictory confidence in external objectivity unfortunately limited the theory's full utility.
- Piaget's (1969) constructivist developmental theory pioneered the idea that cognition emerges through sensory-motor activity, effectively extending the mind to include the body acting in the world. Tversky (2009), a well-known proponent of situated cognition, has stated that "thinking can be regarded as action, internalized" (p. 211). Piaget (1970) expressed a similar sentiment fifty years ago, where he wrote that "intelligence [...] consists in executing and coordinating actions, though in an interiorized and reflexive form" (p. 29). A crucial

failing, however, is Piaget's neglect of the environmental contribution and interaction. This omission may have been a consequence of the prevailing philosophy involving an independent objective world.

- Von Glasersfeld's (1995) radical constructivism attempted to redress this failing by introducing mutual person-environment causal relations. However, the nature of these causal relations still constitutes cognition *within* a pre-given objective world, rather than cognition *with* the environment, through mutual interaction.
- Gibson's (1950) ecological theory of perception similarly failed to fully appreciate the cognitive-environmental interaction central to situated cognition. Ecological perception is direct perception of an independent environment, which effectively precludes sensory-motor enaction of the environment. Nevertheless, Gibson's theory contains the seeds of situated cognition in his notion of environmental affordances; his holistic bodily view of the perceptual system; and his then novel approach to perceptual function, namely perception for action, rather than for awareness.
- Information-processing theory and connectionism offer little to a theory of situated cognition. However, the situated nature of cognition became apparent through the failure in building a successful artificial intelligence model based on these theories. Brooks (1997), a roboticist, admitted to the impossibility of identifying the locus of intelligence in *any* system, since it is the dynamic interactions of many components, with each other and with the world, which produces intelligence. Consistent with his view, and inspired by embodied cognition, Brooks has advocated a bottom-up artificial intelligence design, using subsumption architecture. The resultant reactive behaviour-based robotics stands in contrast to the notion of a system with a central executive processor. Connectionist advancements towards dynamic multi-agent complex systems theory networks with feedback loops have led to a degree of non-linear emergence of certain behaviours. This represents a positive direction for situated cognition. Clark (2011) pursued a bold attempt to incorporate these advancements into his vision of the extended mind. To accomplish this, he extends the network feedback loops from the purely neural to include, for example, objects in the world. He terms this *scaffolding*, whereby aspects of the environment literally become a part of cognitive processing.

Further contribution from the discipline of psychology that is more meaningful to the situated perspective rests upon the increasing recognition of critical dynamic cognitive, biological, environmental, and social interactions. Clarification of the claims that cognition is

embodied, socially embedded, and environmentally extended will serve to consolidate situated cognitive theory for the purposes of this study.

4.2 Terminology

Despite its historical grounding, as a comprehensive science of cognition situated theory is still in its infancy. Consequently, it currently lacks a rigid set of construct definitions. The terminology utilised appears variable and interchangeable, reflecting an overlap of ideas from various disciplines. Key concepts include embodiment, emergence, embedment, extension, and enaction. In accordance with Robbins and Aydede's (2009) stance, situated cognition is the overarching concept encompassing all these elements.

Embodiment is a core concept of the grounding phenomenological philosophy, particularly as articulated by Merleau-Ponty (1962). The premise is that the phenomenal experience of the body is very different from that of the scientific description. In other words, the physiological processes underlying bodily experience are not *directly* experienced. We do not consciously 'feel' the light rays entering the eye, being absorbed by the retinal cells, or the neurons firing; what *is* experienced however, is the presence of light. Yet sensory-motor experiences are more than purely mental phenomena, since without a physiological body there can be no sensation or motor activity. These embodied sensory-motor experiences are the primary data for knowledge. Embodied cognition, therefore, depends on experience specific to having a body with certain sensory-motor capacities.

The neuronal network, however, does not function only in the direction from perception to action. Mutually linked sensation and motor activity yields cognition as an **emergent** phenomenon. This negates the need for a central executive responsible for cognition. A critical aspect of this emergence is that the mental *supervenes* on the physical, allowing an understanding of the relation, without any reductionism. In other words, the constituent physiological sensory-motor contributions cannot predict cognition. This corresponds with the central Gestaltist tenet where the whole is taken as greater than the sum of its parts. Perception, action, and cognition are co-constituted.

Moreover, Varela et al. (1991) have explained that sensory-motor capacities are not only embodied (biologically embedded), but also embedded in a more encompassing

psychological, social, cultural, ecological, evolutionary, and physical context. The embodied mind is environmentally **embedded**. Continual perception of, and participation in the world, confers meaning.

Embodiment and embedment imply an **extension** of cognition from the purely neural into the body and the world. Often, the interpretation is simplistically literal, as can be seen in: *the world as an external memory*. An example commonly given is the use of physical reminders to assist memory. However, extended cognition implies more than just the world as a cognitive tool. In strong opposition to radical constructivism's cognition *within* the world, extended cognition occurs *with* the world. The cognitive boundary encompasses the environment.

Varela et al. (1991) have expounded that cognition *with* the world means that it is not simply constrained by the world, but also contributes to the **enaction** of the world. Through dynamic and mutual interaction, cognition shapes, and is shaped by, the environment.

Situated cognition thus entails an emergent extended mind that is embodied, and embedded in a cognitively enacted environment. An important facet of situated cognition is the **situational context**. Although often used interchangeably with the terms *environment* and *world*, situational context is not intended to be limited to the physical, but is understood to include every aspect of the cognitive environment. It necessarily embraces the physical, psychological, cultural, and social world; including historical influences, for which radical embodied theories are often criticised for overlooking.

4.3 Various Interpretations

The notion of embodiment significantly influenced the conceptualisation of situated cognitive theory. A notable distinction between various theories of embodied cognition is the degree to which the contribution of the social dimension is acknowledged. Generally, a greater degree of acknowledgement establishes cognition more firmly the result of dynamic interaction with the situational context, as opposed to processes occurring exclusively within an individual. Specifically, the contribution of language, as a social practice, extends meaning into the socio-historical context. Recognition of this has led to theories that extend cognition further and deeper into the environment. As a result, the cognitive value of social inventions

such as the written word, or pen, paper and computers, in addition to natural environmental interaction, became apparent. The concept of cognition as embodied action is understood to be tightly bound to histories that are lived, which are themselves bound to natural evolution. The enacted world reflects these embodied histories. This is cognition with the world, in which it is situated.

4.3.1 Autopoiesis

An early embodiment theory dating back to the 1970s can be found in the work of Maturana and Varela (1980). It is rooted in Maturana's (1980) assertion that "living systems are cognitive systems, and living as a process is a process of cognition" (p. 13). The authors later coined the term *autopoiesis* to characterise the organisation of living systems, which loosely translates from its original Greek into *self-creation*. Maturana, a biologist, understood the cell as a closed circular system that maintains homeostasis by self-reorganisation in the presence of *perturbations*, or disturbances, external to the system. Maturana and Varela then expanded on this notion of molecular autopoiesis, extending it into a hierarchy of dynamic systems, where organisms exist as collections of cells, and societies exist as collections of organisms.

In simple terms, a system is autopoietic if its components interact in such a way as to produce and maintain themselves and the relationships between them. Such systems are thus structure-determined; not *pre*-determined, since external perturbations can affect the structure of the system itself. Therefore, at any given moment, the system is determined by its structural organisation in that moment. Structural plasticity of the system, along with that of the environment, enables system-environment structural coupling. Structural coupling is a crucial feature of autopoiesis that, together with system closure, promotes system autonomy. The autopoietic process thus gives rise to an autonomous bounded system that has been shaped by its interactions with its environment over time. Equally, interactions with the system shape the environment. Increasingly complex autopoietic systems are constituted through similar structural coupling between simpler autopoietic systems. Maturana and Varela (1980) affirmed that autopoiesis is both necessary and sufficient in order to characterise a living system. They state that all associated phenomena, such as conceptual cognition, language, and self-consciousness, are subordinate to this process.

An implication of autopoiesis is that language is intrinsically non-informative. Instead of the transmission of information, language is comprehended as behavioural coordination through mutual structural coupling. Congruent structural coupling is a series of circular, transactional, and recurrent events between two autopoietic systems. This equates to the establishment of a dialogue. Maturana and Varela (1980) explained that this consensual domain of communicative interactions is a linguistic domain, which is constitutive of society, and the consequent basis for language. It functions to orient the behaviour of the systems with each other, through their structures, as determined during their coupling. This perspective partially affirms the situated view of language as a social practice, whereby language facilitates the dynamic coordination of individual sensory-motor activity within society, in accordance with societal concerns (expounded in Section 4.3.3). However, whether the linguistic domain is constitutive of society, or alternatively is its derivative, remains debatable.

A more significant implication of autopoiesis is the total absence of cognitive representations and objects of knowledge. In simple organisms, autopoietic interactions are chemical and physical. The existence of a nervous system expands the domain of interactions to include neural states. However, as an autopoietic system, the neural network does not have inputs or outputs. It is a closed system, so is not open to representations of the environment. The nervous system is structurally coupled to the organism, which is structurally coupled to its environment. This hierarchy of systems enables an organism to interact with its own neural states. When such an organism enters a consensual linguistic domain with another organism, it can therefore also interact with its own linguistic states. This is an occurrence of the system functioning as an observer, by generating a self-linguistic domain, which supports self-conscious, or reflective, behaviour. The cognitive domain is the consequence of such a hierarchy of structurally-coupled interactions. Knowledge is relative to this cognitive domain, and is attained through successful autopoiesis. Absolute knowledge is therefore not possible.

Critics focus mostly on the self-referential nature of autopoietic systems. In addition, there is no notion of intentionality. An autopoietic existence is purposeless, determined exclusively by the changing systemic structure in the face of perturbations, until loss of organisation results in death of the system. However, Beer (1980) has noted that in the hierarchy of systems, an autopoietic system may be considered as if it were allopoietic within the context of a more encompassing autopoietic system. In contrast to an autopoietic system, an allopoietic system produces and maintains something other than the system itself.

Nevertheless, divergence in Maturana's and Varela's subsequent work poses the question as to whether the original theory of autopoiesis may require revision. Maturana remained doubtful as to whether a society constitutes a true biological living system, while Varela's (Varela et al., 1991) later work reflected autopoiesis as a sufficient, but not necessary, condition for autonomy. Thompson (2007) is in support of this view and further developed these ideas.

4.3.2 Embodiment

Not all embodied theories are as radical as that of Maturana and Varela (1980). The essence of embodiment is that basic sensory and motor skills are innate. They exist by virtue of an animal having evolved to interact successfully with the world in which it is situated. Mammalian evolution further gave rise to a nervous system that effectively interconnects sensation and motor activity in such a way that dynamic interaction with the structured world extends these basic skills to a higher level of functioning. This simply reflects an extension of basic innate pattern perception to pattern perception of a higher order, both of which utilise the same neural pathways. This is cognitive emergence, which promotes comprehension of the enacted world, the meaning of which rests in direct embodied experience. Roy et al. (1999) have explained that the notion of an experienced body, as sustained by embodied theory, is a fusion of subjectivity and objective givens, in a body that has the capacity for both sensitivity and sensibility. Consequently, the body is not only an object in the world, but also the agency whereby the enacted world comes into being. This is contrary to Cartesian duality.

The embodied cognitive facility that emerges through sensory-motor experience is abstraction. All high level cognition rests upon this emergent cognitive foundation. The degree of abstraction attained determines the level of cognition achievable in a species. In keeping with the basic premise of the four levels of abstraction as identified by Devlin (refer to Section 2.2.6), a possible revision of his descriptions may be beneficial here.

1. The lowest level of abstraction is shared by most animal species. Arising from basic sensory-motor pattern perception, the **differentiation**, and simple **categorisation** of patterns, or objects, in the immediate perceptual environment that directly affect survival, reflects minimal abstraction.

2. As categorisation is furthered and refined, patterns develop within and among the categories. **Conceptualisation** is the perception, or recognition, of these category patterns. Such concrete conceptualisation extends cognition beyond the immediate perceptual environment.
3. Conceptualisation continues and pattern perception involving concepts results in a level of abstraction that supports **symbolic representation** of patterns or objects. This level of abstraction is unique to humans.

Each of these three levels involves some form of analogy and metaphor, since it is perceived that *something is like something else*. These perceptual correspondences arise naturally through a particular manner of sensory-motor action with the world, and may thus be considered as embodied. They thus give rise to embodied concepts. Sheets-Johnstone (1990) referred to such non-linguistic concepts as corporeal concepts, and asserted that they are in no way inferior to linguistic concepts. Embodied conceptual metaphor can thus be considered a perceptual cognitive mechanism.

The symbolic representation evident in the third level of abstraction is nominal. For example, a simple representational object, notch, or grunt. The capacity for symbolic representation is, however, a pivotal cognitive acquisition for the development of more complex cognition, since it supports the development of language, which makes the fourth level of abstraction possible.

4. Language promotes concept definition and sharing within a society, linguistic metaphors, and general cognitive reflection, expression, and discussion. This elevates conceptualisation and abstract cognition to ever-increasing levels of complexity, which distances it from the perceptual world.

However, rather than language per se, it is instead the context of a society of human language-users that enables the development of abstraction.

In addition to innate sensory-motor neural interconnections, Sheets-Johnstone (1990) attributes the seminal cognitive symbolic function to the fact that humans are social creatures, and are uniquely bipedal. She explains how the nature of hominid walking produces mutually reinforcing three-dimensional, tactile-kinaesthetic binary correspondences – mainly through the binate symmetry of the body – which reinforces the perception of one-to-one correspondences, the basis for symbolism. With reference to symbolic and abstract numerosity, in particular, she then presents a complex theory of how this human experience of the binate nature of existence in the world gave rise to the decimal system. In part, she

postulates that an upright posture frees the hands and the body, which promotes counting by symbolic correspondence with the fingers, and other body parts. Sheets-Johnstone thus concluded that socialisation and bipedalism provide a unique and beneficial mode of interaction with the world, which facilitates the emergence of a greater degree of abstraction in humans, as compared to that of other mammals.

Facilitated by language, individual embodied cognitions mutually interact, directing attentions to embodied experiences relative to the norms of cultural practices. A higher level of cognition thus emerges, as the enacted world is elevated to a more general, or abstract, plane. The added meaning rests in the shared social context of embodied experiences. The fourth level of abstraction is therefore a consequence of cognition that is embedded within a society.

4.3.3 Societal Embedment

In harmony with the essential features of embodiment described above, the radical autopoietic theory of Maturana and Varela (refer to Section 4.3.1) has raised an important consideration. The individual, although significant, is also but one component in the encompassing system of its society. In an extension of Gibson's insights (refer to Section 3.5.2), it may then be said that the visual perceptual system includes photoreceptors that are stimulated, activating the eyes, which are part of a dual organ, set in a head that can swivel, and attached to a moveable body, *which is a member of a society, situated in an enacted world*. This reveals that perception, initiated on the individual level, has the potential to become a global concern. The core notion of embodied cognition grounds the regress of meaning in sensory-motor activity with the enacted world. Situated cognitive theory extends this meaning onto a societal level.

The notion of situated cognition emphasises dynamic mutual interaction with the situational context. Humans are social creatures, and live in the context of a society of language-users. This aspect of the human world therefore cannot be ignored, nor can its importance be under-estimated. Preservation of the society, secondary only to reflexive individual survival, instils its members with common concerns, which dictate particular sensory-motor interactions with the world. Language enables the dynamic co-ordination of these interactions within the society. As neural co-ordination of sensation and motor activity gives rise to the comprehension of the meaningful world, so language is the metaphorical

‘nervous system’, in the sense that it interconnects members of a given society. The development of meaning within societies is therefore contingent upon effective communication between its members, for which language, based in embodied symbolic abstraction, is a necessary condition. Linguistic expression and discussion of individual sensory-motor experience imbues it with additional meaning, by extending the frame of reference to the societal level. Embodied meaning is thus extended by means of social conceptual metaphor, which may be considered a linguistic cognitive mechanism.

An animal does not repeatedly re-conceptualise commonly encountered aspects of the world. Rather, certain embodied concepts become experientially entrenched. Similarly, language facilitates societal conceptual inculcation, through common conceptual usage and sharing. Social conceptual reproduction is a feature of this process, whereby concepts are linguistically conveyed, in lieu of individual, direct experience. A socio-historical aspect of cognition is thereby introduced, since previously formed human concepts and abstractions are imparted throughout subsequent generations. Social reproduction eliminates the necessity for individual recreation of every concept, or metaphoric abstraction, from its sensory-motor foundation. It facilitates the development of abstract cognition, since further abstractions are based on those already conceived throughout history. Linguistic social conceptualisation and conceptual reproduction therefore supplement perceptual embodied conceptualisation.

A common socio-historical frame of reference is thus derived through language, elevating the development of meaning to the social contextual level. The added meaningful dimension, however, is simply the extension of the level of abstraction beyond that which is obviously embodied. It is possible that the efficacy of language in facilitating abstraction conceals foundational direct experiences. Social conceptual reproduction and social abstraction through linguistic conceptual metaphor do not detract from the embodied origins of conceptualisation, as sensory-motor interaction with the world still firmly grounds the ultimate regress of meaning.

The meaningful enacted world exists by virtue of our perception, conception, and abstraction, not by our naming of it. An embodied concept is meaningful prior to its linguistic expression, and further linguistic social abstraction relates to the comprehension of the pre-verbal concept. A simple example illustrates this position. We can achieve a pre-verbal embodied concept of *hardness*, as can most animals, by our particular interaction with the world. Linguistic expression, conceptualisation, and abstraction may allow further

comprehension of the concept *hard*. However, in no way does this detract from the meaning inherent in the embodied concept. A simple sentence such as ‘*maths is hard*’, is built upon a host of embodied and social conceptualisations and abstractions. In the context of the example, though, the gist of its meaning may be traced back to the direct embodied experience of *hard*. In essence, it means, ‘the abstraction that is mathematics is metaphorically impenetrable to cognitive conception, like *hard* things are physically impenetrable to the body’. In addition, Presmeg (1997b) has reported that the social conceptual abstractions created by many eminent scientists originate in perceptual conceptualisation. Einstein allegedly described his creative process as imagistic ‘flashes of insight’ that apparently appeared at random. The pattern imagery he described depicted the interplay between concrete perceptual visualisation and abstract symmetry. Only after he had conceived of a core abstraction, was a theory linguistically developed, and thereby shared. Therefore, even in the absence of direct experience, the meaning inherent in linguistic social conceptualisation is nevertheless grounded in the sensory-motor *potential* of its recreation.

Embodied experience with the world thus grounds the regress of meaning. However, effective communication within human society crucially elevates embodied abstract cognition to a higher level. Comprehension of the meaningful enacted physical and social world is thus achieved through direct embodied experience, and in conjunction with language in a society, both of which eliminate the need for mental representation. The social aspect of the world is therefore as important as the physical aspect, to the holistic theory of situated cognition.

4.3.4 The Amalgamated Mind

Rowlands (2010) has been a firm advocate of the new cognitive science. However, he promulgates the alternative term *amalgamated mind* as an apt description for cognition composed of the *amalgam* of neural, bodily, and environmental structures and processes.

In a creative revision of Gibson’s theories, Rowlands (2009) has defined cognition, in part, as the manipulation and exploitation of external structures. These actions supplement the diminished role of ‘representation’. Unlike Gibson, and contrary to the premise of situated cognition, he does not wholly repudiate representation. This is a tendentious attempt, by Rowlands’ own admission, to entice traditionalists into acceptance of the new paradigm. Although, his notion of representation is certainly vastly dissimilar to the concept

traditionally held (refer to Section 3.5). He cleverly connects representation and action by arguing that acting *is* a form of representing. Rowlands contended that manipulative and exploitative action transforms information present within external structures, making it available for direct detection, and thus negating the unqualified need for representation construction and storage.

Although the amalgamated mind theory supports all the key concepts of situated cognition, it prioritises embodiment and extension. Rowlands (2010) has reasoned that these two parallel concepts entail definitive cognitive constitution, whereas embedded cognition entails only the weaker assertion of environmental dependence. Moreover, he argues that neither embedment nor enaction are in contradiction of Cartesianism, and in fact may easily be assimilated into traditional cognitive theories.

The elimination of brain-body and body-environment causal interactions seems to be a particularly problematic notion, since even within the sphere of situated cognition theorists often exhibit difficulty in assimilating this idea. Eliasmith (2009) argued for the validity of his proposed Neural Engineering Framework (NEF) by claiming that it assists in understanding high-level cognition, and “is consistent with traditional boundaries among brain, body, and world” (p. 150). Bechtel (2009) advocated a bounded brain-body, although he does accede to this boundary being permeable to the environment. These instances of resistance to embodiment and extension appear to support Rowland’s (2010) view that these two concepts particularly challenge entrenched Cartesian dualism and classical cognitive science.

4.3.5 Environmental Extension

Clark’s treatise (2011) presented compelling argument for the extension of the cognitive boundary into the environment. However, his tenacious conviction in a connectionist and representational mode of cognitive processing betrays an unfortunate degree of reductionism. Consequently, much of his valuable input is lost in discussion about *where* in the environment the cognitive boundary lies. In other words, he considers which objects, and the interactions with them, may properly be taken as an integral part of cognitive activity. This attempt to distinguish the cognitive domain from the world being conceived reflects a futile search for the locus of intelligence, and reveals the fundamental incompatibility between information-theoretic cognition and situated cognition.

In situated cognition, there is no ‘locus of intelligence’ and there is no determinate boundary, as presumed in Cartesian dualism. Implicit in the phrase ‘*the cognitive boundary encompasses the environment*’ (refer to Section 4.2) is the belief in a holistic brain-body-world system. Cognition is an emergent property by virtue of the dynamic interactions within this system. The exclusion of *any* aspect of the system would render cognition redundant. Moreover, for all practical purposes, the aspects of the system are inseparable, since we are a part of the world, and the world is a part of us. Cognition occurs *with* the world, in its entirety.

4.4 Implications

Initial resistance to any new scientific paradigm is the norm. Acceptance usually occurs as a process over time. Scientific progress is a consequence of society, which might equally be one of the factors that slow this progress down. Analysis of this statement coincidentally also demonstrates the extension of meaning into the societal dimension, and highlights the influence of social tradition and linguistic inculcation.

A new paradigm is often associated with new concepts, as well as with the re-conceptualisation of existing concepts. Ideally, therefore, it should present with its own conceptual terminology. This is in harmony with Sfarid’s (1997) assertion that new knowledge involves the “creation of a new discourse” (p. 358). However, scientific concepts and terms are abstractions. Cognitive comprehension of the abstract is dependent upon a process of mapping onto previously formed concepts. The generation of a new socially accepted lexicon is indicative of the completion of abstraction, and simultaneously renders the mapping as covert. Novel concepts may be easier to assimilate, since the mapping is no more than a comparison of limited features, entailing a similar previous concept and its expansion. Re-conceptualisation, however, involves a novel linguistic description for a concept already accepted through social inculcation. A new process of abstraction is required in order to alter the associated habitual meaning within the societal frame of reference. During this process, the intended new meaning and the established meaning are in cognitive competition.

