Pricing European and American Bond Options under the Hull-White extended Vasicek Model

by

Marc Mukendi Mpanda

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Supervisor: Dr. EM Rapoo

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Dedication

To my parents Theodore Kalombo and Cornelie Kapya Mwadi,
And my great families Kalombo, Bulembi and Mpanda.
Abstract

In this dissertation, we consider the Hull-White term structure problem with the boundary value condition given as the payoff of a European bond option. We restrict ourselves to the case where the parameters of the Hull-White model are strictly positive constants and from the risk neutral valuation formula, we first derive simple closed-form expression for pricing European bond option in the Hull-White extended Vasicek model framework. As the European option can be exercised only on the maturity date, we then examine the case of early exercise opportunity commonly called American option. With the analytic representation of American bond option being very hard to handle, we are forced to resort to numerical experiments. To do it excellently, we transform the Hull-White term structure equation into the diffusion equation and we first solve it through implicit, explicit and Crank-Nicolson (CN) difference methods. As these standard finite difference methods (FDMs) require truncation of the domain from infinite to finite one, which may deteriorate the computational efficiency for American bond option, we try to build a CN method over an unbounded domain. We introduce an exact artificial boundary condition in the pricing boundary value problem to reduce the original to an initial boundary problem. Then, the CN method is used to solve the reduced problem. We compare our performance with standard FDMs and the results through illustration show that our method is more efficient and accurate than standard FDMs when we price American bond option.

Keywords: Term structure equation, Hull-White extended Vasicek model, Coupon bearing and zero-coupon bond option, European and American bond option, Diffusion equation, Finite Difference Methods and Artificial Boundary method.
Declaration

I declare that: “Pricing American and European Bond Options under the Hull-White extended Vasicek Model” is my own work and has not been submitted in any institution. All the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

Mr. Mukendi Mpanda

Date
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# Table of Contents

Abstract ............................................................................................................................... iii  
Declaration ......................................................................................................................... iv  
Acknowledgements ........................................................................................................... v  
Table of Contents ............................................................................................................. vi  
Notations ............................................................................................................................ ix  
List of Figures ..................................................................................................................... xii  
List of Tables ..................................................................................................................... xiii  

0. Introduction ...................................................................................................................... 1  

1. Stochastic processes for the dynamics of one factor short interest rates 5  
   1.1. Stochastic processes and Brownian motion as a specific case......................... 6  
      1.1.1. Stochastic process in continuous time......................................................... 6  
      1.1.2. Brownian motion ...................................................................................... 9  
   1.2. The Stochastic integral and Ito formula ............................................................... 12  
      1.2.1. Definitions and properties ........................................................................ 12  
      1.2.2. The stochastic differential and Ito formula .............................................. 13  
      1.2.3. Evaluation of Ito integral via Ito formulae ................................................. 14  
   1.3. Stochastic differential equations for one-factor short interest rate model  
      1.3.1. Existence and uniqueness result ................................................................ 16  
      1.3.2. Two important SDEs for our work ............................................................. 17  
   1.4. Link between SDEs and parabolic PDEs ............................................................. 22  
      1.4.1. Formula for the generator of diffusion process ......................................... 22  
      1.4.2. Kolmogorov’s backward equation .............................................................. 23  
      1.4.3. Change of measure: The Girsanov Theorem ............................................ 28  
   1.5. Conclusion .............................................................................................................. 31  

2. Derivation of the term structure equation and the bond price process  
   under the Hull-White extended Vasicek model ............................................................ 32  

vi
2.1. Bond prices and interest rate models ................................................. 32
  2.1.1. Bond market model ................................................................. 33
  2.1.2. The money account process ..................................................... 36
2.2. Derivation of the term structure equation ........................................ 36
  2.2.1. Term Structure Equation ......................................................... 36
  2.2.2. The risk neutral valuation formula .......................................... 39
  2.2.3. A brief literature review for the dynamics of one-factor short interest rate. ................................................................. 42
  2.2.4. Affine term structures ............................................................... 44
2.3. Bond prices under Vasicek and Hull–White models ......................... 47
  2.3.1. Bond price under Vasicek model .............................................. 47
  2.3.2. Bond price under the Hull-White model .................................... 52
2.4. Conclusion ..................................................................................... 56

3. Pricing European bond options under the Hull-White extended Vasicek model ................................................................. 57
  3.1. Preparations ................................................................................ 57
    3.1.1. Statement of the problem .................................................. 57
    3.1.2. Valuation of interest rate derivatives in general .................... 59
  3.2. European bond option under the Hull-White model ..................... 62
    3.2.1. European zero-coupon bond option .................................... 62
    3.2.2. European coupon-Bearing bond option ............................... 66
    3.2.3. Illustration and results ......................................................... 70
  3.3. Conclusion .................................................................................. 73

4. Pricing American bond options under the Hull-White extended Vasicek model ................................................................. 74
  4.1. Analytic representation of American bond option ....................... 74
    4.1.1. Formulation ......................................................................... 75
    4.1.2. Analytic representations of the American bond option ............. 77
  4.2. From the Hull-White TSE to the diffusion equation ....................... 78
4.2.1. Elimination of the time independent drift in the Hull-White TSE .... 78
4.2.2. Further transformations to the diffusion equation ......................... 79
4.2.3. A new formulation of a boundary value problem for the obtained diffusion equation ................................................................. 82

4.3. Solution of the obtained diffusion equation through FDMs ............... 83
4.3.1. Explicit, Implicit and Crank-Nicolson scheme ............................ 83
4.3.2. Crank-Nicolson method over an unbounded domain ................. 86
4.3.3. Illustration and results.............................................................. 91

5. General Conclusion ............................................................................. 99

Appendices .............................................................................................. 100

A.1. Matlab codes for European bond option ......................................... 100
A.2. Matlab codes for standard FDMs and the CN method over an unbounded domain ................................................................. 102
    A.2.1. SOR method to Problems (4.22) and (4.38) .............................. 102
    A.2.2. Matlab Codes...................................................................... 108

References .............................................................................................. 115
Notations

General mathematical symbols used in the dissertation:

\[ \mathbb{R} \] Set of real numbers

\[ \mathbb{N} \] Set of positive integers

\[ P, Q \] Probability measures

\[ E^P[\cdot] \] Expectation with respect to the probability measure \( P \)

\[ \text{Var}[\cdot] \] Variance

\[ \text{Cov}[\cdot] \] Covariance

\[ \ln(\cdot) \] Natural logarithm

\[ \Delta t \] Small increment in \( t \)

\[ \mathcal{T} \] Transposed

\[ \mathcal{C}^0[a, b] \] Set of functions that are continuous on \( [a, b] \)

\[ \mathcal{C}^p[a, b] \] Set of functions that are \( p \)-times continuously differentiable on \( [a, b] \)

\( (\Omega, \mathcal{F}, P) \) Probability space where \( \Omega \) is the sample space, \( \mathcal{F} \) is the \( \sigma \)-algebra and \( P \) the probability measure

\[ L^p(\Omega, \mathcal{F}, P) \] Space in \( (\Omega, \mathcal{F}, P) \) such that for any function \( f(t) \in \mathcal{C}^0[a