An attempt to explain the implications of situated cognition exposes the difficulty of re-conceptualisation. Scientific, philosophical, and social progress has rendered many

concepts common to traditional cognitive theories obsolete, in terms of their socially established meanings. For example, the concept of subjectivity, as traditionally conceived, should not hold currency since the abolishment of the notion of duality of the world. Similarly, the concepts of intelligence, rationality, and knowledge, in terms of their previously established meanings, are redundant when it comes to situated cognition. In fact, the concept of a 'concept' is itself misleading. However, comprehension of the new paradigm lies in the mapping process of abstraction. Therefore, the use of traditional terms is necessary until a revised, meaningful terminology appropriate to the current discourse is introduced.

4.4.1 Intelligence

The situated perspective of cognition elucidates why it is so difficult to define intelligence. In the absence of a central executive processor, sensory-motor capabilities are an essential component of intelligence. The difficulty then, presumably, lies in distinguishing between what is intelligence, and what is environmental interaction. Yet, this matter becomes superfluous, with the realisation that any intelligence, to be successful or useful, *must* be situated in a world. Intelligence is the result of dynamic sensory-motor interaction *with* the world. Simultaneously, the world is partly constituted through this sensory-motor activity. This view of situated intelligence is captured in Merleau-Ponty's (1962) observation that although sensory perception *underlies* intelligence, intelligence cannot be *reduced* to perception. It means that the relationship is not characterised by cause-and-effect linearity, but rather involves a dynamic circularity. A firm distinction between intelligence and environmental interaction is, therefore, nonsensical. Intelligence is an emergent phenomenon, which consequently *supervenes* on the physical. All knowledge thus originates in sensory-motor experience.

To appreciate this statement fully, consider the futility in attempting to describe the perceived colour blue to a congenitally blind person. Experience of the colour blue is essential to the knowledge of it. Active, situated perception is the only basis for true knowledge.

4.4.2 Knowledge

Situated cognition successfully negates the need for representations, since the world, both external and bodily, grounds the regress of meaning. There is no need to *represent* a

world *in* which we are already situated. In the words of Brooks (1997), “the world is its own best model” (p. 416). As both Gibson (1950) and Rowlands (2009) have suggested, meaning is presented in the structure of the world, and is directly detected through embodied involvement with the world (refer to Sections 3.5.2 and 4.3.4). In concert with this, co-operative reference to embodied activity, achieved through the social-linguistic aspect of the human world, metaphorically distances cognition from direct bodily experience. This promotes cognitive reflection, which adds contextual meaning to existing experiential meaning. A meaningful world thus constitutes part of our existence.

Knowledge is therefore not constituted by objective facts, but arises conceptually, through dynamic interactive construction, and social reproduction. Clancy (2009) and Millikan (2009) have agreed that this dynamic constraint renders knowledge fallible; but non-arbitrary, and not *wholly* subjective or culturally specific. It is fallible since it is relative to experience, which can be ambiguous, as evidenced by perceptual illusions. It is non-arbitrary by virtue of the distinctive interactions supported by the particular bodily and external world context. It is not wholly subjective or culturally specific, due to the commonalities of human bodily interaction with the world, and thus to the shared cultural concerns of societies composed of such interaction.

4.4.3 Concepts

The conceptual system supports knowledge (Medin & Rips, 2005). Conceptualisation is an important component of the theory of situated cognition. Substantial research in cognitive linguistics has significantly contributed to an understanding of concrete conceptualisation (English, 1997a; Sfard, 1997; Lakoff & Núñez, 1997, 2005). However, further research is required in order to comprehend abstract conceptualisation more fully.

Traditional semantic memory theories of the conceptual system, such as Collins and Quillian’s network model (cited in Barsalou, 2009) and other feature set models, view concepts as detached databases of category specifications. In contrast, Millikan (2009) has alternatively described situated conceptualisation as an ability to interact successfully with the world. Petitot (1999) has described this as a process whereby cognitive interest governs conceptual development. Cognitive interest may reflect that of the individual, or it may reflect interests from the broader context of a society. There is thus no fixed, universal set of perceptual specifications, as has been traditionally suggested. Instead, the facility to respond

to certain perceptual properties, as they manifest through sensory-motor interaction with the environment, determines conceptual content. This perspective provides new insight into the perceptual constancies, explicating, for example, the capacity to recognise a shape under a wide range of viewing conditions. Situated perception results in situated conceptualisation.

Categorisation captures the uniqueness of each perceptual experience, transforming and enriching it, through its incorporation into the meaningful abstract conceptual edifice. Núñez (2008) attributed the meaning within this structure to cognitive mechanisms, such as conceptual metaphor, which govern its inferential organisation. In fact, he argued that categorisation is actually the application of metaphor. Although categorisation is one of the most fundamental cognitive activities, category recognition does not reside solely within the brain. Categorisation is a facility that emerges from mutual embodied interaction with the world, in accordance with particular sensory-motor interconnectivity. It is therefore an evolutionary-derived interactive ability, perfected through situated perceptual development and experience. Categorisation thus contributes to conceptual development.

As a result, conceptual development is not simply the collection of a larger set of category specifications. Rather, as described by Millikan (2009), it is the development of the ability to assimilate new, dynamic perceptual information accurately and efficiently, creating a *richer* conceptual system. This culminates in the progressive evolution of abstract *relational* conceptualisation. In any relational structure, as Hiebert and LeFevre (1986) recognised, the linking relationships are of as much consequence as the nodes themselves. In contrast to the traditional notion of a concept as an inner representation or specification, situated conceptualisation is thus a dynamic process, focused on relationships, which are conceptual patterns. These patterns may present in the embodied or social frame of reference. They may involve perceptions or abstractions, images or language, or any combination of these. Conceptualisation begins with perception, or recognition, of a conceptual pattern. Assimilation of the perceived or socially reproduced relationship is the cognitive comprehension of a ‘concept’.

Experientially and socially entrenched concepts, or conceptualisations, play an important role throughout cognition. Barsalou (2009) elaborated:

- Concepts facilitate probable predictions, enhancing cognitive processing speed, through inferential induction. For example, concepts support efficient visual

perceptual processing of familiar scenes, by way of figure-ground segregation and *filling-in*, which is a form of visual closure.

- In problem solving, concepts provide analogical and metaphorical content essential for deductive reasoning.

The information abstracted by analogy and metaphor remains tightly coupled to the founding situated concrete concept. Barsalou affirms that this critically situates abstract concepts and, by extension, it situates rationality (Millikan, 2009).

4.4.4 Rationality

Situated rationality emerges through the interactive cognitive adaptation to the structural properties of an enacted environment (Brighton & Todd, 2009). This is contrary to the traditional view of a disembodied logic as the governing principle of rational thought.

A world exists in which, and aligned with, cognition evolved so as to distinguish certain parameters and patterns present. Sensory-motor activity enables cognitive perception of these patterns. Perception and further sensory-motor interaction with the world – in tandem with the foundational cognitive ability for categorisation (pattern perception of a higher-order) – initiate conceptualisation. Perception and conception mutually advance, resulting in the enactment of a cognitively comprehensible environment. This unique coupling, where the mind exploits and complements the structurally rich environment, enables simple mental mechanisms to achieve high-level cognition. Rationality is the dynamic interaction between sensory-motor perception, conception, and the meaningful environment.

In simple terms, although not strictly a linear relationship, situated sensory-motor activity enables perception, which leads to concrete embodied conception. Further conception realises a level of abstraction to support language development, which promotes conceptual generalisation into the broader context of the meaningful social environment. Assisted by linguistic social conceptual metaphor, conception attains a level regarded as truly abstract. Mathematical cognition, the epitome of abstract rationality, therefore originates in perceptual ability. In the insightful words of Merleau-Ponty (1964a), “we never cease living in the world of perception, but we go beyond it in critical thought – almost to the point of forgetting the contribution of perception to our idea of truth” (p. 3).

4.5 Situated Visual Perception

With a new appreciation of the perceptual underpinning of cognition afforded by the situated paradigm, a firm grasp on visual perception and its concomitant skills is imperative. Brooks (1997) believed that the design of a successful artificial intelligent system is simple, *if only* the essence of acting and reacting is made available. He construes this essence as the ability to move and sense in a dynamic environment. Brooks' view appears to be a practical application of Maturana's (1980) theory on the biology of cognition. Maturana reasoned that for an organism to be successfully mobile, it requires sense abilities for environmental interaction and orientation. More advanced *cognitive* organisms have developed nervous systems in order to connect and co-ordinate motor and sensory perceptual skills. Visual perceptual skill is thus a necessary basis for the development of intelligence. Yet, this poses even more of a design challenge, since not only is there no explicit representation of the world, but there is an equally frustrating lack of an identifiable location of the 'output' of visual perception.

For similar reasons, a definitive operational definition of visual perception remains elusive. The problem, according to Merleau-Ponty (1964b), is fundamental, namely that there is no physical causality in perception. Perception is not stimulus-produced, but rather results from the behaviour of the perceiver in the presence of dynamic stimuli. It is, therefore, impossible to decompose a perception to a set of basic sensations. In familiar terms, the whole proves greater than the sum of its parts. Merleau-Ponty (1962) further highlighted an ostensibly insurmountable ambiguity that exists in the study of perception. Perception is primary to cognition; however, it is only through cognition that perception is made manifest. This cognitive immersion in perception prohibits *detached* objective perceptual investigation, and favours a more phenomenological approach. Scientific thinking removes us from the perceptual experience, severing the link between the physical world and the embodied mind.

Notwithstanding the difficulties in providing a concise definition, situated cognitive theory understands that situated visual perception is neither the idealist projection of an internal system, nor the realist recovery of external properties. Rather, visual perceptual skill entails substantially more within the situated paradigm than has been suggested by previously accepted definitions. Rosner (1982) defined visual perceptual skill as "the ability to identify

the task-pertinent concrete features of a visual array” (p. 288). A more apt description, however, may be that visual perceptual skill is *the dynamic interactive ability to co-ordinate and render visual-motor activity with an enacted environment*.

4.5.1 Colour Perception

Varela et al. (1991) have provided a detailed discussion of colour perception, as a representative microcosm of cognitive science, in an attempt to explain their embodied enactive approach. Likewise, the relatively concrete nature of colour perception aids in grasping the complex concept of situated visual perception.

There is no disputing that retinal cells absorb certain wavelengths of radiation. Varela et al. (1991) have suggested that this may possibly occur in accordance with Hurvich and Jameson’s opponent-process theory, despite debate as to the exact physiology involved. Nevertheless, common human brain-body structure ensures a common enacted environment. Hence, a different common anatomy would yield a vastly different enacted world of colour. The bee’s visual system, for example, detects ultra-violet radiation, and therefore a ‘white’ daisy appears fluorescent violet to the bee. In essence, radiation physically exists in the external, but colour does not. As an even more concrete analogy, *sweetness* is not a property of honey; it is purely our experience of honey.

In addition, surface reflected radiation does not faithfully correspond to colour perceived, as manifested by the phenomena of approximate colour constancy and chromatic induction. Perceived colour is therefore context-dependent. Kay and Berlin (cited in Varela et al., 1991) further found that the experience of colour is not only perceptual, but also cognitive. They claim that among the many colours linguistically named – eleven, covering the entire visible spectrum – are perceptually universal. The balance is culturally specific. Colour is thus an emergent property of neuronal activity, constituted by virtue of the structure and function of the body, and the physical and social world.

The case of colour effectively demonstrates the coalescence of perceptual embodied conceptualisation and linguistic social conceptualisation. The appreciation of colour therefore depends on the biological evolution of the body, and its interaction with the enacted world, including the physical and social aspects in which it is embedded, and extended.

4.5.2 Perceptual Context

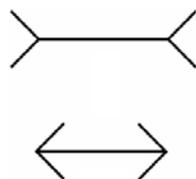
The bodily sense organs detect external patterns, a process open to central neural influences that add non-physical contextual meaning to the patterns. The perceptual world is thus constructed in collusion with the physical world.

The particular way in which the body interacts with the environment determines sensory stimulation by the external patterns. Visual perception is the process of interpretation of the visual patterns. This is not a conscious intellectual analysis, however, but simply the apprehension of meaning inherent in the physical and contextual patterns. It is a primitive capability, attained from having evolved in the world, and developed through experience. The inherent meaning is not transcendent; it does not lie beyond perception. The body and its behaviour endow the sensory stimuli with added significance. This view is redolent of Gibson's concept of affordances. Simply put, an overhang is 'seen', but is perceived as shelter.

Contextual significance, however, extends far beyond this simple example. In a dynamic and elusive visual world, situational context shapes visual perception, determining the potential meaning actually apprehended. The perceptual world thus admits a degree of ambiguity intolerable to scientific thinking. The Muller-Lyer illusion (Figure 4.1, reproduced from Merleau-Ponty, 1962) provides an impressive example.

Figure 4.1

Muller-Lyer Illusion



The perception – and inherent meaning – of two lines is altered by the addition of contextual arrowheads; this despite the fact that the lines prove to be equal when measured. It is from this perceptual basis that situated knowledge (including that of mathematics) is rendered fallible, yet non-arbitrary.

4.5.3 Perception and Action

Due to bodily movement and a dynamic world, external patterns detected and perceived are not static. The eyeball, in particular, is in constant motion. Visual perception is therefore experienced *in action*, as previously suggested by Gibson (refer to Section 3.5.2).

This is a more intimate link than a mere causal relationship between perception and movement. Embodied sensory-motor coupling dictates a particular sensitivity to the physical world, through which the perceptual environment is enacted. Varela et al.'s (1991) enactive approach to perception and the sensori-motor model of visual perception, developed by O'Regan and Noë (2002), both reflect this position. The retina is seen to simply be the probe that interfaces with the environment. As such, it cannot, and does not, provide an internal representation of the external. Nor is there any reason for it to do so, since objects are not perceived by the visual recovery of external properties. The perceptual paradox is that a perceived object exists only in perception, with visual perception being the dynamic visual-motor interaction between the body, mind, and world. Object perception is thus realised through visually guided action. Guided action, imperative to survival, is therefore argued to be the true, original, and primary function of visual perception.

4.5.4 Visuo-Spatial Perception

Visual perception for the primary purpose of action, rather than for object identification, endows visuo-spatial perception with special significance; since, by its nature, movement involves a spatial component. Merleau-Ponty (1962) also unequivocally advocated the importance of spatiality in his statement, "spatial existence is the primary condition of all living perception" (p. 126). Every visual percept, and in fact every sensation, is spatial.

Just as visual perception is situated, so is spatiality. Space is not the realistic, objective setting in which things are positioned, but is rather egocentrically experienced, as the potential action of the body. Movement, directionality, and ultimately space and even spatial form, are relative – that is – in relation to a pivotal embodied perspective. As early as 1905, Poincaré (cited in Myin & O'Regan, 2009) embraced the connection between visuo-spatial perception and action, in his explanation that the potential movements in reaching an object, constitute its position. The notion of three-dimensional space is therefore inferred from sensory-motor possibilities. Heidegger (cited in Gallagher, 2009) presented a similar

conception of situated spatiality, from which its apparent objectivity is subsequently derived, through metaphor.

Paradoxically, spatial conception derives from sensory-motor action, and yet equally enables it. By virtue of this mutual dynamic coupling, visuo-spatial perception is not only essential for physical survival, but also forms a foundation for the emergence of higher cognition (Gattis, 2001; Montello, 2005). Tversky (2009) identified three notable cognitively constructed situated spaces:

1. **Space of the body** supports conceptual development, by encouraging the inference of functional categorisation, from categorisation based on perceptual similarities.
2. **Space around the body**, the space of actual, or potential, perception and action, institutes the conception of directional axes, through the asymmetries of the body and through gravitation.
3. **Space of navigation** extends beyond the limits of the immediate visual perceptual field. This necessitates its cognitive construction from separate, situated perceptions, which introduces systematic perceptual errors. These distortions are apparent in perceived space, and are carried through, via metaphor, into abstract space. The asymmetry of the mental number line provides a patent example.

In general, these perceptual spaces, through metaphorical mapping, provide for abstract conceptual cognition. Natural and spontaneous correspondences between visuo-spatial perceptual and abstract concepts, possible only in a non-random structured world, support situated cognition. This is evidenced in the revealing peculiarities of language use, gestures, symbolisms, and visualisations (Lakoff & Núñez, 1997; Sfard, 1997; Gattis, 2001; Kita, Danziger & Stolz, 2001; Tversky, 2005a). Visuo-spatial cognitive abilities, such as mental transformations, are simply common experiential perceptual-motor actions applied to the abstract, which provides a spatial basis for imagination and creativity. The body and the world, therefore, do not limit imagination; they enable it.

Abstract spatial cognition is thus shaped by both visual perception of, and interaction with, the enacted world. It is a product of embodied and social conceptualisation. Embodied experience is encompassed within the more general, or abstract, social realm, by means of language. Abstract cognition therefore emerges through being embodied, embedded in, and extended into the enacted physical and social world. Abstract cognition is situated.

4.5.5 Visual Perception and Temporality

Expanding on the importance of three-dimensional visuo-spatial perception, a fourth dimension may be argued as equally critical. Just as motor activity is spatial by nature, so do movement and change naturally introduce an element of time. Cognitive psychology already robustly affirms embodied spatial relativity (refer to Section 4.5.4). In comparison, temporality has received minimal attention, although progress in the development of a comprehensive cognitive science has regenerated interest in it.

Gentner (2001) attributed the embodied relativity of time to the metaphorical mapping of the observable spatial domain onto the abstract domain of time. The temporal structure thus constituted by space-time metaphors is therefore relational. Núñez (2008) alternatively maintained that temporality, like spatiality, is a fundamental component of our experience. Yet, he also contends that the only possible human conceptualisation of time is in the abstract. This is due to the lack of a dedicated sensory system to perceive it *directly*, analogous to the visual system, for example.

Linguistic expressions and *embodied spontaneous gesture*, believed to be evidence of the inferential structure underlying the metaphorical mapping, indicate that time is commonly conceived in the spatial form of a future in front of us and the past behind us. This social conception manifests in linguistic expressions such as ‘*moving forward into the future*’. However, in a study of the Andean Aymara culture, Núñez (2008), in collaboration with Sweetser, argued for this conception as cultural, rather than universal. These authors originally attributed the reversed Aymaran conception to a temporal, rather than an ego, reference point. However, they later concluded that the reversal was instead a consequence of the spatial-temporal mapping relating to visual perception, rather than to movement, of the ego. In effect, what lies behind an observer is outside their visual field and is thus unknown, as the future is unknown. Similarly, the knowable past maps to the front of the observer, which is a visually knowable space. The existence of these various culturally specific, opposing, yet internally consistent, metaphorical mappings highlights the relativity of the cognitive conception of time in the enacted environment. It is reminiscent of the varying cultural orientations in the conception of the spatial number line (refer to Section 2.2.4). Irrespective of the particular reference point or cultural spatial orientation, Núñez (2008) inferred the general conceptual temporal metaphor to be “time events are things in sagittal uni-dimensional space” (p. 338).

The phenomenological philosophy of Husserl and Merleau-Ponty (1962), from which situated cognitive theory partly grew, probes temporality at a deeper and more fundamental level. Metaphorical spatial-temporal mapping may well explain human social conceptualisation of time, which critically permits reflection on, and linguistic communication about, time. These abilities are imperative for social coordination. Yet, this philosophy claims that a basic *sense* of temporality, inherent in most animal species and analogous to the number sense, is a primitive, embodied cognitive phenomenon. It thus supervenes on situated sensory-motor experience, of which temporality is a fundamental dimension, as Núñez (2008) maintained (see above). This is the result of a natural and inseparable association between motion and both space and time, since movement from A to B occurs *in time*. Temporality, as a dimension of the enacted world (like colour), is therefore relative, rather than transcendently universal. Phenomenologically, a sense of temporality cognitively emerges from spatial actions, while temporality is equally essential to the structure of spatial actions. Such dynamic coupling is characteristic of situated cognition.

Gallagher (2009) has interpreted Husserl's analysis of time-consciousness as showing that experience is not momentary. Rather, the cohesion of the *just past* to the *just about to be* forms a temporally extended flow. Time therefore exists in the past, present, and future relations of an embodied cognition with the enacted environment. Thompson's (2007) simple example of walking past a tree provides clarification. Despite the continually different perspectives of the tree, it is still known to be the same tree. Perceptual synthesis thus involves a temporal synthesis. Therefore, just as visual perceptions and sensations are essentially spatial (refer to Section 4.5.4), Merleau-Ponty (1962) equally asserted that spatial objects are also temporal.

Merleau-Ponty (1962) further suggested that this *retentional-protentional* temporal structure crucially enables cognition within the enacted spatial environment. Therefore, temporality is as essential to cognition as is spatiality (refer to Section 4.5.4). This is situated cognition; situated not only spatially, but also temporally.

4.5.6 Cognitive Consequences

Within the situated cognitive paradigm visual perceptual skill may tentatively be defined as the dynamic interactive ability to co-ordinate and render visual-motor activity in

an enacted environment (refer to p. 82). Due to the nature of motor activity, visual perceptual skill thus defined necessarily incorporates visuo-spatial and visuo-temporal elements.

However, situated visual perception is clearly not an isolated phenomenon. Through its primitive function of guided action, and the evolutionary development of its cognitive structures, visual perception gives rise to the emergence of knowledge, namely conceptualisation. The abstract symbolic function therefore relies on the visual. This is not a reduction of knowledge to sensation; vision is not its *cause*. Abstract concepts are integrated with concrete perceptual concepts until the abstract emerges as a means in itself. This is a process, as Merleau-Ponty (1964b) describes, whereby visual perception assists “at the birth of knowledge, to make it as sensible as the sensible” (p. 25).

Situated abstract cognition thus emerges through the dynamic sensory-motor interaction with the enacted world. Originally, sensory-motor experience was presumed to be predominantly spatial. In contrast, within the *space-time* fabric of the enacted world, sensory-motor experience equally incorporates a temporal element. By extension, both visuo-spatial and temporal elements are integral to abstract cognition. However, at this stage in the development of situated theory, the focus has been constrained to three-dimensional visuo-spatial perception. Nevertheless, the implication is that situated dynamic sensory-motor activity grounds abstract mathematical objects, concepts, and cognition. This is evidence for the plausibility of a relationship between visual perceptual skill and mathematic ability.

4.6 Situated Mathematics

The situated cognitive science paradigm clarifies the nature of mathematics and mathematical cognition, namely *what* mathematics is and *how* we do it. Situated mathematics is an embodied social abstraction. It is therefore grounded in situated sensory-motor activity. Mathematical cognition is the comprehension, or conceptualisation, and utilisation of abstract mathematical objects. It is an instance of general abstract cognition, which emerges through dynamic mutual interaction with the enacted physical and social world, in which it is embedded and extended.

4.6.1 The Nature of Situated Mathematics

The nature of situated mathematical objects and their relationships is contrary to all other prevailing contemporary mathematical philosophies. Mathematics does not objectively exist in the world, nor is it a set of disconnected abstract symbols and formalist rules. Mathematics is a mental creation. It is not, however, an arbitrary creation *within* the world, as propounded by constructivist and other postmodern philosophies. Mathematics is cognitively created *with* the world. Physical and social evolution, culture, structure and function of the body and the brain, along with everyday environmental interaction, contribute to its formulation and development. Mathematics is thus an emergent phenomenon of situated cognition. From this perspective, the mathematical objects that constitute mathematics are cognitive abstractions.

Unique human biology and mutual human-environmental interaction inherently constrain conceptualisation. It is this mutual interaction that crucially differentiates situated and constructivist philosophies. It elevates abstract conceptualisation beyond concept construction as simply a matter of social convention. In other words, situated experience determines basic conceptual mathematical content, and foundational mathematical cognition, such as perceptual subitization. As an evolving discipline, however, mathematics is equally a consequence of the socio-historical aspect of the human environment. Over time, individuals express and share relevant situated experience, and conceptualisation thereof, through language and cultural mathematical symbols, themselves a consequence of certain situated social practices. As a result, the abstracted linguistic concepts are communal, and incorporated into the public domain of mathematics. In addition, cultural and societal concerns direct the evolutionary path of mathematical progression. Lakoff and Núñez (2000) have described mathematics as “the greatest product of the collective human imagination” (p. 377).

As a product of the imagination – a creative concept or cognitive phenomenon – mathematics is largely grounded in sensory-motor activity, visuo-spatial perception in particular (refer to Section 4.5.4; Sternberg, Lubart, Kaufman & Pretz, 2005). This entails an important consequence for the nature of mathematics: it is fallible. Just as perceptual experience can be contradictory, epitomised by visual illusions, so too can the cognitive conceptualisations arising from it. Mathematics is continually evolving, often discrediting its

own previous 'truths'. This is not a declaration of the speciousness of mathematics. Rather, it properly highlights the embodied, social relativity and abstract status of mathematics.

Classical Newtonian mathematical laws of motion offer a prime example. Conceived of in order to describe the observable macroscopic world, they have stood the scientific community in good stead. In fact, their scientific value persists in spite of having been proven within the context of very fast moving microscopic particles, to be only approximations. Mathematical laws are created simply to enable the comprehension of empirically experienced regularities of the world. Brunshvicg (cited in Merleau-Ponty, 1964b) has put it concisely that "the law is not truer than the fact; the law is conceived exclusively to make the fact intelligible" (p. 20).

Mathematics is, therefore, not universal or transcendent. Yet, mathematics is also not a wholly 'subjective' or culturally specific creation. Common human brain-body structure, physical and social environmental interaction, as well as shared cultural concerns, ensure that mathematics is non-arbitrary (Lakoff & Núñez, 1997; Sfard, 1997). General cognitive mechanisms, such as those listed by Lakoff and Núñez (2000), which include categorisation, spatial-relations schemas, conceptual metaphors, and subitizing, produce mathematics. Núñez (2008) has asserted that mathematical truth is intrinsic to the inferential organisation of metaphorical mappings, and the apparent objectivity of mathematics arises through the combined and co-operative use of these mechanisms within a society. The only mathematics is human-situated mathematics.

Similarly, the effectiveness of mathematics in the world is a result of the interactive connection between situated cognition and the enacted environment. That is, a result of the fact that cognition is biologically and environmentally embedded and extended. Mutually interconnected sensory and motor neural capacities are inherited through evolution in the world. These support a basic number sense and primitive spatial relations. More advanced mathematics arises through general cognitive mechanisms, already evolved as suitably effective in the world as we experience it. Mathematical concepts, through conceptual metaphor, have thus been created in the perceived world – the enacted environment – in order to explicate this world, as it is perceived. Situated mathematics is metaphorical.

4.6.2 Situated Mathematical cognition

The situated perspective asserts that from a base of innate sensory-motor capacities, which supports the basic number sense, all mathematics is experientially learned. It is not ‘discovered’ in the external world. Experiential learning involves conceptual metaphor, which may be embodied or social, perceptual or linguistic. Social conceptual reproduction aids conceptualisation (refer to Section 4.3.3). Mathematics is thus created and sustained by a society of human, embodied language-users. The emergent cognitive capacity for metaphor rests upon the simultaneous recognition of structural similarities and differences (Presmeg, 1997a), a by-product of basic pattern perception, which relies heavily on visual perceptual skill.

Sawyer and Greeno (2009) claimed that a number of studies analysing everyday problem solving support the argument that logico-mathematical reasoning is better understood as emerging from person-environment interactions, rather than as a product of mental symbol manipulation. In particular, Lave’s (cited in Sawyer & Greeno, 2009) analysis of the reasoning displayed by grocery shoppers revealed that the traditional conception of the ‘problem space’, as a representation of a task, is not a stable cognitive structure. It is therefore not amenable to problem solving as traditionally conceived of, in terms of a logical search from the initial state, through a series of sub-goals, towards the final goal. Instead, the implication was that the problem space was dynamically co-constructed by the shopper, the physical environment, other sources of information, and other people in the situation. Sawyer and Greeno thus concluded that learning is always situated.

Learning involves the acquisition of new, or the enhancement of existing skills and knowledge. To this end, a seminal cognitive capacity is categorisation, dependent upon the proficient assimilation of dynamic perceptual information, leading to richer conceptualisation. A rich conceptual base offers greater diversity from which to draw, to support effective metaphor and analogy instantiation, thereby facilitating further learning and reasoning (Davis & Maher, 1997; English, 1997b). Analogy, the comparison of existing source and target concepts, is an effective adjunct to reasoning (Novick & Bassok, 2005). Metaphor, on the other hand, is a constitutive part of the process of conceptualisation. Metaphors create meaning by mapping a known source domain to a new contextual domain, generating a novel target concept (Sfard, 1997).

The field of cognitive linguistics provides substantial evidence that metaphorical mapping is the basis of ordinary cognition and language (Davis, 1984; English, 1997a; Sfard, 1997; Lakoff & Núñez, 2000). Abstract concepts are conceptualised in concrete terms, vestiges of which are evidenced in certain quirks of language use and gesture (refer to Section 4.5.4). Mathematics, which utilises general cognitive structures, thus originates analogous to all abstract concepts in the metaphorical mapping of its grounding sensory-motor inferences. Further systematic, non-arbitrary metaphorical mapping, among the abstract concepts themselves, promotes mathematics beyond its direct sensory-motor grounding. The layering of metaphors explicates the hierarchical nature of mathematical concepts, where the more indirect the grounding, the more abstract the mathematics.

Lakoff and Núñez (2000) defined conceptual metaphor as “a neurally embodied fundamental cognitive mechanism” that allows the use of “the inferential structure of one domain to reason about another” (p. 351). It is the inherent preservation of inferential structure between metaphorical correspondences, originating in concrete experiential concepts, which imbues mathematical objects with their apparent objective reality. Johnson (cited in Lakoff & Núñez, 2000) justified metaphorical mapping via the process of cognitive conflation. Simultaneous activation of two distinct, experientially unconnected brain regions potentially causes cross-domain neural links to develop. This often naturally and automatically results in conceptual metaphor.

Habitual use of a conceptual metaphor, whether at the individual or the societal level, renders it covert. This is a necessary pre-condition for abstraction completion, whereby the concept attains its own identity (Sfard, 1997). Bruner (cited in Sfard, 1997) has provided a simple metaphor for this process as it pertains to mathematics, thus:

Metaphors are crutches to help us get up the abstract mountain. Once up, we throw them away (even hide them) in favour of a formal, logically consistent theory [...] The formal models that emerge are shared [...] and prescribe ways of life for their users. The metaphors that aided in this achievement are usually forgotten (pp. 347-348).

In other words, over time, societal establishment of mathematical concepts and a cultural notational system, permit mathematical reasoning without recourse to experiential substantiation. The transfer of mathematical ideas thus becomes socially mediated. So, learning formal mathematics is, in effect, simply the unconscious use of conceptual metaphorical blends. However, gesture studies (cited in Núñez, 2007) in the field of cognitive

linguistics show that even *dead* metaphorical expressions, which have long been lexicalised and rendered covert, are spontaneously enacted during speech. This serves as evidence that the metaphorical conception of situated experience is *essentially constitutive* of overlying abstractions and, by extension, of abstract cognition such as that of mathematics.

Lakoff and Núñez (2000) have pioneered a comprehensive theory of mathematical cognition, firmly entrenched within the situated paradigm. Beginning with the fundamental cognitive capacities and basic arithmetic, the theory extends to a clear portrayal of the development of the full gamut of mathematical concepts, symbolic logic, and their relationships. The conceptual complexity of deceptively innocuous concepts, such as zero and equal, is uncovered, whereas complex mathematical concepts, such as infinity, limits, and calculus, are reduced to simple layers of metaphorical mappings. For example, Núñez (2007) has contended that, socio-historically, the relative difficulty with the concept of zero stems from the fact that *nothing*, or the empty set, is not directly perceivable, in the same way that *one thing* is. They do accede, however, that their innovations may require some refinement.

4.6.3 Sensory-Motor Grounding of Mathematical Cognition

The cornerstone of Lakoff and Núñez's (2000) theory of mathematical cognition lies in the core, situated sensory and motor cognitive schemas, in which the inferential structure of the most basic sensory and motor experiences are made manifest.

- *Cognitive image schemas* arise from the primitive spatial relations concept, which is dependent on visuo-spatial perception. Image schemas are exceptional in that they are both perceptual and conceptual in nature, providing a link between vision and cognition. By virtue of their image-schematic structure, complex image schemas display an inherent spatial logic, since the logical inference preserved in metaphorical mapping to the abstract is, in fact, a spatial inference. They may therefore be used directly in spatial reasoning, which is an essential component of mathematical reasoning (English, 1997a; Presmeg, 1997b).

The principle image schema in mathematical cognition is the **Container schema**, a gestalt composed of a boundary, an interior, and an exterior. Núñez (2008) has speculated that this schema arises by virtue of the neural topography of the retina, with its centre-surround visual receptive fields and gating circuitry. Sheets-Johnstone (1990) has alternatively attributed such

a schema to the experience of having a body that contains bounded internal cavities. The spatial logic of this cognitive conceptualisation of a physical container is metaphorically applied in the concept *categories are containers*, facilitating a natural understanding of Venn diagrams and Boolean logic.

- **Cognitive aspect schemas** relate to motor control, with inferences metaphorically conveyed to the structure of events. The principle schema concerned with motion is the **Source-Path-Goal schema**. In mathematics, the conceptualisation of the number line, directed graphs, and functions in a Cartesian plane as *motion along a path* are evidence of the logic of this schema.

Emergent cognitive mechanisms, such as conceptual metaphor, extend these schemas of basic sensory-motor experience to the abstract. For example, preserving the Container image schema structure, numbers are conceptualised as collections. The extension of innate perceptual subitization to arithmetic is therefore achieved, through the capacity for conceptual metaphor. Key to this process is the fact that conceptual metaphors convey, and preserve, the inferential structure of cognitive schemas.

Lakoff and Núñez (2000) have theorised that the entire abstract mathematical edifice rests upon four critical grounding metaphors, which are:

1. **Arithmetic is Object Collection:** the most basic grounding metaphor. It arises naturally from regular use of subitizing, innate arithmetic, while interacting with small collections of objects. Basic arithmetic operations neurally link to the physical actions of adding objects to, and taking objects away from, the collections.
2. **Arithmetic is Object Construction:** a more specific metaphor that carries the inferences of the Object Collection metaphor. It is the conceptualisation of numbers as wholes, while at once composed of other numbers.
3. **Measuring Stick metaphor:** a metaphorical blend (which may account for the deviation from the linguistic metaphoric format). It is a blend of physical segments and the abstract numbers specifying their lengths. Segments are particular object constructions.
4. **Arithmetic is Motion Along a Path:** a natural metaphorical correlate to the Measuring Stick metaphor, where the path of motion forms a physical segment. This metaphor provides a natural extension to negative numbers.

Each of these metaphors emerges, through the embodied cognitive comprehension of the enacted world, from the ordinary experience of collecting, constructing, manipulating, and moving. Subsequently, the domain is gradually extended via conceptual blending and linking. For example, the Arithmetic is Object Collection metaphor entails only addition and subtraction. Blending numbers with this metaphor extends it to permit multiplication and division. Lakoff and Núñez (1997) have submitted that the number line, being an amalgam of numbers and points on a line, is a product of the *arithmetic is geometry* blend. This metaphorical blend also functions as a linking metaphor that enables the spatial conceptualisation of arithmetic.

Accompanying these cognitive processes, natural metaphorical entailments arise from basic truths of the grounding domain, an adjunct of the stable and structured world we inhabit. These entailments become mathematical truths that constitute the laws of arithmetic, which are consequently not axiomatic. For example, a basic truth of the Object Collection metaphor is that the experience of adding collection A to B is equivalent to the experience of adding collection B to A. The natural metaphorical entailment of this is the mathematical truth that numerical addition is commutative. Arithmetic is thus extended beyond that which is innate, while preserving its basic innate properties. Symbolising capacity maps numerical concepts to the cultural mathematical notational system, and the common cognitive mechanism of conceptual metonymy generalises arithmetic, to allow for algebra.

Higher mathematics constitutes a similar concept to that of metaphorical basic arithmetic, except in this case, metaphor is built upon metaphor. In addition, each branch of mathematics is metaphorically linked to other branches. Lakoff and Núñez (2000) explicitly detailed the metaphorical linking and blending underlying nearly all complex mathematical concepts, an accomplishment that dispels the enigma of mathematics. They claim that the flawless metaphorical grounding contributes to misinterpretation of the metaphorical nature of the abstract mathematical edifice.

Arithmetic taken exclusively as symbol manipulation dissociates it from higher mathematics. Arithmetic involving conceptualisation, on the other hand, divulges the hierarchical nature of mathematics. The grounding metaphors are simply a mapping from everyday experience onto abstract concepts, which constitute arithmetic. This process is so natural and spontaneous, that it requires very little instruction. Higher mathematics is simply the conceptual extension, the increasing abstraction, of arithmetic. Linking metaphors

connect arithmetic to other branches of mathematics. This is still a simple process, although being less grounded, it requires more explicit instruction, and yields concepts that are more abstract. Within this context, mathematical instruction ideally involves social conceptual reproduction.

4.7 Implications for Mathematics Education

In terms of situated cognitive theory, mathematics emerges from a base of foundational cognitive abilities. Then, in conjunction with normal cognitive development, it is education which extends these foundational abilities to higher levels (refer to Sections 2.2 and 4.6.3).

With this idea in mind, the 2012 South African Annual National Assessments for mathematics depicts an interesting scenario. The following table shows the national averages per grade, as reported by Jansen (2012).

Table 4.1

National Assessments for Mathematics (SA 2012)

	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5	Grade 6	Grade 9
Ave %	68%	57%	41%	37%	30%	27%	13%

In Grade 1, the mathematics is presumably basic enough to be accomplished mostly intuitively, by virtue of foundational cognitive abilities. This grade obtained a reasonable national average of 68%. The mathematics in successive grades must surely require progressively higher levels of cognition, the development of which is in part the responsibility of formal education. Therefore, the concomitant steady decline in national average may be construed as evidence for something amiss in the broader landscape of mathematics education in the country.

Research (cited in Bernstein et al., 2013) by the Centre for Development and Enterprise (CDE) has shown that mathematics education in South Africa is among the worst in the world. Bernstein et al. (2013) claimed that this is partly a consequence of poor mathematics teaching in South African public schools. In particular, the CDE study revealed that Grade 6 mathematics teachers in South Africa were the least competent when compared

with teachers in eight other countries (Jansen, 2013). Bernstein et al. (2013) reported that poor mathematics teaching in primary schools was considered the greatest contributor to poor scholastic mathematics performance in general. The South African government has apparently begun to realise the necessity of urgent intervention to address this crisis in education. However, there is no clear plan of action yet.

According to situated theory, education for the extension of foundational cognitive abilities entails situated learning, through embodied experience, and social conceptualisation and reproduction (refer to Sections 4.3.3 and 4.6.2). Alone, rote learning of formal mathematical facts and formulae can therefore be considered insufficient to a proper mathematics education. This is not to say that previously established mathematical truths are not beneficial. They are in fact vital to the continued expansion of mathematics. However, in tandem with this type of rote learning, a simultaneous attempt at explaining the derivation of these truths is argued as indispensable. In other words, Bruner's (cited in Sfard, 1997) metaphorical crutches, used to assist getting up the abstract mountain, must be retrieved (refer to Section 4.6.2). This could involve teaching strategies such as providing appropriate learning experiences, and social conceptual reproduction. The following examples clarify these ideas:

- ***Rote learning of formal mathematics.***

Meaningless rote mathematics learning is epitomised by the arithmetic rules for complex addition, subtraction, multiplication, and long division. The following provides a rough example of the addition algorithm:

To add $119 + 7$, write the first figure, and then vertically align the second figure beneath the first, right side justified. Add the units' column: $9 + 7 = 16$. Write the right-hand digit in the units' column, and carry the left-hand digit to the tens' column. Add the tens' column: $1 + 1 \text{ carried over} = 2$. Write the answer in the tens' column. Add the hundreds' column: $1 + \text{nothing} = 1$. Write the answer in the hundreds' column. Thus, the answer is 126.

(*Note:* The step involving $9 + 7 = 16$ may be accomplished through finger counting, although rote learning of the addition tables presents a likely alternative.)

These rules are not difficult to memorise, and can be applied to sums of virtually any complexity. Their undisputed value therefore lies in their versatile application. However, these algorithmic rules represent mere cognitive shortcuts. Learning such formal mathematics is an imperative, practical skill, particularly for those times when an electronic calculator is unavailable. However, following a learned sequence of steps is insufficient in the promotion of situated mathematical cognition. Teaching the rules in isolation, divorced from cognitive comprehension of the underlying mathematical ideas, is counter-productive from the situated perspective, where conceptual understanding of each level of mathematics develops the abstract building blocks necessary for exploring ever-higher reaches of mathematics.

Recognition of the conceptualisation of addition often manifests in alternative and more natural strategies for arriving at the correct answer. One such strategy is:

To add $119 + 7$, let '7' give 1 to '119', then $120 + 6$ equals 126.

Intuitively, by virtue of the base-10 number system and the syntax of linguistic number words, $120 + 6 = 126$ seems less cognitively demanding than $119 + 7 = 126$. In fact, expert calculators often report that their speed and efficiency of calculation is attributable to these types of strategies. Regrettably, it is impossible to convert such an addition procedure to an uncomplicated universal algorithm. Not only is it restricted to certain maths problems, but this particular example also pertains only to certain languages. Various other languages feature vastly different number word syntax, which nevertheless support different unique strategies. Yet the ability to convert the sum spontaneously in such a manner is an indication of superior, situated mathematical cognition. The level of abstraction clearly endorses the numbers as distinct entities, in their freedom to give and receive pieces of themselves, which rests on a clear understanding of the relationships between separate number entities. Moreover, it implicitly shows the conceptualisation of numbers as wholes, yet composed of various combinations of other numbers. This is a crucial foundational mathematical concept, described by Lakoff and Núñez (2000) as the *Arithmetic is Object Construction* grounding metaphor (refer to Section 4.6.3).

- ***Appropriate learning experiences.***

Davis and Maher's (1997) 'pebbles-in-a-bag' classroom activity poses a good example of what constitutes appropriate situated learning experiences. Beginning with a bag of pebbles, the learners then put another five pebbles into the bag. Generally, the class

unanimously agrees that there are now five *more* pebbles than before. They then remove six pebbles from the bag. Again, there is consensus that there is now one pebble *less* than the original amount. All the while, the teacher notes the pebble movements on the board. The learners have no trouble comprehending these obvious conclusions, but it is only after ensuring this comprehension that the formal notation of $5 - 6 = -1$ is introduced. This conceptual introduction of negative integers is essentially the exploitation and extension of Lakoff and Núñez's (2000) *Arithmetic is Object Collection* grounding metaphor (refer to Section 4.6.3).

- ***Social conceptual reproduction.***

To show the possibilities of conceptual reproduction in mathematics teaching, Núñez (2007) provides the example of the multiplication of two negative numbers, which yields a positive product. He claims that this is too often taught as a mathematical fact, devoid of any *meaningful* explanation. Although a formal mathematical proof exists, it is equally divorced from human reality. The algebraic proof is, in any case, too complex for the level of mathematics at which the fact is first learned and required. In actuality, though, embodied direction is the conceptual basis for the fact. It is an extension of the grounding metaphor of *Arithmetic is Motion Along a Path* (refer to Section 4.6.3). Various orientations and forward and backward steps along a number line can therefore explain it, simply and meaningfully. Such conceptual reconstruction promotes proper abstraction and enhances mathematical understanding. This type of teaching, together with appropriate learning experiences, has the added advantage of re-connecting mathematics with everyday experiences, potentially improving learners' interest and motivation.

In all three of the above examples, mathematics teaching invoked Lakoff and Núñez's (2000) grounding metaphors. This shows the potential for the practical application of their high-level theory of the origins of mathematical concepts. Poynter and Tall (2005) have wholly embraced the relation of Lakoff and Núñez's theory to practical mathematical teaching. They offer a broader view by integrating the development of mathematical concepts with human conceptual development. In their view, cognitive development proceeds from perception and action, and leads to abstract conceptions through reflection. Focusing on this development, they theorise that the shift from a procedural action to object conception is particularly problematic for a child to master. A deficiency in this shift results in learning *how* to do mathematics, without the associated understanding. They suggest that a slight

alteration in attention is required, to focus on the embodied *effect* of the action. They believe that this will foster mathematical learning that is conceptual, rather than procedural.

Ideally, therefore, mathematics teaching should encompass both factual and conceptual learning, accomplished with teachers facilitating a degree of conscious conceptual reconstruction for learners (Sfard, 1997). The benefit of this would be the promotion of proper abstraction, and enhanced mathematical understanding. Devising a syllabus based on situated metaphorical mathematics, including conceptual metaphors rather than solely pre-given axiomatic mathematical facts, has the added potential to relieve the current tension between an absolutist view of mathematics and the progressive education ideology (refer to Section 2.3.2). These insights raise reservations as to whether mathematics is intrinsically difficult to learn, or whether teaching methods are simply inappropriate.

If the prediction of situated cognitive theory that a relationship exists between visual perceptual skill and mathematic ability proves to be true, then it would be imperative, on school entry, to ensure that learners possess adequate visual perceptual skill for ensuing abstract cognitive development. Building from a base of innate sensory-motor abilities, perceptual skill is experientially learned (refer to Sections 1.1, 3.2 and 4.5). Like all cognitive skills, appropriate learning experiences must therefore be effective in enhancing visual perceptual skill (Halpern & Collaer, 2005). After attaining an adequate level of visual perceptual skill, the focus should then shift to the application of this skill in learning mathematics. This is a crucial point, which perhaps requires more emphasis in the didactics of mathematics.

Under the influence of the current theoretical paradigm, Kavale (1982) may be considered to have been remiss in his exclusion of this significance of learning through visual perceptual experience in his list of reasons pertaining to the continued controversy over the relationship between mathematic ability and visual perceptual skill (refer to Section 3.4.1). The claim can be made that adequate visual perceptual skill, possessed by the majority of normally developed children, is insufficient. It can be argued that, in addition, learners need to be taught how to use this skill in their mathematics learning. That is, learners need to know how to effectively implement imagery and metaphor in mathematical learning and problem solving.

In support of the claim that the mere possession of the necessary visual perceptual skills is insufficient for conceptual learning in mathematics, Presmeg's (1997b) personal

teaching experience in a South African high school provides substantiation. Two particular learners, shown to have superior visual perceptual skill, regularly scored over 80% for mathematics under her imagistic and visuo-spatial mode of tuition. When the author left the school, the replacement teacher unfortunately taught mathematics algebraically and non-visually. Thereafter, both learners failed mathematics. Possession of the cognitive tools is superfluous without appropriate guidance and teaching.

According to Lakoff and Núñez (2000), abstract mathematical cognition is particularly dependent upon the visuo-motor and visuo-spatial experiences of collecting, constructing, manipulating, and moving. An early mathematics class, therefore, should resemble 'supervised mayhem', with the learners counting, collecting, constructing, manipulating, and moving, preferably in a wide-open space. With solid sensory-motor grounding, mathematics education thereafter entails exposition of successive metaphorical layers. This demands the continued utilisation of visual perceptual skill, to enable the abstraction required for further metaphoric creation, and comprehension of existing social metaphoric conceptualisations.

In summary, mathematics education with a focus on rote learning of arithmetic tables, formulae, and algorithms amounts to teaching exclusively how to do mathematics, but is deficient in the requisite associated understanding. This disregards both progressive educational psychology (refer to Section 2.3.2) and the cognitive science view of situated mathematics and cognition. The situated cognitive paradigm grounds mathematical cognition in sensory-motor activity. Consequently, the current cognitive scientific emphasis in the development of foundational mathematical cognition, rests on the ability to move, manipulate, collect, and construct. It is therefore fundamentally incongruous for mathematics to be rote-learned from a textbook, sitting at a desk. This is particularly relevant in primary school mathematics learning, before extensive metaphorical layering substantially distances the mathematics from its sensory-motor foundations.

4.8 Summation

Situated theory presents a holistic theory that reveals an intimate connection between perception and cognition. Situated cognitive scientific insights, in concert with neuropsychological data on innate visuo-motor capacities, thus lend theoretical plausibility to

the hypothesis of an intimate relationship between visual perceptual skill and mathematic ability.

Substantial evidence (refer to Sections 2.2.1 – 2.2.5) reveals an indisputable degree of foundational mathematical cognition, in which the highly evolved visual system plays a pivotal role. Visual discrimination, visual memory, visual-motor integration, and spatial relations are crucial visual perceptual component skills that enable the innate number sense, and its extension to arithmetic and beyond. Spatial relations are particularly critical. This may possibly be due to the inherent cognitive number-space association, evidenced by the spatial nature of the mental number line, which is the natural conceptualisation of number. The correlation between spatial abilities and mathematic ability has been widely published (Dehaene, 1997; Guay & McDaniel, 1997; Wheatley, 1997; Bryant & Squire, 2001; Fias & Fischer, 2005; Halpern & Collaer, 2005; Hegarty & Waller, 2005).

Dehaene (1997) and Giaquinto (2007) have both been cited (refer to Section 2.3.5) as acknowledging the essential visuo-spatial nature of arithmetical cognition, exceeding that of superficial numerical decoding. However, they then revert to a formalist rule-based arithmetic in their explanation. Overall, Giaquinto has presented an insightful theory on the role of visualisation as a pertinent means of mathematical discovery; within which lies a paradox. The assessment of mathematical philosophies presented in Section 2.1 reaches the conclusion that mathematics is not *discovered*, but is cognitively constructed. Furthermore, although his theory aptly rests on phenomenal perceptual concepts, Giaquinto later reduces these to lists of ‘category specifications’ reminiscent of the information-processing theories of the conceptual system. A philosophical shift from entrenched Cartesian dualism, information-processing, and formalism, may benefit many existing theories of visual perception and mathematical cognition.

Lakoff and Núñez (2000) appear to have accomplished this shift, with their tentative theory of mathematical cognition, situated within the situated cognitive scientific paradigm. The theoretical crux is that mathematical conception is metaphorical, and based on perception. Their theory thus discloses a firm relationship between visual perceptual skill and mathematical cognition. Operation of the visual system is not restricted to vision; it includes mental imagery, and therefore contributes to abstract conceptualisation. Additionally, the prefrontal cortex neurally links the visual system to the motor system, allowing the bodily utilisation of cognitive motor schemas in order to trace cognitive image schemas. This

provides the sensory-motor grounding, and basic inferential structure from which mathematics originates, and derives its meaning. A base of neuropsychologically confirmed innate sensory-motor skills support numerosity and perceptual subitization, from which Lakoff and Núñez provide a convincing argument that all mathematics is constructed through conceptual metaphor, with increasing abstraction as metaphor is layered upon metaphor.

Findings related to the mental number line provide evidence for the metaphorical nature of mathematics. Originally believed to be the result of number-detecting neurons, due to its spontaneity in development, neurophysiologic evidence (cited in Fias & Fischer, 2005) has since shown the absence of any such topographical neuronal map. The mental number line is better conceived as a basic conceptual metaphor, capturing and preserving the inherent spatiality of number. Further up the mathematical hierarchy, calculus also clearly reveals the metaphorical nature of mathematics. Kaput (cited in Núñez, 2007) explained that basic calculus attains its meaning through a number of motion metaphors, which patently manifest in the notation and social discourse vocabulary associated with calculus. Unfortunately, the formal mathematical definitions capture only limited aspects of the metaphorical concepts. Núñez (2007) contended that limitations such as this may contribute to the naïve disregard for the underlying meaningful metaphors accompanying formal mathematics learning. The disconnection of formal definitions from human-situated motion may account for the ostensible counter-intuitiveness of calculus.

The metaphorical mathematics of Lakoff and Núñez (2000) fosters appreciation for Dehaene's (1997) and Devlin's (2000) insistences for the necessity of proper abstraction of number beyond that of the basic number sense, in successful higher mathematics (refer to Section 2.2.6). Sound metaphorical abstraction of number provides an essential foundation, a metaphorical grounding metaphor, on which to base the next hierarchical layer of metaphor: the linking metaphors, connecting higher mathematics to arithmetic. Indeed, each of Devlin's levels of abstraction may be comparable to conceptual metaphorical layering (refer to Sections 2.2.6 and 4.3.2). The lowest level equates to Lakoff and Núñez's grounding metaphors, with levels two and three each incorporating an additional metaphorical layer, the kind of abstraction possibly sufficient to allow basic arithmetical cognition. The final level, representative of mathematical cognition, equates to an entire metaphorical network, including layers, links and blends.

Despite the shortcomings in their theories, both Piaget's (1963, 1969) and Giaquinto's (2007) ideas also resonate with those of Lakoff and Núñez (2000), providing further theoretical support for Lakoff and Núñez's theory of mathematical cognition.

- Piaget's belief (refer to Section 3.2) in the grounding of abstract mental operations in embodied sensory-motor actions, albeit within an objective world, may nevertheless be regarded as a prototype for Lakoff and Núñez's theory of the same, but which is embedded within an enacted environment. Despite concern over the timeline, Piaget's theoretical account of self-constructed mathematical conceptual knowledge passes consecutively through Devlin's (2000) levels of abstraction, which are comparable to the development of Lakoff and Núñez's network of conceptual metaphor.

- In Giaquinto's theory of the epistemologically visual nature of mathematical cognition (refer to Section 2.3.5), as opposed to pure logic, he posits that abstract general truths can be discerned through specific visual images. Giaquinto described this process as occurring 'via inference, generalisation, and analogy', although he may reasonably have substituted *via metaphor*. The 'weak intrinsic connection' he draws between analytical and perceptual concepts – which he claims poses the challenge in calculus – is analogous to calculus being more metaphorical and abstract than geometry. Giaquinto's reasoning, albeit expressed in an alternative terminology, is thus in line with that of Lakoff and Núñez.

Giaquinto's (2007) concern, however, over abstract concepts, which he termed analytic concepts, extending the calculus domain to the unvisualisable, is unfounded, since this is precisely the purpose of conceptual metaphor. Visual perceptual concepts provide the basis for conceptual metaphor, which may be layered ad infinitum in the development of abstract concepts. This often metaphorically distances abstract concepts from their visual perceptual base, rendering this base unrecognisable, yet preserving the original visual inferences. The root of Giaquinto's confusion may lie in the nature of mathematics itself.

Situated mathematics does not exist in the physical world. It is an abstract cognitive creation, and therefore cannot be observed. However, nor is it a formal analytical creation resting on a set of assumptions. It is a creative abstract conceptual construction of a situated cognition. Specifically, it is created by abstracting general patterns from concrete sensory-motor instantiations. Situated sensory-motor experience therefore grounds the meaning of mathematics. Visual perception, which is directly connected to motor activity, is a major component of sensory-motor experience. If mathematics is grounded in visual perception, it

sensibly follows that the same neural structures, and their concomitant cognitive functions, are involved in actual mathematical cognitive activity. This authenticates the plausibility of an intimate relationship between visual perceptual skill and mathematic ability.

The fundamental natures of situated visual perception and situated mathematical cognition are comparable. Neither visual perceptual properties, such as colour, nor mathematical objects, namely numbers, exist externally; they are instead a product of situated cognition. There are a limited number of perceptually universal colours, the balance being culturally specific. Similarly, from the limited embodied sense of numerosity, mathematics is socially created according to cultural concerns. Visual perception is experienced in action, with object discrimination and position composed through potential action. Likewise, the motor metaphorical actions of collecting, constructing, manipulating, and moving, establish mathematical cognition. Visual perceptual skill is the dynamic and interactive co-ordination of situated sensory-motor activity with the enacted environment. This same sensory-motor activity grounds the regress of mathematical meaning. This common relation to action naturally endows both visual perception and mathematical cognition with a strong spatial component.

Situated theory consistently emphasises visuo-spatial perception, supported by the spatial relations visual perceptual component skill. In particular, perceptual space, via metaphor, provides for abstract conceptual cognition, the epitome of mathematical cognition (refer to Section 4.5.4). This, together with the inherent spatiality associated with the primitive number sense, suggests the possibility of a relative importance for spatial relations in mathematical cognition. Nevertheless, the theory of situated cognition equally implies the broader relation between visual perception and mathematical cognition. The situated cognitive paradigm therefore provides a coherent alternative to traditional cognitive theories. The holistic and dynamic brain-body-world conception eradicates the Cartesian duality of the world, and its associated limitations. Situated theory clarifies the natures of visual perception, cognition, and mathematics, offering a viable explanation for a possible relation among them. Sharing more than a mere causal relationship, situated visual perception and mathematical cognition are co-constituted emergent cognitive capacities, through embodied sensory-motor activity embedded in, and extended into the enacted physical and social environment.

A relationship between visual perceptual skill and mathematic ability is therefore a reasonable expectation, from the perspective of situated cognition. In further exploration of

this relationship, a small investigation was undertaken to test the hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement. Determining the relative contribution of visual perceptual component skills is expected to reveal the comparative significance of spatial relations.

CHAPTER 5: Research Methodology

5.1 Conceptualisation

The plausibility of arguments from current cognitive theory, combined with empirical and developmental evidence intimates a link between perceptual skills, predominantly visual, to all types of learning, through conceptual development. This is particularly evident since the advent of situated cognitive theory. The existence of the more specific relationship between visual perceptual skill and mathematic ability may be inferred from this broader link.

To explore this possibility further, an empirical investigation was designed to test the hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement. In the context of the poor status of scholastic mathematics achievement across all grades in South African schools, this specific hypothesis was selected over the broader relation of visual perceptual skill and mathematic ability. For clarification, and to assist in developing practical strategies to enhance mathematics education, the relative contributions of various visual perceptual component skills were also determined.

In the absence of agreement on construct definitions, for the purposes of this exploratory investigation, visual perceptual skill was conceptualised in terms of assessment of the set of component skills that constitute visual perception as measured by the third edition of the Test of Visual Perceptual Skills (TVPS-3). This instrument specifies seven component skills essential for visual perceptual identification and spatial perception, including visual discrimination; visual memory; spatial relations; form constancy; sequential memory; figure-ground; and visual closure. Scholastic mathematics achievement was operationalised as the performance on a grade-appropriate mathematics test.

5.2 Research Design

Investigation into the veracity of the hypothesis required a one-group correlational study. The visual perceptual skill and mathematics achievement of a representative sample of Grade 6 learners were assessed. This implied the collection of primary, empirical, numerical data, rendering the study as quantitative. The significance of correlation between the visual

perceptual component skills, overall composite skill, and scholastic mathematics achievement, was determined.

The focus of the hypothesis on *scholastic* mathematics achievement, rather than pure mathematic ability per se, suggested a natural classroom setting for the data collection. Such a natural setting had the advantage of decreasing experimenter and laboratory effects, and increasing the generalisability of the results. While this design does allow for statistical analyses of the results, the main limitation of a natural setting is the lack of control of extraneous factors (Mouton, 2001). Usually, care to control context effects is called for in this type of design. However, the classroom context, and its effect, seemed to be an important factor in the measurement of scholastic mathematics achievement, particularly within the situated cognitive theoretical framework.

The study may be further categorised as fundamental exploratory research, in its aim of providing empirical support for the hypothesis derived from recent cognitive theory. It is hoped that this may subsequently assist in improving practical educational methodologies.

5.3 Sampling

5.3.1 Technique and Criteria

A natural correlational design generally calls for simple random sampling. However, due to the sheer scale of the possible research population of schoolchildren, this was deemed impractical. Convenience sampling, restricted to a defined area, was thus utilised. Every effort was taken to eliminate experimenter bias, thus improving the accuracy of sample representativeness.

Based on the principle of quota sampling (Van Vuuren & Maree, 1999), two important sub-groups of the population were identified: those of high and low socio-economic status, respectively. It was taken that economic status would possibly influence the life-style of the participants, and the quality of education received, thereby potentially affecting the measurable variables. To ensure representative diversity, non-random samples were selected from each sub-group.

The non-random selection of samples involved first approaching all the English and dual medium primary schools in the Stellenbosch area of South Africa, to request permission to conduct the study in their school. The idea was to select two government schools; one situated in a wealthy area, and the other in a previously disadvantaged area. This was seen to ensure an unbiased mix in terms of gender, racial, cultural, and socio-economic factors. However, when only one school showed interest, the sample area had to be widened to include Somerset West. Ultimately, the sample consisted exclusively of the only two schools within this area that granted permission for the request, plus a sister-school in the Southern suburbs of Cape Town. One was a government school in a previously disadvantaged area, and the two sister-schools were private schools in affluent areas. Fortunately, the number of participants from these three schools was, roughly equally, divided between the two sub-groups based on socio-economic status. Since there was a minimal active role in deciding sample inclusion, negligible bias was introduced. Hence, there was no danger of invalidating probability statistics.

All three schools were co-educational. Therefore, it was reasonable to suppose that the proportion of male and female learners were already representative of their respective sub-populations. In addition, all three schools were dual medium, offering classes in either English or Afrikaans. One active selection criterion was to include only the learners from the English stream, since I was not sufficiently confident to conduct the study in any other language. Yet, this could be construed as an advantage, in the elimination of language as a variable. To curb further bias, *all* English stream Grade 6 learners were invited to participate in the study. All the parents of those invited agreed to allow their child to participate (see Section 5.5).

Despite this support from parents, the target of between 100 to 150 participants was not reached. In these dual medium schools, there was invariably only one English class per grade, and these classes were exceptionally small. A sample of only 70 participants was achieved. This however fell within the size range of previous similar studies. Although the more recent studies of the twenty-first century, cited in Section 3.4.1, comprised between 100 and 200 participants, the 221 older studies incorporated in Larsen and Hammill's (1975) review and Kavale's (1982) meta-analysis, ranged from as low as 20 participants. Furthermore, a sample size of 70 was well in excess of the required 20 observations necessary for the statistical significance testing utilised in this study. It was also in line with the acceptable ratio of participants to independent variables recommended for multiple

regression analysis. Generally, approximately 10 times more participants than independent variables, is considered acceptable (Statsoft, 2011).

Only one grade, Grade 6, was elected from which to select participants to ensure that all participants were of an equivalent educational level. A range of grades would have necessitated the compilation of several mathematics tests of varying levels. The difficulty in ensuring inter-test equivalence posed a potential threat to validity. Grade 6 was partly an arbitrary designation, but several reasons supported the decision:

- Mathematics becomes increasingly abstract in secondary school, and is therefore more dependent on explicit instruction (refer to Sections 4.6 and 4.7), potentially confounding a possible relationship with visual perceptual skill.
- The human brain physically matures between 11 and 13 years of age (Louw, Van Ede & Ferns, 1998), facilitating the transition from primarily concrete to increasingly abstract cognition. This is a crucial period, therefore, in mathematical cognition. It is conceivable that it is at this critical juncture that a child first begins to lag mathematically. I was loath, however, to disrupt Grade 7 classes, which is an important foundational grade for secondary school, and Grade 6 was chosen as a suitable alternative.
- All the empirical studies cited in Section 3.4.1 were conducted with participants from primary school or kindergarten. Of the 60 studies reviewed by Larsen and Hammill (1975) 15% included Grade 6. Guay and McDaniel's (1977) study incorporated Grades 2 through 7. The average age of the balance of the studies was only slightly lower than the present study, ranging from around 8 years to just under 11 years of age. Growth of any body of comparable empirical evidence enables more robust conclusions to be drawn.

It was decided not to implement a vision screening for the identification of those participants with visual disorders, thereby ensuring their inclusion in the study. Indeed, sub-optimal visual acuity would adversely affect the visual perceptual skill assessment, through its direct negative effect on visual discrimination, which is the most basic underlying visual perceptual component skill. However, the naturalistic and scholastic focus of the study demanded participants' *habitual* visual acuity during testing. Low mathematics achievement due to poor visual acuity, or visual discrimination, presents an important correlate for the purpose of this study. The elimination of candidates with vision problems would effectively equate to the selection of a biased sample of participants with superior visual perceptual skill.

In consideration of the above, the sampling methods were considered to adequately address the two main concerns of sample selection: representativeness, and size, with the introduction of very little bias.

5.3.2 Sample Profile

The sample consisted of 70 Grade 6 learners from the greater Cape Town area, of which 32 were male and 38 were female. The age of the participants ranged from 10 years 7 months to 13 years 8 months, with an average age of 11 years 8 months. The government school provided 34 of the participants, classified as low socio-economic status, to the study. The two private sister-schools contributed 21 and 15 participants, respectively, totalling 36 for the high socio-economic sub-group. The home languages of the participants were unknown, but the sample was obtained wholly from the English streams of bilingual schools. Remedial extra-maths lessons, based on the school curriculum, were attended by a total of 10 learners, 6 of whom were classified as low socio-economic, and 4 as high socio-economic. This equated to 14.3% of the sample, distributed unevenly across all the schools, with the majority attending the public government school.

5.4 Measurement

5.4.1 Visual Perceptual Skill

The commercially available TVPS-3 was utilised to assess participants' visual perceptual strengths and weaknesses. This was the most suitable of the perceptual tests that I could ethically and competently utilise.

- ***The TVPS-3***

The TVPS-3 is a revision of a test originally authored by Dr. Gardner (see Martin, 2006). It is an untimed test, intended for administering to individuals between the ages of 4 years and 18 years 11 months. Development and standardisation of the test, reliability and validity issues, and test norms are detailed in the test manual (Martin, 2006).

In general, the TVPS-3 is considered internally and temporally consistent. The values of coefficient α , the Spearman-Brown coefficient, and the test-retest correlation, are all equal

to 0.96, with the TVPS-3 relatively free from error (95% CI = 1.88) and comprising highly homogenous content. Although test validity requires continuous re-evaluation, at the time of publication the TVPS-3 displayed moderately strong content validity, criterion-related validity, and construct validity. Correlation with the visual subtest of the Developmental Test of Visual-Motor Integration produced a coefficient of 0.67.

Although visual perceptual component skills are not independent, seven skills are functionally described and assessed by the TVPS-3:

1. **Visual Discrimination:** While a design is displayed, the child must choose the matching design from a series of designs.
2. **Visual Memory:** A design is shown for five seconds. From memory, the child must then choose the matching design from a series of designs.
3. **Spatial Relations:** A series of designs is displayed, and the child must choose the one that is partly or wholly different in detail or rotation.
4. **Form Constancy:** While a design is displayed, the child must find the same design, although it may be larger, smaller or rotated, within a series of designs.
5. **Sequential Memory:** A design sequence is shown for five seconds. From memory, the child must then choose the matching sequence from a series of sequences.
6. **Figure-Ground:** While a design is displayed, the child must find the same design within a series of complex background designs.
7. **Visual Closure:** While a design is displayed, the child must choose the matching design from a series of incomplete designs.

The test comprises two example items and 16 tasks for each sub-skill, with a multiple choice answer format. No basals are imposed, but a subtest ceiling is set as three consecutive incorrect answers. All task designs are black and white, and arranged in order of difficulty.

The chronological age of the child is calculated, and used to convert the subtest raw scores to scaled scores. These range in value from 1 to 19, with a mean of 10 and standard deviation of 3. The overall score and the basic, sequencing, and complex index scores are derived from the sum of the subtest scaled scores. These are presented as standard scores, with a mean of 100 and standard deviation of 15.

The purported advantages of the TVPS-3 are the flexible response mode, the clarity of the black on white test designs, and the non-language related tasks (Martin, 2006), which afford all children equal opportunity. The neutral test patterns also exclude a cultural bias.

The test is as motor-free as possible, so as not to confound the visual perceptual measure. However, the omission of visual-motor integration precludes a comprehensive assessment claim.

- *Administering the Test*

For the purpose of this study, the test administration verbal instructions and scoring procedures, as per the test manual (Martin, 2006), were followed. However, four important test considerations were violated:

- The test was not administered individually, as prescribed. Instead, it was administered simultaneously to all participants within each school, as a group.
- The group setting necessitated participants writing their answers down, as opposed to the suggested verbal or pointing communication mode.
- The test manual sets subtest ceilings at three consecutive items answered incorrectly, after which the test moves on to the next subtest. However, practical constraints in determining subtest ceilings, a consequence of group testing, demanded that each group complete the entire test.
- The specified test environment is to be quiet, comfortable, and free of any distractions, including other people. The normal classroom situation in which the test was administered represents the antithesis of this description.

A child learns and practices mathematics in a classroom, with its attendant disruptions. Furthermore, slower learners are forced to keep pace with the class. If visual perceptual skill is a factor of mathematic ability, and so contributes to learning mathematics, then it is the visual perceptual skill available within *this* context that is of importance. The TVPS-3 was thus administered collectively at each school, in the classroom. The test plates were projected onto a screen, and the participants wrote down the number associated with the correct multiple-choice option. When approximately half the class had completed a task, the rest of the learners were encouraged to guess at an answer, after which the next task was presented. This procedure emulated a contextual environment similar to that of a typical mathematics class.

Disregard for the precise test conditions would undeniably affect the reliability and validity of the resulting measure. However, for this study, the assessment of the participants' habitual cognitive resources, within the natural classroom environment was of greater import

than pure visual perceptual ability. Furthermore, the adaptations to the test instructions generated testing conditions similar to that of the subsequent mathematics assessment. In this case, the TVPS-3 test-retest reliability would be low, yet it would simultaneously demonstrate high construct validity.

5.4.2 Scholastic Mathematics Achievement

Performance on a grade-appropriate mathematics test served as a measure of scholastic mathematics achievement. The use of an available standard mathematics achievement test was rejected, because of their reputation for an emphasis on computational skill (Wheatley, 1997).

The difficulty in garnering unmitigated support for the study, from the majority of the schools approached, hindered the acquisition of actual class maths tests, from which to form an integrated unbiased grade-appropriate test. Their overt lack of enthusiasm in allowing access to their learners was a deterrent to requesting a copy of a maths test. A standard mathematics test was therefore created, in accordance with the national curriculum guidelines. This proved beneficial as the guideline criteria ensured that the test conformed to clearly identified standards of ability. A copy of the mathematics test is appended as A1.

The Outcomes-Based Education (OBE) system, in place in 2010, identified certain skills, or learning outcomes, which needed to be mastered in each grade. Since data collection took place early in the academic year, the mathematics test created was based on the learning outcomes pertaining to the Grade 5 mathematics syllabus. This ensured that all the participants had theoretically covered the relevant class work. Mandy Facer (2008), in the *Oxford Blitz Maths* series, describes the learning outcomes appropriate for Grade 5 as follows:

- **Numbers, operations and relationships:** Recognise, describe and represent numbers and their relationships, and to count, estimate, calculate and check with competence and confidence in solving problems.
- **Patterns, functions, and algebra:** Recognise, describe and represent patterns and relationships, as well as to solve problems using algebraic language and skills.
- **Space and shape:** Describe and represent characteristics and relationships between two-dimensional shapes and three-dimensional objects in a variety of orientations and positions.

- **Measurement:** Use appropriate measuring units, instruments and formulae in a variety of contexts.
- **Data handling:** Collect, summarise, display and critically analyse data in order to draw conclusions and make predictions, and to interpret and determine chance variation (p. 7).

Questions comprising the mathematics test were adapted from a broad spectrum of the practice activities presented in *Oxford Blitz Maths Grade 5* (Facer, 2008). The first 11 questions involved number and pattern recognition and representation, counting, common and decimal fractions, and basic arithmetic calculation. Question 12 required solving number sentences in algebraic language. Questions 13 to 16 concerned estimation and measuring, which necessitated an understanding of measuring units and three-dimensional objects. Questions 17 to 23 were 'story sums', epitomising contextual problem solving. Question 24 presented a graph for interpretation, an aspect of data handling. The total mark allocation for the test was 101.

The *Oxford Blitz Maths* series of books were designed for both teachers and parents to assist a child in mastering the mathematics required for the South African school curriculum. The teacher of one class involved in the study admitted to routinely relying on this series in his teaching. The recognition, by educators as well as the Department of Basic Education, of the *Oxford Blitz Maths* series served to attest that the derived mathematics test was reliable, stable and valid. There was therefore no further reliability and validity testing, save pilot testing with two Grade 6 children, in order to gauge their reaction and performance. Validity of the test was confirmed by the teachers' assertions that their learners' test scores were closely aligned with their typical performance in class maths tests.

5.5 Data Collection

The first step in collecting the data was the identification of potential participatory schools. A letter (Appendix A4) requesting their participation was delivered to viable schools, between March 2009 and February 2010, within an ever-increasing area. Appointments were made with those willing to assist. A week before the scheduled test date, a letter of consent (Appendix A5) was sent home with suitable learners, for their parents. Should permission be granted, the letter was to be signed, and returned the day before assessment.

As a courtesy to all those involved, the results of the assessments were made available to both the school and parents of the learners involved. School principals received the letter appended as A6, together with their learners' mathematics test results and original TVPS-3 scoring sheets. Simplified visual perceptual skills evaluation forms (Appendix A7) were sent to the parents. The participants each received a rainbow pencil to thank them for their effort.

The testing was devoid of any physical or psychological harm to the participants. The required tasks were similar to activities they perform in the normal school context. Furthermore, confidentiality of the names of the participants, as well as the schools involved, was maintained. There were no ethical violations.

5.5.1 Context

I administered and scored all tests myself, in order to minimise experimenter effects. At each school, both tests were conducted on the same day, in the morning when the learners were mentally fresh. The TVPS-3 was administered first and then the mathematics test thereafter, with a 10 to 15 minute break between them to prevent mental fatigue. During the break, the participants were encouraged to run around outside. The TVPS-3 took between 40 and 50 minutes to complete, and one hour was given for the completion of the mathematics test.

Testing took place at School 1, the low socio-economic sub-group, on 26 January 2010, from 9 am. The classroom did not have provision for a projector and screen, so the learners were provided desks and chairs in the library. The teacher was not present. The majority of the class was unruly and disinterested. Unfortunately, they were not allowed to leave the venue, so between the tests, we expended some energy by dancing and stretching.

School 2 was visited on 28 January 2010 at 8 am. The classroom had state of the art projection equipment, and the teacher was so excited about the study that he also completed the mathematics test. He asked to borrow their papers, once I had finished marking them, so that he could go through them with his learners, as a supplementary maths lesson.

With the unexpectedly small class size at School 2, it was necessary to source further participants towards the high socio-economic sub-group. I was referred to their sister-school. Testing at School 3 therefore took place a little later, on 9 March 2010, from 9 am. The teacher was present, and the class displayed a high level of discipline. As the children

completed the mathematics test, they either selected a reading book from the bookshelf in the classroom, or quietly did other schoolwork, while waiting for the rest of the class to finish. This appeared to be standard practice.

5.5.2 Collection Procedure

- *Visual perceptual skill*

The TVPS-3 was administered, contrary to test instructions (refer to Section 5.4.1), to each class as a group. The test plates were projected onto a screen, maintaining the advantageous black on white clarity, but introducing a distance vision component, which is an important facet in a classroom situation. All verbal instructions given to the participants were issued as per the test manual.

Pointing to the correct option, however, was not a possible response mode within the group setting. In addition, the authentic response forms provided with the TVPS-3 reveal the correct answers. Therefore, an alternative answer sheet was designed (Appendix A3), on which the participants had to write down the number associated with the correct multiple-choice option. This introduced a motor component, contrary to the aims of the TVPS-3, but was deemed advantageous in simulating a natural mathematics class context.

Another deviation from the test instructions was the completion of the entire test. Normally, the ceiling system is utilised in order to curb the child's wasted effort and frustration in test items beyond their level of competence. However, within a group, composed of varying levels of competencies, there could be no immediate awareness of when subtest ceilings were reached. However, since each participant had to complete the entire test, there was consequently no bias.

The responses were later transcribed onto genuine TVPS-3 scoring forms, and scored as per the instruction manual, with the application of subtest ceilings. For the purpose of the parents' simplified evaluation forms only, the visual perceptual skill subtest and overall scores were categorised, based on the associated TVPS-3 percentile rank. A percentile rank of between 38 and 62 was categorised as average. Below average and above average corresponded with 16 to 37 and 63 to 84, respectively. A percentile rank of below 16 was categorised as inferior, and above 84 as superior.

- ***Mathematics achievement***

The participants were given one hour to complete the mathematics test. The test incorporated a cover page, so that participants could begin only when told to do so. Many finished within the hour, and were encouraged to check their work and re-attempt those questions that they had omitted. At the end of the hour, all participants were instructed to stop, and put down their pencils.

It was explained, before beginning, that the test did not become progressively more difficult. They should, therefore, do what they could first, leaving anything with which they were struggling. If they had time at the end, they could revisit any problematic questions. They were also shown the blank pages opposite each typed page of the test, and encouraged to use this space for any jotting or ‘working out’ which may have been necessary to assist them in solving the problems. I explained that I would be interested in seeing how they arrived at their answers. These instructions were re-iterated throughout the duration of the test. Despite the constant reminder, participants were still frequently observed to surreptitiously erase their jottings.

The mathematics test was marked as per the schedule appended as A2. Detailed mark allocation instructions were devised to enhance reliability. The test actually totalled 101, but for the sake of simplicity, participants’ totals out of 101 were treated as their percentage achievement score.

5.6 Data Capturing

After scoring the TVPS-3, and marking the mathematics tests, the results were carefully tabulated on a Microsoft EXCEL[®] (2007) spreadsheet.

The participants had filled in their name, date of birth, and school name on the TVPS-3 response forms. The date of birth provided was used to calculate each participant’s age in months. All this information was preserved during the transcription onto genuine TVPS-3 scoring forms. Likewise, the cover page of the mathematics test included the following

information: participants' name, gender, school name, and whether they had ever attended extra-maths lessons. The categorical data options were assigned a numerical code.

Consequently, the TVPS-3 scoring forms and the mathematics test manuscripts together enabled the construction of a comprehensive data table. The original data table thus incorporated fifteen columns: name, age, gender, extra-maths lesson attendance, school, socio-economic status, raw math percentage, overall visual perceptual skill (VPS) score, and a column for each of the seven visual perceptual skill subtest scores. The test manuscripts were ordered alphabetically within each school group, and the data entered into the table in that order. This facilitated the subsequent verification of the data entered. Once the accuracy of the data was ensured, the column listing the participants' names was amended to a case number.

To assist with the choice and understanding of statistical analyses, the data was clarified by means of a data specification sheet, appended as A8. During the course of the data analysis, a decision was taken to standardise and scale the raw math test results. Hence, a scaled math score column was added to the data table, and the additional data specified.

Having the data captured on an EXCEL[®] spreadsheet was advantageous, because statistical analyses were possible with no further data transcription. Analysis was either executed within EXCEL[®] or, where an alternative statistical package was utilised, EXCEL[®] compatibility permitted direct importation of the data table. This served to minimise possible transcription errors.

5.7 Data Analysis

Before investigating the correlation between visual perceptual skill and scholastic mathematics achievement, the data was described and it was determined whether any other known variables had a bearing on the test results. The MoonStats CC (© 2001, 2002) software package was used for the basic descriptive statistics, the Pearson's product-moment correlations, and the *t* tests. It was selected for its user-friendly interface and clarity of output. However, it does not extend to statistical analyses that are more complex, such as ANOVA, and multiple regression analysis, which were therefore accomplished using EXCEL[®] (2007).

5.7.1 Data Description

Basic statistics elucidated the sample constitution, and provided a summary depiction of the data collected.

Frequency counts were carried out for each category of the nominal categorical school group variable and the nominal dichotomous variables of gender, extra-maths lesson attendance, and socio-economic status. The nature of the sample was further described by cross-tabulation with the test results.

Participants' age was the only sample variable continuous in nature. The mean and range were calculated, and the data from a grouped frequency count was plotted on a graph to show the distribution of the age variable. Skewness and kurtosis were calculated to assess the degree of normality.

The mean, standard deviation, minimum, and maximum descriptive statistics were calculated for each of the test results. For each test, the data from a grouped frequency count plotted on a graph, aided visualisation of the distribution of the results. Again, skewness and kurtosis were calculated to assess the degree of normality.

5.7.2 Influence of Known Extraneous Variables

To enhance the validity of subsequent statistical analyses of the test data, it was determined whether any of the known sample variables affected the test results.

- *Dichotomous variables*

There were three categorical sample variables of concern. Gender and socio-economic status were tested for significance against both sets of test results. Extra-maths lesson attendance was tested only against the mathematics test results, since there was no reason to suppose any influence on the visual perceptual skill measurement.

The categories of the nominal dichotomous independent variables were themselves also independent, due to random sample selection. The dependent variables of raw math and overall VPS scores consisted of ratio and interval data respectively. The particular configuration of the variables and data involved, prescribed the use of the 2-sample independent t test. Five t tests were thus performed. In each case, the t test, with its associated

probability value, determined whether the difference between the two categories' mean test results were of statistical significance. Since directionality was not an issue, all the t tests were treated as 2-tailed, and the p values therefore read as presented, and unmodified. To add meaning to the data, Cohen's d indicated the effect size of significant relations.

- ***Categorical variable***

It was subsequently decided to perform a one-way analysis of variance for the raw math variable, over the nominal categorical school variable. This involved comparing the mean mathematics test percentages of the three schools. To facilitate the test, descriptive statistics for the raw math results were calculated for each school.

In testing for a significant difference amongst the three means, the F test yielded the ratio of the 'between groups' estimate to the 'within groups' estimate. The effect size was computed by means of η squared. The Multiple Range Tests then determined which group means were significantly different from which others, by applying a multiple comparison procedure based on Fischer's least significant difference of 95%. With this method, there was only a 5% risk of mistaken significance of difference between each pair of means.

- ***Continuous variable***

The sample age variable was also of concern. The continuous nature of both the sample variable and the test variables sanctioned Pearson's product-moment correlation for determination of the association. The strength of influence was indicated by the resultant r value. The preceding sign of the correlation co-efficient r signified the direction of the correlation, these both being either positive or negative. The associated p value indicated whether the correlation was statistically significant within the wider population. Two separate correlations, the independent age variable versus each dependent test result, were performed at the 5% probability level.

5.7.3 Standardisation

The statistical testing to this point prompted the standardisation of the raw math test results, before proceeding with the hypothesis testing.

Standardisation normalised each individual raw math test percentage, within each school group, through their transformation to z scores. Consequently, each participant's

mathematics test percentage was expressed in terms of the number of standard deviations that their score deviated from that of the mean of their particular school. The conversion yielded a standard normal distribution, with a mean of zero and standard deviation of one.

For ease of data visualisation and comparison, the z scores were reconverted to a common scaled score. The scaling was based on the mean of the three schools' mean mathematics test percentages, and their mean standard deviation. This produced a scale with a range of possible values from 1 to 99.

Standardisation permitted just comparison of the mathematics test results, despite the significantly different means and ranges across schools. Ancillary to this process was the effective elimination of the socio-economic status variable. The entire standardisation process, with its simple formulae, was executed directly on the Excel[®] (2007) data sheet. The new scaled math variable was added to the data specification sheet, and described in terms of its mean; standard deviation; minimum; maximum; and frequency distribution. The participants' scaled math scores were thereafter utilised as the measure of their scholastic mathematics achievement.

5.7.4 Testing the Hypothesis

The principal concern of the study was whether or not visual perceptual skill positively correlates with scholastic mathematics achievement. This was determined by Pearson's product-moment correlation. Secondary to this was the assessment of the relative contributions of the seven visual perceptual sub-skills, by means of a multiple regression analysis, and a correlation matrix.

- ***Pearson's product-moment correlation***

For the purposes of statistical analysis, two mutually exclusive hypotheses were formulated:

Null hypothesis (H_0): There is no relationship between visual perceptual skill and scholastic mathematics achievement.

Alternative hypothesis (H_1): Visual perceptual skill positively correlates with scholastic mathematics achievement.

The continuous nature of both sets of data dictated the use of Pearson's product-moment co-efficient in order to calculate the strength and direction of the correlation between the independent overall VPS variable and the dependent scaled math variable. One-sided testing on Pearson's correlation was performed for significance. The statistical hypothesis would be $H_1: r > 0$, since a positive correlation is predicted. The effect size of the correlation was indicated by r squared.

- ***Multiple Regression Analysis***

Supplementary information regarding the relative contributions of the various visual perceptual sub-skills to scholastic mathematics achievement was afforded by means of a multiple regression analysis.

The seven independent visual perceptual subtest scores, in addition to the dependent scaled math scores, were all interval measures, and therefore amenable to multiple regression. What posed a concern was the number of independent variables in relation to the small sample size. However, it is advised (Statsoft, 2011) that the minimum number of participants be five times that of the number of independent variables; although a higher ratio is preferred, for a stable replicable regression. A ratio of 10:1, as submitted by this study, is considered acceptable, especially for descriptive or exploratory research such as exemplified by the present study. Co-linearity – correlations amongst the independent variables – was also a threat to stability; particularly since the visual perceptual sub-skills are merely functionally described elements of a wider composite complex facility. Although, an accurate prediction formula was superfluous, as sovereign causality is not supported within the situated cognition theoretical framework.

The regression generated an R value, the measure of correlation between the observed and predicted scaled math scores. An F test determined the statistical significance of the multiple correlation co-efficient R , and R squared indicated the proportion of variance. However, this value tends toward over-estimation, due to optimal weighting of highly correlated independent variables. Adjusted R squared accounted for the number of participants, and is thus more realistically applicable to the population.

The strength of the correlation between each visual perceptual subtest and the scaled math scores was ascertained through their respective β regression co-efficients, measured in standard deviations. The results of the TVPS-3 subtests were all expressed as scaled scores

with a mean of 10 and standard deviation of 3, so the β values were directly comparable. Their significance was established by t tests at the 95% confidence level. Previous statistical testing substantiated a directional positive correlation, yet the statistical software default is to report p values as two-tailed. Hence, the reported probabilities were halved to confirm significance at $p < .05$.

- ***Correlation Matrix***

Since inter-correlations amongst the independent variables were a potential threat to the accuracy of the regression analysis, a correlation matrix was plotted to assess the degree of co-linearity. In addition, it established the individual correlations between each visual perceptual subtest and the scaled math score, with the balance of the set ignored.

5.8 Concluding Remarks

The research design, sampling, and general methodology adhered to accepted principles of psychological research. Overall, the data collected was believed to be of good quality. Experimenter bias was minimised, there was no missing data, and the data capturing procedures instilled a high level of confidence in the prevention of transcription and coding errors. The statistical analyses performed were considered appropriate for the data.

These considerations promote the production of sound results, which would provide steadfast empirical evidence for, or against, the hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement. This, in turn, would assist in clarifying the broader relation between visual perceptual skill and mathematic ability. The results are presented in the following chapter.

CHAPTER 6: Results

6.1 Data Description

6.1.1 Sample Data

Table 6.1 shows the distribution of participants over the categorical variables, and describes the nature of the sample by cross-tabulation with the test results. The labels are further explicated in the data specification sheet (Appendix A8).

Table 6.1

Sample Description

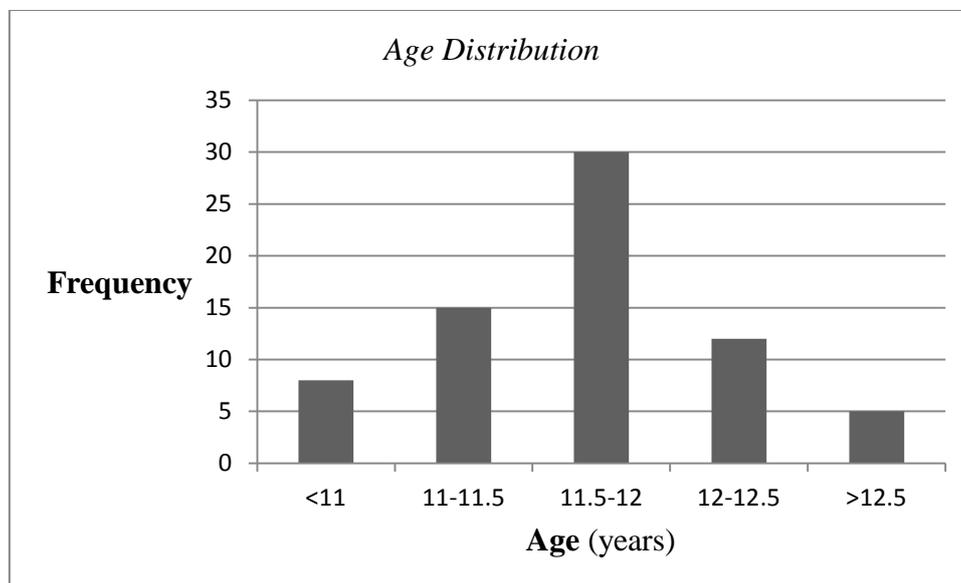
Variable	Category	<i>n</i>	Raw Math		Overall VPS	
			<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>
Gender	1 (male)	32	37.19	11.76	98.97	11.99
	2 (female)	38	52.66	18.03	100.82	12.55
EML	1 (no)	60	45.87	16.67	100.70	11.73
	2 (yes)	10	43.90	21.20	95.60	14.90
S-E	1 (low)	34	37.40	11.74	99.41	11.76
	2 (high)	36	52.92	18.35	100.50	12.82
School	1 (School 1)	34	37.69	12.30	99.41	11.76
	2 (School 2)	21	58.31	17.40	102.67	12.76
	3 (School 3)	15	45.37	17.45	97.47	12.72

Although not strictly a necessity for statistical testing, the instances of both gender and socio-economic status (S-E) represented a fairly even dichotomous split. The male to female ratio equated to 46:54, which deviates slightly from an exact split, but was representative of the population because of the sampling methods. Extra-maths lessons (EML) were attended by 14.3% of the sample, with the majority from School 1, and thus within the low socio-economic group. The small size of this EML sub-group posed a potential obstacle to subsequent *t* testing utilising the EML variable, should the variances prove unequal.

The age of the participants ranged from 10 years 7 months to 13 years 8 months, with a mean of 11 years 8 months. A grouped frequency count plotted on a graph (Figure 6.1) shows the distribution to be approximately normal, with a positive skew (0.7). However, the

large number of participants near the mean age resulted in a leptokurtic curve (kurtosis = 0.98).

Figure 6.1



6.1.2 Test Data

The test results are summarised in Table 6.2.

Table 6.2

Descriptive Statistics for the Test Results

Test Variable	<i>N</i>	<i>M</i>	<i>SD</i>	Min	Max
Overall VPS	70	99.97	12.24	72	133
Raw Math	70	45.38	17.26	11	94

Overall VPS is the standard score as measured by the TVPS-3, and raw math is the percentage score achieved on the mathematics test. For both sets of data, grouped frequency counts were plotted on a graph (Figures 6.2 and 6.3). This aided visualisation of the distribution of the results. Skewness and kurtosis were calculated to determine the degree of normality.

Figure 6.2

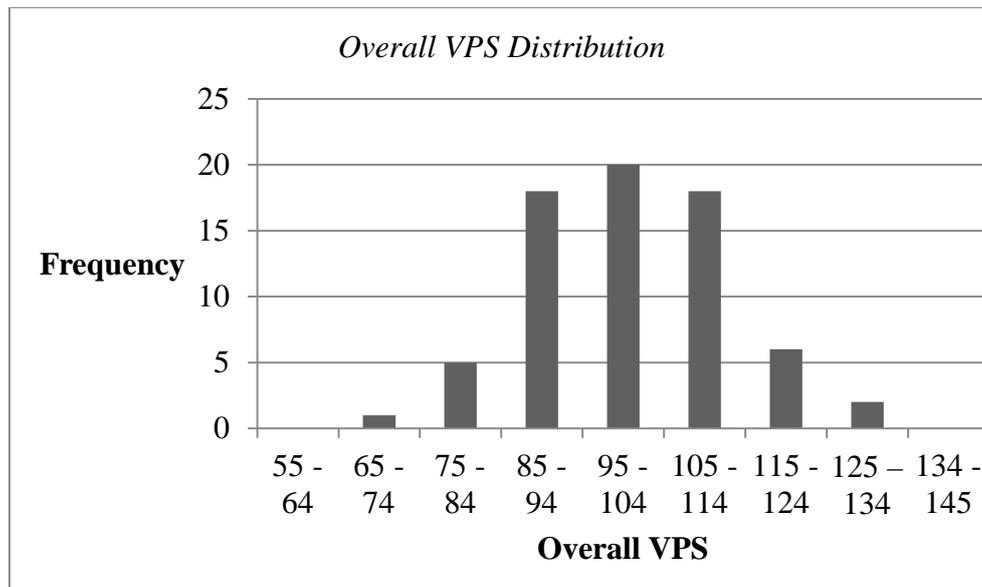
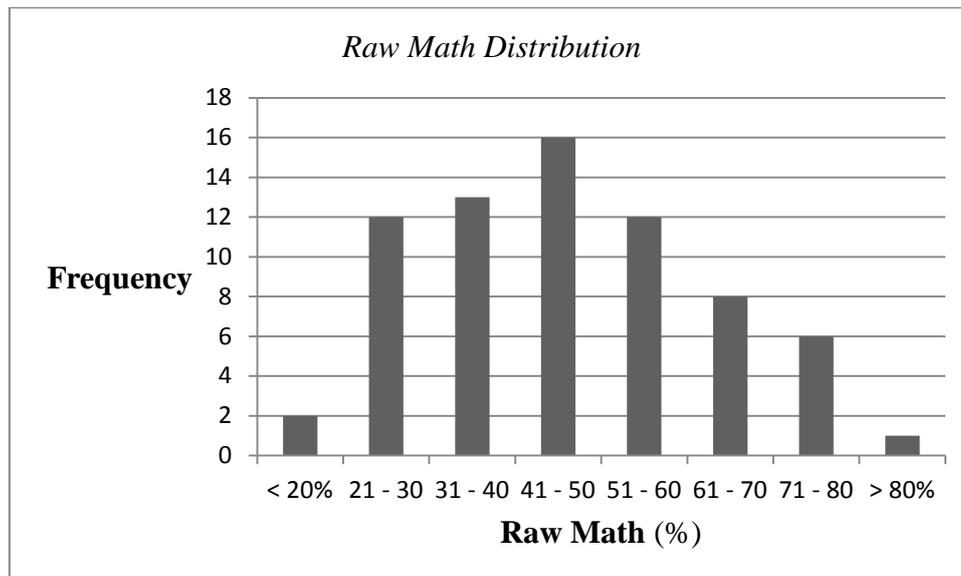


Figure 6.3



The distribution of the overall VPS standard scores across the sample (Figure 6.2) demonstrates a near perfect normal distribution (skewness = 0.10; kurtosis = 0.17). In contrast, the distribution of the raw math percentages (Figure 6.3) was bimodal and positively skewed (0.45). Kurtosis was calculated to be -0.20. A bimodal distribution often implies the superimposition of two normal distributions. However, separate plotting of the distributions

of the raw math percentages for each pair of the dichotomous variables revealed no evidence of such a superimposition.

6.2 Influence of Known Extraneous Variables

6.2.1 Dichotomous Variables

Five 2-sample independent 2-tailed t tests were performed, the results of which are summarised in Table 6.3.

Table 6.3

Summary t Test Results of Extraneous Variables

Independent Variable	Dependent Variable	Test No.	t	p	df
Gender	Raw Math	t_1	-0.77	0.445	68
	Overall VPS	t_2	-0.53	0.601	68
EML	Raw Math	t_3	1.42	0.160	68
S-E	Raw Math	t_4	-4.24	0.000	60
	Overall VPS	t_5	-0.37	0.713	68

Note: The values reported for tests t_1 , t_2 , t_3 , and t_5 are those based on the assumption of equal variance between the two categories, since the associated F tests proved not significant. In contrast, the t_4 values assume unequal variance ($F = 2.44$, $p = 0.006$).

The small number of participants in category 2 of the EML variable did not affect the test t_3 (refer to Section 6.1.1), since the variances between the two categories were assumed to be equal (see above *Note* for Table 6.3). The actual difference between the means of the two EML categories (refer to Table 6.1) was, in any case, very small ($|M_1 - M_2| = 1.97$).

Tests t_1 , t_2 , t_3 , and t_5 all produced p values greater than 0.05, which suggested that there was no statistically significant difference between the means of each of their two categories. In other words, neither gender nor socio-economic status influenced the overall VPS scores. Similarly, neither gender nor extra-maths lesson attendance impacted on the raw math percentages. Gender and extra-maths lesson attendance were therefore entirely non-influential variables.

The only test that produced a significant result, with $p \leq .01$, was t_4 , which analysed the effect of socio-economic status on the mathematics test results. Such a p value indicated

that the raw math means of the two S-E categories were statistically significantly different at the 1% level, suggesting a greater than 99% probability of a similar difference within the population. Low socio-economic status ($M = 37.40$, $SD = 11.74$) yielded significantly lower raw math percentages than high socio-economic status ($M = 52.92$, $SD = 18.35$). The effect size of the statistically significant difference of 15.52 between the mean raw math percentages of the two S-E categories was calculated by means of the following formula:

$$Cohen's\ d = \frac{|M_1 - M_2|}{s_p}$$

This yielded a value of 1.03, which meant that the difference between the two categories was far greater than would be expected from random variance in the data. The statistical significance and effect size was large enough for the relationship to prove significant even when allowing for a Bonferroni correction.

6.2.2 Categorical Variable

Section 6.2.1 revealed that socio-economic status had a large and significant effect on the mathematics test results, while it had no significant effect on the visual perceptual skill measurement. With this in mind, it was tentatively surmised that the impact on the mathematics results was not directly related to socio-economic status, but rather a derivative of a bias in the quality of education apparent between the schools. This rationale for the negative bias within the low socio-economic group was supported by the subjective observations, within each school, of both the teachers' and learners' relative attitudes and behaviour.

Potential corroboration of this conjecture involved the performance of a one-way analysis of variance, to determine the extent of influence of the categorical school variable on the raw math percentages. This tested for any significant difference amongst the means (column 3 of Table 6.4) of the three schools.

Table 6.4

Descriptive Statistics for Raw Math Results of the 3 Schools Groups

School	<i>n</i>	<i>M</i>	<i>SD</i>	Min	Max	Range
1	34	37.69	12.30	17.5	67.0	49.5
2	21	58.31	17.40	25.0	94.0	69.0
3	15	45.37	17.45	11.0	76.0	65.0

The results, presented in Table 6.5, proved significant at the 99% confidence level.

Table 6.5

ANOVA table for Raw Math by School

Source	Sum of Squares	df	Mean Square	F	p
Between groups	5519.24	2	2759.62	12.08	0.000
Within groups	15309.00	67	228.49		
Total	20828.20	69			

The effect size was calculated by means of the following formula:

$$\eta^2 = \frac{SS_{\text{Between groups}}}{SS_{\text{Total}}}$$

There was thus a statistically significant difference between the schools' mean raw math percentages, $F(2,67) = 12.08$; $p = 0.000$, although the size of the effect ($\eta^2 = 0.27$) proved to be rather small.

The Multiple Range Tests further determined exactly which school means differed from which others. In Table 6.6, the third column shows the estimated difference between each pair of means.

Table 6.6

Multiple Range Tests for Raw Math by School

Contrast	Significant	Difference	+/-Limits
1 - 2	yes	-20.62	8.37
1 - 3	no	-7.68	9.35
2 - 3	yes	12.94	10.20

Only the contrast between School 1 and 2, and between School 2 and 3 exhibited statistically significant differences at the 95% confidence level. This exposed School 1 and School 3 as an homogenous group, contrary to their original classification into distinct socio-economic categories. It may have been this distinction between School 1 and 3 versus School 2 that resulted in the bimodal distribution of the raw math percentages (refer to Section 6.1.2).

Schools 2 and 3 represented the high socio-economic group, and School 1 comprised the low socio-economic group. Yet, mathematics scores from School 2 were significantly higher than were those from both the other two schools. This indicates the probability that the differences in the mathematics test results may not be related specifically to socio-economic status. Other factors, such as attitudes, discipline, available technologies, and teaching methods, might be of consequence. Nearly 30% of the variance may be attributed to such bias in the quality of education.

6.2.3 Continuous Variable

Pearson's product-moment correlations were used to investigate the possible influence of the age variable on each of the test results. Both the correlation between age and overall VPS score ($r = -0.18$, $p = 0.14$), and that between age and raw math ($r = -0.20$, $p = 0.09$) proved not significant. Age, therefore, was not statistically relevant to either of the test results.

6.3 Standardised Measure of Mathematics Achievement

The only influence of known extraneous variables was that of socio-economic status and school group, on the raw math percentages. The one-way ANOVA performed (refer to Section 6.2.2) highlighted the effects of a possible educational quality bias amongst the schools. Consequently, to fortify the validity of ensuing hypothesis testing, this bias was neutralised by standardising the mathematics test results.

The raw math percentages were standardised through their transformation to z scores, characterised by the corresponding school group's raw math mean and standard deviation in the formula,

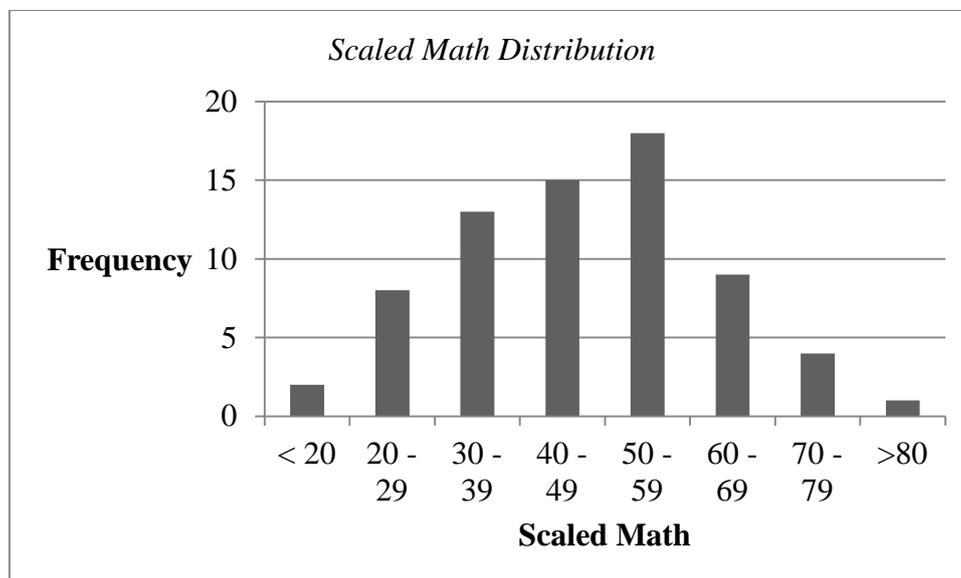
$$z = \frac{X - M}{s} .$$

The z scores were then reconverted to a common scaled score, based on the raw math mean and standard deviation across all three schools. Each individual score was calculated by means of the following formula:

$$X = z * s + M .$$

The new scaled math variable ($M = 47$, $SD = 15.29$) ranged from a minimum of 16 to a maximum of 86, and its distribution is portrayed in Figure 6.4.

Figure 6.4



The distribution of the scaled math scores was less skewed than that of the raw math results (skewness = 0.18, as opposed to 0.45 for the raw math distribution; kurtosis = -0.31). It was also noticeably less bimodal, which seems to confirm that it was the educational bias amongst the schools that resulted in the bimodality of the raw math distribution (refer to Sections 6.1.2 and 6.2.2).

6.4 Testing the Hypothesis

6.4.1 Pearson's Product-moment Correlation

Pearson's product-moment correlation co-efficient was used to test the statistical hypotheses:

$$H_0: r = 0; H_1: r > 0.$$

A moderately strong positive correlation was shown to exist between the overall VPS and scaled math scores, $r = 0.38$, $p = 0.001$ (2-tailed). The conversion to a one-tailed correlation, to accommodate the directional alternative hypothesis, effectively halved the probability value, $p < .001$. The correlation was therefore statistically significant at the 1% level. However, the effect, due to sample size, was small ($r^2 = 0.14$).

This meant that a greater than 99% probability existed that the sample result was not achieved by chance, but indicative of an occurrence, albeit it small, in the wider population. Consequently, the null hypothesis (H_0) was rejected in favour of H_1 . It was thus concluded that visual perceptual skill positively correlates with scholastic mathematics achievement.

6.4.2 Multiple Regression Analysis

A multiple regression described the relationship between the visual perceptual sub-skills, as measured by the seven TVPS-3 subtests (the independent variables), and scholastic mathematics achievement represented by the scaled math scores (the dependent variable). It revealed four unusual residuals, with studentized values between two and three. However, two of these were positive which offset the two that were negative. A general under or over prediction was thus refuted. A mean absolute error of 10.68 was recorded.

The relationship proved significant ($R = 0.48$, $F(7,62) = 2.62$, $p = 0.019$), with 23% of the variance ($R^2 = 0.23$) in scholastic mathematics achievement, within the sample, attributable to the visual perceptual sub-skills. The adjusted R squared value, more applicable to the population, equated to 14%. The individual subtest contributions are summarised in Table 6.7.

Table 6.7

<i>Individual Parameters of Independent Regression Variables</i>					
Variable	Co-efficient	Std. Error	<i>t</i>	<i>p</i> (1-tailed)	
DIS: visual discrimination	0.79	0.52	1.51	0.068	
MEM: visual memory	1.11	0.58	1.91	0.030	
SPA: spatial relations	0.17	0.64	0.26	0.397	
CON: form constancy	0.68	0.50	1.36	0.090	
SEQ: sequential memory	0.72	0.59	1.22	0.114	
FGR: figure-ground	-0.16	0.53	-0.31	0.380	
CLO: visual closure	-0.84	0.56	-1.49	0.071	

The only t test of significance was that involving visual memory ($\beta = 1.11$, $p = 0.03$). This means that visual memory significantly contributes to scholastic mathematics achievement, beyond that already accounted for by the other six visual perceptual sub-skills. Figure-ground and visual closure both displayed negative correlations.

6.4.3 Correlation Matrix

It was possible, however, that high co-linearity of the independent variables affected the regression results (refer to Section 5.7.4). A correlation matrix, Table 6.8, was therefore produced to investigate the extent of correlations amongst the visual perceptual sub-skills. It also determined the correlation between each sub-skill and the scaled math score, while ignoring the influence of the other independent variables.

Table 6.8

Correlation Matrix

	Scaled Math	DIS	MEM	SPA	CON	SEQ	FGR	CLO
Scaled Math								
DIS	0.31**							
MEM	0.33**	0.38**						
SPA	0.26*	0.33**	0.29*					
CON	0.29*	0.32**	0.33**	0.36**				
SEQ	0.28*	0.30*	0.26*	0.33**	0.27*			
FGR	0.16	0.34**	0.30*	0.27*	0.47**	0.38**		
CLO	0.08	0.42**	0.49**	0.10	0.34**	0.22	0.45**	

Notes: * significant at 5%; ** significant at 1%

The correlation matrix showed that the independent variables were highly inter-correlated. Of particular importance was the fact that there were no negative correlations. Since high co-linearity existed, individual correlations between each visual perceptual sub-skill and the scaled math score clarified the results provided by the regression analysis.

Visual discrimination (DIS), the basis for all the other sub-skills, and visual memory (MEM) most significantly correlated with the scaled math score. Spatial relations (SPA), form constancy (CON), and sequential memory (SEQ) significantly correlated, to a lesser degree. Figure-ground (FGR) and visual closure (CLO) were not significantly correlated with the scaled math variable. However, FGR and CLO were shown to significantly correlate with other visual perceptual component skills, and may therefore be regarded as not independent of them. The negative correlations displayed by FGR and CLO in the regression analysis are likely to have been the result of these inter-correlations. Highly inter-correlated independent

variables make it difficult to determine which variables are actually influencing the dependent variable.

In general, spatial relations (SPA), form constancy (CON), sequential memory (SEQ), figure-ground (FGR), and visual closure (CLO) all significantly correlated with visual discrimination (DIS) – the most basic underlying component skill – and visual memory (MEM). Therefore, the balance of the component skills may be considered as not independent of the two important sub-skills of visual discrimination and visual memory.

6.5 Key Findings

The study sample consisted of 70 participants, with an average age of 11 years 8 months. The sample variables of gender, extra-maths lesson attendance (EML), and age, all proved irrelevant to both sets of test data. Socio-economic status (S-E) also had no effect on the overall VPS score. The overall VPS scores were approximately normally distributed, whereas the raw math distribution was positively skewed and bimodal.

There was a significant difference, with a large effect, between the low S-E and the high S-E groups' raw math scores. However, an ANOVA test revealed School 2 (high S-E) to have obtained statistically significantly higher raw math percentages than either School 1 (low S-E) or School 3 (high S-E). These results cast doubt on the validity of socio-economic status as a variable. Subsequently, the standardisation of the raw math percentages essentially removed the covariance contributed by socio-economic status, and nullified socio-economic status as a viable variable. Based on the data, this seems to be a reasonable statement. However, further comprehensive research, with more representative samples of data, will be needed to clarify this.

The standardised scores were converted to scaled math scores, the distribution of which was less skewed, and less bimodal than that of the raw math percentages. The scaled math scores were hereafter used as the measure of scholastic mathematics achievement.

Pearson's product-moment correlation between the overall VPS scores and scaled math scores proved statistically significant. The null hypothesis was thus rejected in favour of the alternative hypothesis, namely that visual perceptual skill positively correlates with scholastic mathematics achievement. A multiple regression analysis involving the seven

TVPS-3 subtests revealed that visual memory was the only visual perceptual sub-skill to contribute significantly to scholastic mathematics achievement, beyond that contributed by the entire set. However, the significance of the regression analysis was minimised by a correlation matrix that showed high co-linearity amongst the visual perceptual subtests. Independently, visual discrimination and visual memory most significantly contribute to scholastic mathematics achievement; however, the other five component skills are not independent of these two important component skills.

In conclusion, the results support the hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement. This constitutes empirical evidence in support of the broader relation between visual perceptual skill and mathematic ability, as implicated by situated cognitive theory. Spatial relations, however, proved less consequential than the theory had predicted. The following chapter will consider the significance of the results in more detail, and reflect on limitations of the study.

CHAPTER 7: Discussion

In terms of the original goals and hypotheses of the study, statistical analyses performed on the data were a partial success. The results provided some clarification as to the relationship between visual perceptual skill and mathematic ability. Specifically, the analysis supported the hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement. In addition, the relative contribution of a set of component visual perceptual sub-skills was determined, although this proved contrary to the particular prediction of the situated cognition theoretical framework.

7.1 Theoretical Significance of the Results

The empirical evidence supports the possibility of a relationship between visual perceptual skill and mathematic ability. In the preceding theoretical discussion, the claim was made that the theory of situated cognition is best suited to explain this relationship. As substantiation, the significance of certain key results is considered in terms of this theoretical paradigm.

7.1.1 Educational Context

Prior to the hypothesis testing, a one-way ANOVA revealed that the quality of mathematics education received was an influential variable on the mathematics test results. While socio-cultural factors may undoubtedly contribute to scholastic mathematics achievement, statistical testing of the nominal sample variables against the test results indicated that educational differences are of greater significance than life-style experiences relevant to socio-economic status. In addition, since the schools supposedly follow a common national syllabus, the differences in the quality of education must relate more to context than to content. To a limited extent, the descriptive statistics presented in Table 6.4 depict the consequences.

The teacher at School 2 holds a PhD in mathematics education. It is likely that he may have transmitted the passion for his subject, confirmed by his eager participation in the testing, to his learners. Their achievement of the greatest mean, and highest maximum, raw

math results can be seen to reflect this possibility. School 3, with an experienced, dedicated teacher, and a disciplined class, achieved an average mean raw math result. On the other hand, in School 1, the young teacher was absent and the class was ill-disciplined and disruptive. This school obtained the lowest mean raw math results and the lowest maximum score. Even prior to formally analysing the data, it was intuited that a moderate achiever within this kind of educational context, may indeed possess the potential for outstanding achievement under more favourable circumstances. Standardisation of the raw math scores corroborated this suspicion. The highest raw math score achieved in School 1 was 67, and in School 2, a result of 94 was obtained. These equate to scaled math scores of 86 and 79, respectively. After countering for the inequality in educational context, the top achiever in School 1, with a mediocre raw result of 67, obtained the highest score across the entire sample.

Furthermore, a curious result was the small range, and low standard deviation of raw math scores exhibited by School 1, in comparison to the other schools (see Table 6.4). Considering the number of participants within this sub-group, almost half the entire sample, a greater spread of mathematic ability would perhaps have been expected. There is a possibility that this incongruity may also be a consequence of educational context. The more favourable learning environment and manner of teaching provided in Schools 2 and 3 might have been conducive to encouraging those learners with an interest in mathematics to reach their full potential, and excel; thereby inducing a wider range of results.

These observations are in accordance with the importance of the situational context of situated cognition (refer to Section 4.2). Moreover, they underscore the diverse and multitudinous aspects of the cognitive environment. Factors such as the encumbrance of ambient noise; the behaviour of classmates; motivation provided by teachers; teaching style; peer pressure and attitudes towards maths, and school life in general, are only but a few of the contributory aspects. An embodied cognition embedded in, and extended into the environment, is partially a product of every aspect of the learning situation, both past and present.

7.1.2 The Visual Perceptual Component Skills

The situated cognitive paradigm predicts the special significance of the spatial relations visual perceptual component skill to mathematic ability. This derives from the

sensory-motor grounding of mathematics, since movement is spatial by nature. A well-developed sense of three-dimensional spatial relations, derived from concrete object constructions and other motor activities, enables mathematic conceptualisation.

Contrary to this prediction, a regression analysis performed (refer to Section 6.4.2), along with a correlation matrix (Table 6.8), disclose visual memory as the foremost contributor to the positive correlation between visual perceptual skill and scholastic mathematics achievement. Many previous comparable studies have also yielded similar results (refer to Section 3.4.1). The nature of the results is a credible reflection of the vestiges of formalist mathematics, where algorithmic mathematics is rote learned.

The prominence of the visual memory component skill suggests this educational focus on rote learning, since rote learning *is* memorising, usually involving visual and auditory stimuli. Naturally, the other six component skills are not superfluous, since they are simply functionally described inter-dependent elements of the complex visual perceptual facility, as stated in the TVPS-3 manual (Martin, 2006), and as reflected in the correlation matrix (refer to Section 6.4.3). Nevertheless, in a predominantly rote-learning style, they are useful only in so far as to perceive the stimuli to be visually memorised. Although Dehaene (1997) and Giaquinto (2007) have posited that algorithmic arithmetic is visuo-spatial in nature (refer to Section 2.3.5), this is a very limited two-dimensional spatiality. The cognitive burden in remembering the sequence of steps and rules in an algorithm far outweighs that of the simple spatial relations involved in each step. It may be this type of rote mathematical learning, and the over-reliance on visual memory, which quite conceivably leads to subsequent mathematical stagnation. It does not support the development of conceptual mathematical cognition essential to the novel problem solving of higher mathematics (refer to Section 4.7).

Situated mathematics, grounded in sensory-motor activity, is metaphorical. The metaphorical, or conceptual, nature of mathematics, and the analogical reasoning required for novel problem solving depends upon the ability to perceive similarities and differences. Generally, this entails deep structural similarities and differences, in addition to that of merely visual features. However, emergent cognition grounded in sensory-motor capabilities extrapolates concrete perceptual concepts to the abstract. This means that the cognitive facility for metaphor and analogy originates in the differentiation of perceptual surface features, which gives rise to conceptual development. This in turn, promotes the perception of structural feature similarities and differences (refer to Section 4.6.2). Further experiential

interaction with the world advances perception and conception co-development (refer to Section 4.4.4). Conceptual differentiation thus represents the development of pre-existing skills, rather than the creation of novel cognitive aptitudes. As such, the cognitive facility for conceptual metaphor, as the cornerstone of situated mathematics, is contingent upon those same skills that enable the perception of visual similarities and differences.

Visual memory certainly plays a role in this perception. As the correlation matrix shows, none of the visual perceptual component skills are independent of visual memory. However, all seven of the TVPS-3 subtests explicitly require either the matching of designs or discerning design differences. Visual discrimination, form constancy, figure-ground, visual closure, and spatial relations as well as visual and sequential memory are therefore all presupposed in situated mathematical cognition.

7.1.3 Composite Visual Perceptual Skill

The recognition of the importance of *all* the visual perceptual component skills may initially seem to contradict the situated theory of mathematical cognition. However, despite the *relative* importance bestowed upon spatial relations, situated cognition is a holistic theory.

In contrast to the reductionism of previous paradigms, there is a wariness of reducing any cognitive activity to a set of distinctly defined abilities. In terms of the theory, there is little benefit in dividing visual perceptual skill into arbitrary functionalities. Notwithstanding the tenacious conviction in reductionism in the past, this is not in fact an innovative insight. As early as 1961, Frostig (cited in Martin, 2006) showed that the subtest scores of the Developmental Test of Visual Perception are highly inter-correlated. Similarly, the present study revealed high co-linearity amongst the seven visual perceptual sub-skills (refer to Section 6.4.3). In the TVPS-3 manual, Martin (2006) admitted to the artificiality of the definition of component skills.

Spatial relations better describes the perception of a perceptual property of the enacted world, such as colour. It is not an isolated skill. In order to perceive spatial relations (SPA), objects need to be visually discriminated (DIS), even when fragmented (CLO) or disorganised (CON), and distinguished from their irrelevant background (FGR), and previous visual information may need to be recalled (MEM, SEQ). Likewise, the execution of each individual component skill necessitates the participation of other component skills. The seven

functionally described visual perceptual component skills are in fact dynamically inter-dependent and functionally inseparable.

7.2 Consequences for the Didactics of Mathematics

In general, the study supports the hypothesis that visual perceptual skill positively correlates with scholastic mathematics achievement. This lends credibility to the existence of a relationship between visual perceptual skill and mathematic ability, which seems best explained by situated cognitive theory (refer to Chapter 4). This encourages consideration of the didactics of mathematics in terms of situated theory. Situated theory suggests that in mathematics education, especially of a foundational level, formal mathematics can serve only as a necessary but secondary adjunct, if rote learning devoid of understanding is to be avoided (refer to Section 4.7). A viable cognitive science of mathematics, such as that of situated theory, on which to base the development of useful, conceptual educational content and technique, has the potential for a significant positive impact on scholastic mathematics achievement.

During the present study, the participants' constant erasing of everything except their final answers was of concern. This gave the impression that, in the normal course of classroom mathematics, the teachers preferred clean, simple answers with little concern for the cognitive process behind them. In this case, even if educators are teaching mathematics conceptually, not being privy to the thought processes of their learners nullifies the benefit. It is impossible to evaluate, by the assessment of a final answer, whether a child has fully grasped a concept, or the level of abstraction and understanding that has been achieved. Basic mathematics is, to a degree, metaphorically intuitive by virtue of embodiment. As a result, children often develop their own strategies for mathematical problem solving. The onus is on the teacher to encourage such mathematical cognition, by modifying erroneous techniques and supporting those that are effective. Yet a cognitive strategy can only be determined through insight into the cognitive processes interim to the final answer.

From the situated perspective, a conceptual focus in mathematics education that capitalises on natural basic visual perceptual skill will favour the development of abstract mathematical cognition, and imbue mathematics with meaning and relevance. Such a pedagogy might therefore reflect in improved scholastic mathematics achievement.

Mathematics teaching within the context of the shifting scientific attitude towards holistic, situated cognition, may yield a stronger relationship between mathematics achievement and visual perceptual skill in future studies.

7.3 Reflections on the Study

Although not strictly an inadequacy, a larger sample size would have lent greater significance to the results. In addition, the sample allocation into the socio-economic subgroups was arbitrarily applied, with no supporting evidence or confirmation of family income and standing. It was executed under the feeble assumption that government school attendance entails low status, while only those of high socio-economic standing attend private schools. As it happened, though, socio-economic status was ultimately rejected as a variable, after statistical analyses revealed it to be non-influential. The greatest limitations of the study, however, were the choice, design, and use of the measuring instruments, and the context in which they were used.

The TVPS-3 was the most suitable available test choice, providing separately normed subtests. However, the high level of co-linearity amongst the subtests shown in the correlation matrix (refer to Section 6.4.3) suggest that the visual perceptual component skills that the TVPS-3 purports to assess, do not constitute a set of clearly distinguished constructs. In addition, the TVPS-3 lacked a visual-motor integration component. Furthermore, the fact that the test instructions were deliberately altered (refer to Section 5.4.1), for which apologies are extended to the developers, undeniably perverted its reputed reliability and validity.

Similarly, the focus on scholastic mathematics achievement inevitably introduced reliability and validity issues. A diversity of mathematics syllabi over place and time would necessitate a distinct mathematics test for replication of the study. In addition, quality of education was revealed to have had a significant impact on the mathematics test results, a source of serious threat to external validity of the study. The priority for a randomly selected sample representative of the population prevented the eradication of this threat, although it was moderated by standardisation of the raw math test results.

In general, the naturalistic setting of the data collection induced significant uncontrollable context effects. Together, these context and measurement concerns directly

influenced the external reliability and validity of the study. However, within the situated framework, situational context is of the utmost importance. The results were therefore considered internally reliable and valid, within the situation of this particular study.

7.3.1 Visual Perceptual Skill Testing

The limited mode of testing composite visual perceptual skill may have contributed to discrepancies between the empirical evidence and the situated cognitive theoretical framework, in terms of a relationship between visual perceptual skill and scholastic mathematics achievement. Larsen and Hammill (1975) were quoted in Section 3.4.1 as stating that the results of their research review suggested, “that measured visual-perceptual skills are not sufficiently related to academic achievement to be particularly useful” (p. 287). The crux of the issue lies in the word *measured*. Its inclusion negates an outright denial of a useful relationship. Instead, it subtly hints at the inadequacy of the measures of visual perceptual skills.

Gestaltist principles form the basis of all current tests of visual perceptual skill (refer to Section 3.1). Gestaltism promotes visual perception beyond simple object detection, however, Gestalt theorists’ thinking has perhaps remained too dualistic, which constrained the full development of the gestalt (refer to Section 4.1). Consequently, realisation of the importance of the visuo-spatial component to visual perception was limited to the organisation of the perceived object or scene. Visual perceptual skill test designs reflect this limitation.

Since Gestalt laws of perceptual organisation (refer to Section 3.1), and the conventionally accepted definition of visual perceptual skill (refer to pp. 81-82) inform current test designs, their focus is on identification of features of a visual array. Converging with Birch and Lefford’s (cited in Leonard, 1986) three levels of visual perceptual skill, Rosner (1982) has described the general thrust of visual perceptual skill test measurement thus:

1. Recognition: gross visual discrimination of the concrete features of a visual array.
2. Analysis: the ability to respond to selected aspects of the visual array, by separating it into structural components.
3. Synthesis: the ability to re-order the components of the visual array, by mapping their inter-relationships.

The TVPS-3 realises this measurement in the identification of static designs limited to two-dimensional spatiality. Although there is no claim of comprehensive visual perceptual assessment, this is only in the context of its intentional omission of visual-motor assessment, in order to make the test as motor-free as possible. Yet, even purported comprehensive motor tests of visual perception only allow for limited consideration of movement and spatiality in testing. The Beery Developmental Test of Visual-Motor Integration, and its Supplemental Developmental Test of Visual Perception, involves copying tasks of comparable two-dimensional designs. Visual-motor assessment is thus minimal, restricted to hand-eye coordination. This disregards anthropological (e.g., Sheets-Johnstone, 1990) and neuroscientific (e.g., Moore, 1997) evidence for the intimate link between the development of movement in the body as a whole, and the development of the visual perceptual process in particular.

The narrow focus of such testing pays insufficient attention to the holistic and dynamic characterisation of situated visual perception. An approach, such as situated cognition, proceeding from the consideration of dynamic contextual factors in cognition, can only regard current tests as purely partial measures. Historically, the simple assessment by current tests were perhaps deemed adequate in researching the relation between visual perceptual skill and mathematic ability, since mathematics was previously believed to be limited to formal symbolic manipulation. However, recent cognitive scientific theory shows mathematical cognition to be grounded in sensory-motor experience. Visual perceptual skill testing that is more appropriate to this conception, may yield an enhanced correlation between visual perceptual skill and mathematics achievement, favouring spatial relations.

The theoretical review suggests that mathematics is not simple numerical calculation; it is the science of patterns, including those pertaining to motion and change. Pattern perception, in conjunction with innate subitization, develops the conception of mathematical objects and relationships. Situated mathematical perception and conception originates in sensory-motor activity, particularly that of collecting, constructing, and moving in an enacted world. This type of complex, interactive, dynamic perception in action requires far more sophisticated testing than is currently available in commercial tests of visual perceptual skill.

Tversky's (2005b) partial catalogue of the capacities entailed in perception could well proffer a basis for a new generation of visual perceptual skill tests:

- determining static properties of entities,

- determining relations between static entities,
- performing transformations on entities,
- determining relations of dynamic and static entities,
- performing transformations on self (p. 216).

The first three of these correlate with that which the TVPS-3 measures. The last two are the spatial and motor components critical to comprehensive visual perceptual skill testing. In describing the first of these, Tversky uses words such as direction, speed, acceleration, manner, collision; and in describing the second, uses words such as movement, perspective, enaction. These descriptions overtly expose the additional elements of spatiality and temporality (refer to Sections 4.5.4 and 4.5.5) crucial to visual perception. In essence, though, they are the situated perception of motion and change in a wide, three-dimensional enacted space. The visuo-motor and visuo-spatial capabilities specifically wanting in conventional visual perceptual testing are implicated, and are the two vital aspects accentuated in situated mathematical cognition.

Therefore, in this study of the relation between visual perceptual skill and scholastic mathematics achievement, the TVPS-3 may have been inadequate in the assessment of visual perceptual strengths and weaknesses. The constrained two-dimensional test context ineffectually accounts for the boundless factors that contribute to such a dynamic and interactive facility. The flexibility of visual perception indeed renders its operationalisation and measurement problematic. Notwithstanding these difficulties, a test devised to accommodate the motor and space-time aspects might at least typify a more realistic measure, eradicating the need for qualified statements such as that made by Larsen and Hammill (1975), quoted above. That is, the inclusion of visuo-motor and four-dimensional visuo-spatial skills may sustain a more confident assertion that visual perceptual skill positively relates to academic achievement.

7.3.2 Methodological Concerns

Cognition that is situated, and spatially-temporally structured dynamic action, poses enormous challenges to systematic scientific study. In an attempt to align with the situated theoretical framework, every effort was made to mimic the habitual classroom situation. Nevertheless, the quantitative correlational nature of the study was possibly too restrictive to account fully for the holistic and dynamic cognitive aspects.

Comparability with previous similar studies motivated the selection of this design, and the unsuitability of alternative designs enforced the choice. For example, ontologically, constructionist research directly contradicts the premise of situated cognitive theory. In addition, it is unclear how a methodology based on the interpretative epistemology might penetrate the subconscious dynamic interactions underlying emergence of situated cognitive facilities. However, the design utilised for the study was equally not ideal. The root of the problem perhaps lies in the resolute goal of objectivity typical in such quantitative correlational designs. The only means to this end is the reductionism of cognition to individual constructs for quantification, along with the context, to controllable factors. Yet, within the situated framework, these practices may actually conceal that which is being investigated, rather than reveal it.

- *Constructs.* The emergent nature of situated cognitive facilities confounds the isolation of relevant constructs from the dynamic interactive network of contributory facilities. Certainly, operationalising a situated cognitive facility is a daunting task. The attempt to measure visual perceptual skill clearly illustrates this situation. Martin (2006) has admitted, in the TVPS-3 manual, that “the relationship of perceptual abilities to performance of everyday activities is subtle, but all-encompassing [...] Even moving through space requires perception in that one needs to be able to judge distance and discern objects” (pp. 14 - 15). Yet the test content discloses nothing of the inclusivity of this relationship, nor its dynamic interactivity. Expansion of a visual perceptual test (refer to Section 7.3.1) to include visuo-motor and four-dimensional visuo-spatial aspects, may increase the validity of the test construct, but it would still be unable to capture the dynamic flexibility of the situated facility.

- *Context.* A correlational research design generally requires some control of extraneous factors to enhance validity and reliability of the results (Mouton, 2001). However, the natural situational context is paramount in situated cognition. Consequently, interference with any associated factors simultaneously detracts from the full manifestation of the construct. In defence of the present study, this rationale justifies the flouting of certain validity and reliability concerns (refer to Chapter 5). Quintessentially, situated cognitive investigation is definitive only within its unique, situated context. This may be an unattainable ideal; nevertheless, development in cognitive understanding rests upon alternative research and assessment techniques befitting the current theoretical paradigm.

In general, cognitive facilities lack the apparent structure and predictability of the physical world, which diminishes the effectiveness of conventional scientific methodologies of the natural sciences. Objective measurement of the external world, and inner constructivist cognitive representations and processes may have sufficed under a dualistic philosophy. However, cognition as a situated process eradicates notions of objectivity and subjectivity. Cognitive facilities emerge through situated experiences; they are phenomenal. Situated cognitive science, informed by a phenomenological philosophy, necessitates a more holistic research methodology. Ideally, a phenomenological understanding of visual perception and mathematical cognition should precede the scientific examination of a relationship between visual perceptual skill and mathematic ability. This is in line with Merleau-Ponty's (1964c) statement that "empirical psychology must be preceded by an eidetic psychology" (p. 58).

7.4 Recommendations for Further Research

Growth of any body of positive empirical evidence, such as that provided by the present study, bolsters specific hypotheses. However, the preceding discussions disclose a number of considerations that may potentially lend greater credence to prospective research on the relationship between visual perceptual skill and mathematic ability. Based on these, the following recommendations may improve the quality of future empirical evidence.

- It is possible that some high mathematics achievers in primary grades do so by relying on visual memory. Correlational longitudinal studies are required to assess their ability to retain their level of achievement in higher grades and tertiary levels, at which stage mathematics accomplishment is beyond rote memorisation.
- Long-term studies are necessary to corroborate the efficacy of teaching mathematics conceptually, as opposed to factually. However, such research would need to be sensitive to ethical considerations, so as not to disadvantage certain learners. The most ethically sound approach may be to assign an experimental group to extramural mathematics lessons that are conceptual in nature, and then to compare their level of mathematics achievement over time with a control group receiving mathematics lessons solely within the existing education system.

- Visual perceptual skill comprehended as the dynamic interactive ability to co-ordinate and render visual-motor activity in an enacted environment (refer to p. 82) calls for the expansion of visual perceptual skill tests, to include as a minimum motor and four-dimensional spatial aspects argued for as crucial in this study.
- The holistic, dynamic, and situated nature of current theories of cognition calls for alternative research methodologies that afford consideration to these aspects.

7.5 Conclusion

The poor status of scholastic mathematics achievement, across all grades in South African schools (refer to Chapter 1), demands urgent multi-disciplinary intervention. Historically, psychology has contributed not only to the didactics of mathematics, but to all teaching and learning. Theories from Gestaltist, behavioural, developmental, connectionist, and cognitive psychology have all proven influential to educational philosophy (Fey, 1994). It is imperative that education should continue to benefit from evolving cognitive scientific theories.

Piaget (1970) has opined that “if experimental pedagogy wishes to understand what it is doing [...] it is obvious that it will have to employ a precise psychology, not merely that of common sense” (p. 23). A possible argument, however, might revolve around the definition of a *precise psychology*. In terms of precision, the discipline of psychology has faced an embattled process for recognition by the scientific community. Indeed, it is still often criticised for the lack of an overarching theory and methodological rigour, as found in the mature natural sciences. The evolution of a comprehensive cognitive science might address the concern for an overarching theory. However, further meaningful contribution to situated cognitive science solicits cognitive research to surmount the constraints imposed by conventional methodologies of the natural sciences, thereby potentially eliciting further condemnation.

Nevertheless, situated cognitive science demands a methodology appropriate to its phenomenological grounding philosophy. A phenomenological approach has the potential to redirect the constructivist course of psychology, and to clarify the true nature of perception, cognition, and mathematics. In addition, a common philosophical foundation may clarify the relations between them. Previous difficulties in providing a definitive description of a

relationship between visual perceptual skill and mathematic ability may stem partly from the fundamental philosophical incompatibility between formalist mathematics and constructivist cognitive psychology.

The literature intimates the possible existence of a relationship between visual perceptual skill and mathematic ability. However, the present study presented the argument that traditional psychological theory displays inherent inadequacies in the conception of perception and cognition, and consequently lacks a robust perceptual-cognitive framework, within which to explore the possibility of a relationship further. Of primary concern is the reliance on an independent objective world, and the notion of mathematics as the manipulation of formal symbols, which instil the problem of representation interpretation. The cognitive scientific paradigm of situated cognition was introduced as a viable alternative.

Situated cognition has been explained to begin with select innate sensory-motor abilities, a result of evolution in, and with, the world. Mutual interconnectivity between sensation and motor activity is taken to give rise to the facility of perception. Situated perception is thereby argued to be the dynamic co-ordination and rendition of sensory-motor activity with the enacted environment, through which cognition emerges. Embedment in a society with an effective communication system is theorised to extend cognition to higher levels of abstraction. Abstract mathematical cognition, facilitated by language, is thus argued to emerge through situated perceptual activity with the enacted world. This account grounds mathematical meaning in situated sensory-motor activity, and exposes visual perception as a component in mathematical cognition. Situated mathematics, as described, is an abstract cognitive conceptualisation. Mathematical objects are realised through particular human biological and social characteristics, through situated perception, conception, and cognition. This is, however, in conflict with the predominant belief in our culture that mathematics is transcendent and axiomatic. In order to overcome this entrenched belief, the radically alternative paradigm requires consolidation.

In contrast to traditional psychological theories, situated cognitive theory thus clarifies several aspects of mathematics, perception, and cognition, and simultaneously provides a more coherent perceptual-cognitive model, to investigate the possibility of a relationship between visual perceptual skill and mathematic ability. The situated cognitive paradigm further predicts a viable positive relationship between these two constructs. In support of this prediction, the exploratory study performed verified the hypothesis that visual

perceptual skill positively correlates with scholastic mathematics achievement.

Unfortunately, the exploratory nature of the present study provides limited empirical evidence in support of situated cognitive theory. However, the positive results may encourage in-depth quality investigations into mathematical cognition and perceptual skill, including auditory, tactile, and kinaesthetic modalities. Further research may thus assist in the refinement and development of theories within the situated paradigm.

Within this theoretical framework, the study further suggested that mathematics ought perhaps to be taught conceptually, rather than factually. In addition, it highlighted various influences on mathematics education. These were not limited to the didactics and curriculum, but extended to the learning context, including beliefs, attitudes, motivation, and behaviours of both the teachers and the learners. In general, situated cognitive science theory carries the potential for making substantial contribution to developing an appreciation of various aspects of learning, and the practical application of these for the enhancement of education. Núñez (2007) proposed that a comprehensive cognitive science of mathematics, such as that of the situated paradigm, can inform mathematics education on the implementation of teaching methods and learning environments that are compatible with human cognition. Formal algorithmic mathematics education disregards cognitive conceptualisation, from which abstract mathematical cognition arises. In contrast, conceptualistic mathematics education may rest on the natural cognitive process. Furthermore, situated theory suggests that cognition rests on a perceptual foundation. Since perception is a dynamic facility that develops through experience, enhanced perceptual skill is consequently possible through training. Improvement in foundational perceptual skill may credibly enhance supervening cognitive function.

The striving towards a comprehensive cognitive science is argued here to have thus far been a worthwhile endeavour. The involvement of diverse disciplines, with specialised knowledge contributions, has shifted basic attitudes. Philosophically, phenomenology unequivocally embraces the dynamism between the enacted environment and the embodied emergent cognition embedded within it. An extensive amount more work in cognitive science is required in order to achieve a comparable amalgam. Yet the theory of situated cognition is certainly a valuable precursor to such an amalgam. Already, the complex dynamic interactions that have been recognised confound previously accepted distinctions between sensation and perception; individual component perceptual skills; perception and memory; mind and body; cognition and the world; individual and collective consciousness, and so on.

With numerous disciplines contributing to situated cognitive science, a common theoretical framework amongst them may prove beneficial to further progress. Cohesion may possibly be promoted if the various theories were all informed by a phenomenological philosophy, understood as dedicating focus to the dynamic interaction between the mind, body, and world.

The cognitive science theory of situated cognition is thus presented here to hold a promising future. In the present study, it has facilitated edifying exploration into the relationship between visual perceptual skill and mathematic ability. There is a sensible balance between holism and reductionism, operating in concert. Effective communication among the various disciplines involved will ultimately ensure its successful development, providing valuable insight into human cognition, the world, and our place in it.

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Appendices

- A1. Mathematics Test
- A2. Mathematics Test Schedule
- A3. TVPS-3 Response Sheet
- A4. Letter Requesting School Participation
- A5. Letter Requesting Parental Consent
- A6. Letter Accompanying Results for School Principal
- A7. Results Form for Parents
- A8. Data Specification Sheet

A1. Mathematics Test

Name: _____

School: _____

Gender: _____

Do you attend any **extra**-maths lessons? _____

1. Write the word numbers below in digits:

1. Seven hundred and forty-six thousand, six hundred and thirty-eight

2. Six hundred and twenty-four thousand, nine hundred and fifty-three

2. Complete the patterns:

1. ○ △ ○ ○ ↑ ○ ○ ○ △ ○ ___ ___ ___

2. ○ △ ○ ↑ ↓ ○ △ ○ ___ ___ ___

3. Complete the pattern of numbers:

1. 640 ; 630 ; 620 ; _____ ; _____

2. 250 ; _____ ; _____ ; 400

- 3.
- $4^4/8$
- ;
- $4^3/8$
- ;
- $4^2/8$
- ; _____ ; _____

- 4.
- $2^1/4$
- ; _____ ;
- $2^3/4$
- ; 3 ; _____

5. 18 750,5 ; 18725,5 ; _____ ; _____

4. Arrange from biggest to smallest:

1. 76 367 ; 76 674 ; 76 764 ; 67 647

2. 12 639 ; 21 839 ; 21 863 ; 12 689

5. Complete:

1. $120 + 120 =$ _____
2. $5165 + 30 + 21 =$ _____
3. $825 - 75 =$ _____
4. $9650 - 8350 =$ _____

6. Write as common fractions:

1. $0,5 =$ _____
2. $0,68 =$ _____

7. Write as decimal fractions:

1. $\frac{3}{10} =$ _____
2. $\frac{4}{5} =$ _____
3. $\frac{12}{20} =$ _____

8. Arrange from smallest to biggest:

1. $\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{6}$; $\frac{3}{6}$
-

2. $\frac{5}{8}$; $\frac{3}{4}$; $\frac{1}{8}$; $\frac{7}{8}$
-

9. Fill in the spaces in the brackets:

1. $\frac{6}{6} = \frac{12}{(\quad)}$

2. $\frac{3}{4} = (\quad) / 100$

3. $(\quad) / 12 = \frac{4}{6}$

4. $\frac{4}{(\quad)} = \frac{8}{10}$

10. Complete:

1. $2 + \frac{1}{4} = \underline{\hspace{2cm}}$

2. $\frac{2}{3} + \frac{1}{7} = \underline{\hspace{2cm}}$

3. $\frac{3}{10} + \frac{4}{5} = \underline{\hspace{2cm}}$

11. Complete:

1. $\frac{2}{3}$ of 6 = $\underline{\hspace{2cm}}$

2. $\frac{3}{5}$ of 15 = $\underline{\hspace{2cm}}$

3. $\frac{1}{6}$ of 18 = $\underline{\hspace{2cm}}$

12. Find the values of y:

1. $y + 12 = 25$ then $y = \underline{\hspace{2cm}}$

2. $y - 25 = 75$ then $y = \underline{\hspace{2cm}}$

3. $y \times 15 = 45$ then $y = \underline{\hspace{2cm}}$

4. $64 \div 8 = y$ then $y = \underline{\hspace{2cm}}$

5. $y + y = 26$ then $y = \underline{\hspace{2cm}}$

6. $y \times y = 49$ then $y = \underline{\hspace{2cm}}$

13. Round off:

1. 465 to the nearest 10. _____
2. 91 to the nearest 5. _____
3. 2839 to the nearest 1000. _____

14. Round off to the nearest Rand:

1. R 34,06 = _____
2. R 544,95 = _____
3. R 362,50 = _____

15. Complete these measurement sums:

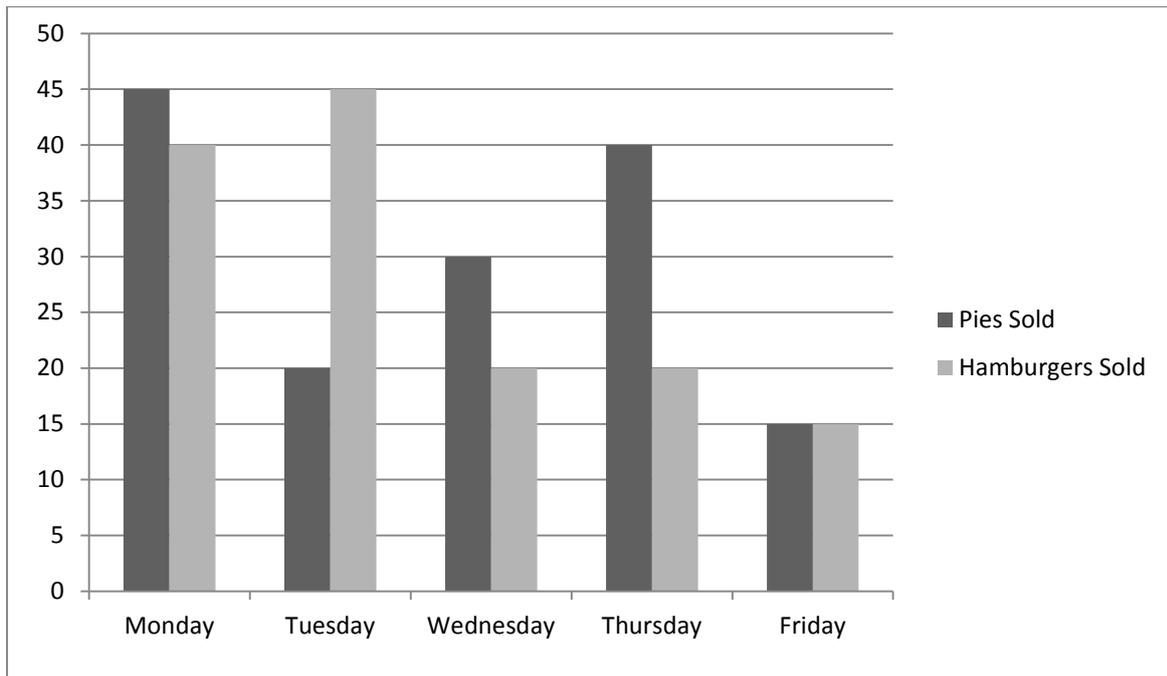
1. 5 mm + 6 mm = _____ cm _____ mm
2. 14 mm + 13 mm = _____ cm _____ mm
3. 65 cm + 6 mm = _____ cm _____ mm
4. 40 cm + 60 cm = _____ m _____ cm
5. 900 m + 700 m = _____ km _____ m

16. Complete these mass problems:

1. 45 g - 24 g = _____ g
2. 1 kg - 500 g = _____ g
3. 60 kg - 45 kg = _____ g

17. What number is 4 times bigger than $6 + 8$? _____
18. If a dog walks 8 km in 1 hour, how far will he walk in 8 hours? _____
19. A cup holds 250 ml of milk. How many cups of milk can you pour from 1 litre? _____
20. A taxi travels at 120 kilometres per hour. How far will it travel in $2\frac{1}{2}$ hours? _____
21. There are 1200 spectators at a soccer match. Girls occupy a quarter of the seats. How many boys are at the match? _____
22. The 30° angle for a wheelchair ramp is 3 times steeper than the builder required. What angle did he want? _____
23. I bought 8 Fizz Pops, 6 Kit-Kats and 10 Gummi Bears from the shop. On the way home, I ate 1 Kit-Kat, and gave my friend 2 Fizz Pops. When I got home, I had to share all the sweets equally with my two brothers.
1. How many Gummi Bears did we each get? _____
 2. How many Kit-Kats did we each get? _____
 3. How many Fizz Pops did we each get? _____
 4. How many of each sweet was left over for Mom? _____

24. Use the graph to answer the following questions:



1. On what day was the most pies sold? _____
2. On what day was the fewest hamburgers sold? _____
3. How many more pies than hamburgers were sold on Thursday? _____
4. On what day were equal number of pies and hamburgers sold? _____
5. How many pies were sold during the whole week? _____

YOU ARE DONE! !!

Thank You

A2. Mathematics Test Schedule**Math Test Schedule**

1. (2) 1. 746 638 (1)
2. 624 953 (1)
2. (2) 1. 000 or $0 \uparrow 0$ (1)
2. $\uparrow \downarrow 0$ (1)
3. (10) 1. 610 (1); 600 (1)
2. 300 (1); 350 (1)
3. $4^{1/8}$ (1); 4 (1) if $4^0/8$ then $(^1/2)$
4. $2^{2/4}$ (1); $3^{1/4}$ (1) if $3^{1/5}$ then $(^1/2)$
5. 18 700, 5 (1); 18 675, 5 (1)
4. (2) 1. 76 764 ; 76 674 ; 76 367 ; 67 647 (1)
2. 21 863 ; 21 839 ; 12 689 ; 12 639 (1)
5. (4) 1. 240 (1)
2. 5 216 (1)
3. 750 (1)
4. 1 300 (1)
6. (4) 1. $1/2$ (2) if $5/10$ then (1)
2. $17/25$ (2) if $68/100$ then (1)
7. (3) 1. 0, 3 (1)
2. 0, 8 (1)
3. 0, 60 (1)
8. (2) 1. $1/6$; $1/3$; $3/6$; $2/3$ (1) if biggest to smallest then $(^1/2)$
2. $1/8$; $5/8$; $3/4$; $7/8$ (1) if biggest to smallest then $(^1/2)$
9. (4) 1. 12 (1)
2. 75 (1)
3. 8 (1)
4. 5 (1)
10. (4) 1. $2^{1/4}$ (1)
2. $17/21$ (1)
3. $1^{1/10}$ (2) if $11/10$ then (1)
11. (3) 1. 4 (1)
2. 9 (1)
3. 3 (1)
12. (12) 1. 13 (2) if $25 - 12$ then (1)
2. 100 (2) if $75 + 25$ then (1)
3. 3 (2) if $45/15$ then (1)
4. 8 (2) if $64/8$ then (1)

5. 13 (2) if $13 + 13$ then (2) if $\frac{26}{2}$ then (1) if $x + y$ then (1)
 6. 7 (2) if 7×7 then (2) if $\sqrt{7}$ then (1)

- 13. (3)** 1. 470 (1)
 2. 90 (1)
 3. 3 000 (1)

- 14. (3)** 1. R 34,00 (1)
 2. R 545,00 (1)
 3. R 363,00 (1)

- 15. (15)** 1. 1 cm 1 mm (3) if 11 mm then (1)
 2. 2 cm 7 mm (3) if 27 mm then (1)
 3. 65 cm 6 mm (3) if 65,6 cm then (1)
 4. 1 m 0 cm (3) if 100 cm then (1)
 5. 1 km 600 m (3) if 1 600 m then (1)

- 16. (3)** 1. 21 g (1)
 2. 500 g (1)
 3. 15 000 g (1)

- 17. (2)** 56 if $4(6+8)$ then (1)

- 18. (2)** 64 if 8×8 then (1)

- 19. (2)** 4 if $1\,000 / 250$ then (1)

- 20. (2)** 300 if $120 \times 2\frac{1}{2}$ or 240 then (1)

- 21. (2)** 900 if $1\,200 - \frac{1}{4}(1\,200)$ or 300 then (1)

- 22. (2)** 10 if $30 / 3$ or 90 then (1)

- 23. (8)** 1. 3 (2) if $10 / 3$ then (1)
 2. 1 (2) if $5 / 3$ then (1)
 3. 2 (2) if $6 / 3$ then (1)
 4. 1G; 2K; 0F (2)

- 24. (5)** 1. Mon (1)
 2. Fri (1)
 3. 20 (1)
 4. Fri (1) if Mon & Tues then ($\frac{1}{2}$)
 5. 150 (1)

A3. TVPS-3 Response Sheet

Name: _____

School: _____

Date of birth: _____

SUBTEST 1

Item #	Response	Score
DIS Ex A		
DIS Ex B		
DIS 1		
DIS 2		
DIS 3		
DIS 4		
DIS 5		
DIS 6		
DIS 7		
DIS 8		
DIS 9		
DIS 10		
DIS 11		
DIS 12		
DIS 13		
DIS 14		
DIS 15		
DIS 16		
Total Subtest 1		

SUBTEST 2

Item #	Response	Score
MEMExC		
MEMExD		
MEM 17		
MEM 18		
MEM 19		
MEM 20		
MEM 21		
MEM 22		
MEM 23		
MEM 24		
MEM 25		
MEM 26		
MEM 27		
MEM 28		
MEM 29		
MEM 30		
MEM 31		
MEM 32		
Total Subtest 2		

SUBTEST 3

Item #	Response	Score
SPA Ex E		
SPA Ex F		
SPA 33		
SPA 34		
SPA 35		
SPA 36		
SPA 37		
SPA 38		
SPA 39		
SPA 40		
SPA 41		
SPA 42		
SPA 43		
SPA 44		
SPA 45		
SPA 46		
SPA 47		
SPA 48		
Total Subtest 3		

SUBTEST 4

Item #	Response	Score
CONEx G		
CONEx H		
CON 49		
CON 50		
CON 51		
CON 52		
CON 53		
CON 54		
CON 55		
CON 56		
CON 57		
CON 58		
CON 59		
CON 60		
CON 61		
CON 62		
CON 63		
CON 64		
Total Subtest 4		

SUBTEST 5

Item #	Response	Score
SEQ Ex I		
SEQ Ex J		
SEQ 65		
SEQ 66		
SEQ 67		
SEQ 68		
SEQ 69		
SEQ 70		
SEQ 71		
SEQ 72		
SEQ 73		
SEQ 74		
SEQ 75		
SEQ 76		
SEQ 77		
SEQ 78		
SEQ 79		
SEQ 80		
Total Subtest 5		

SUBTEST 6

Item #	Response	Score
FGR Ex K		
FGR Ex L		
FGR 81		
FGR 82		
FGR 83		
FGR 84		
FGR 85		
FGR 86		
FGR 87		
FGR 88		
FGR 89		
FGR 90		
FGR 91		
FGR 92		
FGR 93		
FGR 94		
FGR 95		
FGR 96		
Total Subtest 6		

SUBTEST 7

Item #	Response	Score
CLO Ex M		
CLO Ex N		
CLO 97		
CLO 98		
CLO 99		
CLO 100		
CLO 101		
CLO 102		
CLO 103		
CLO 104		
CLO 105		
CLO 106		
CLO 107		
CLO 108		
CLO 109		
CLO 110		
CLO 111		
CLO 112		
Total Subtest 7		

A4. Letter Requesting School Participation

<DATE>

The Principal

I am conducting a Masters research project on the relationship between visual perceptual skill and mathematics achievement in children. The commercially available Test of Visual Perceptual Skills (3rd edition) and a standardized class mathematics test will be used for assessment.

I kindly ask for your assistance in this regard, by allowing access to your Grade 6 pupils. Two sessions, of approximately 1 hour each, will be required.

Letters will be provided that may be sent to the parents requesting their permission. Although the results will be utilized for research purposes, all test information will be viewed as strictly confidential. A report on each child, concerning their visual perceptual strengths and weaknesses, will be issued to their parents and teachers. No fee will be charged for the testing.

With your permission, I will contact your secretary to arrange a meeting with you, to discuss the details and practicalities.

Your assistance would be greatly appreciated.
Yours sincerely

Lynn Freeguard
021 – 883 9399
cdwolf@iafrica.com

A5. Letter Requesting Parental Consent

<DATE>

Dear Parents

Your child's school has agreed to assist in a Masters research project I am conducting. The project concerns the relationship between visual perceptual skill and mathematics achievement. A commercially available Test of Visual Perceptual Skills and a standardized class mathematics test will be used for assessment.

Although the results will be utilized for research purposes, all test information will be viewed as strictly confidential. However, you will receive a report detailing your child's visual perceptual strengths and weaknesses. No fee will be charged for testing.

I kindly ask that you allow your child to participate in the study. Should you agree, please complete the attached permission slip, to be returned to your child's class teacher before <DATE>.

Thank you

Lynn Freeguard
 Wolfswinkel Optometrists 021 – 8839399
 7 Simonsrust Centre, Stellenbosch

 Full name of child: _____

Grade: _____

I hereby grant permission for my child to participate in the above study.

Name of parent / guardian: _____

Signature: _____

Date: _____

A6. Letter Accompanying Results for School Principal

Dear <NAMES>

Thank you for your assistance with my research project, and apologies for any disruption I may have caused.

Visual perception enables us to understand, analyze and interpret what we see. It is a composite skill encompassing a number of sub-skills; and is a primary factor in cognitive development, learning, and many of our daily activities.

Herewith are your students' results of the TVPS-3.

To provide clarification and aid in interpretation:

1. The **Raw Scores** range from 1 – 16.
2. The **Subtest Scaled Scores** range from 1 – 19, and are based on a population distribution having a **mean of 10** and **standard deviation of 3**. They are scaled to account for age.
3. The **Indices** and **Overall Standard Scores** are standard scores which are based on a population distribution having a **mean of 100** and **standard deviation of 15**. Since they are based on sums of scaled scores, age is accounted for already.
4. The **Percentile Ranks** reflect one person's performance relative to the normative population. (E.g.: a percentile rank of 80 indicates the student scored as well as or better than 80% of the same-aged students in the normative population.)
5. The score graphic is a visual summary of the numeric scoring.

Please note:

1. The TVPS-3 does **not** represent a comprehensive assessment of visual perceptual skills.
2. Any test result is an instance of performance at one point in time on one particular set of tasks.
3. The test has been designed to be administered individually; however, for the purpose of the research project habitual classroom ability rather than absolute ability is of greater relevance.

Some tasks to enhance visual perceptual skills:

Dot-to-dots; mazes; find hidden images in a picture; spot the differences between 2 similar pictures; 'memory' card game; 'eye-spy'; 'whirly-words'-(circle words in a grid of letters); playing outdoors.

A useful website: www.eyecanlearn.com

Please do not hesitate to contact me should you require additional information.

Yours Sincerely

Lynn Freeguard
 Wolfswinkel Optometrists 021 – 883 9399
 Simonsrust Centre, Stellenbosch

A7. Results Form for Parents

Dear Parents

Thank you for allowing _____ to participate in my research project.

Visual perceptual skills test results:

	superior	above av.	average	below av.	weak
Visual Discrimination					
Visual Memory					
Spatial Relations					
Form Constancy					
Sequential Memory					
Figure Ground					
Visual Closure					
Overall VPS					

Visual perception enables us to understand, analyse and interpret what we see. It is a composite skill encompassing a number of sub-skills; and is a primary factor in cognitive development, learning, and many of our daily activities.

Please note:

1. The TVPS-3 does **not** represent a comprehensive assessment of visual perceptual skills.
2. Any test result is an instance of performance at one point in time on one particular set of tasks.
3. The test was designed to be administered individually; however, for the purpose of the research project habitual classroom ability rather than absolute ability is of greater relevance.

Some tasks to enhance visual perceptual skills:

Dot-to-dots; mazes; find hidden images in a picture; spot the differences between 2 similar pictures; 'memory' card game; 'eye-spy'; 'whirly-words' -(circle words in a grid of letters); playing outdoors.

A useful website: www.eyecanlearn.com

Please do not hesitate to contact me should you require additional information.

Yours sincerely

Lynn Freeguard

Wolfswinkel Optometrists 021 – 883 9399

Simonsrust Centre, Stellenbosch

A8. Data Specification Sheet

Data Specification

Hypothesis:	Visual perceptual skill positively correlates with scholastic mathematics achievement. Determine relative contribution of component visual perceptual skills to mathematics achievement.
Null Hypothesis:	There is no relationship between visual perceptual skill and scholastic mathematics achievement.
Operationalisation:	Raw Math = percentage score achieved on maths test Scaled Math = scaled score derived from standardised Raw Math Overall VPS = standard score as measured by TVPS-3 DIS, MEM, SPA, CON, SEQ, FGR, CLO = scaled scores as measured by TVPS-3 subtests
Data Analysis Techniques:	Descriptive statistics <i>t</i> Tests Standardisation / <i>z</i> scores Pearson's product-moment correlations Multiple regression analysis Correlation Matrix
Abbreviations:	
DIS	visual discrimination
MEM	visual memory
SPA	spatial relations
CON	form constancy
SEQ	sequential memory
FGR	figure-ground
CLO	visual closure
S-E	socio-economic status
EML	extra-math lesson attendance

Data Description:

Variable Name	Range of Values	Measurement Level	Variable Type	Values / Labels
Age	127 - 164	ratio	x	months
Gender	1 - 2	categorical	x	1 = male; 2 = female
EML	1 - 2	categorical	x	1 = no; 2 = yes
S-E	1 - 2	categorical	x	1 = low; 2 = high
School	1 - 3	categorical	x	1 = Stellenbosch government school 2 = Somerset West private school 3 = Southern Suburb private school
Raw Math	0 - 100	ratio	y	higher score = superior math achievement
Scaled Math	1 - 99	interval	Y	higher score = superior math achievement
Overall VPS	55 - 145	interval	X / y	higher score = superior VPS
DIS	1 - 19	interval	X₁	higher score = superior visual discrimination
MEM	1 - 19	interval	X₂	higher score = superior visual memory
SPA	1 - 19	interval	X₃	higher score = superior spatial relations
CON	1 - 19	interval	X₄	higher score = superior form constancy
SEQ	1 - 19	interval	X₅	higher score = superior sequential memory
FGR	1 - 19	interval	X₆	higher score = superior figure-ground
CLO	1 - 19	interval	X₇	higher score = superior visual closure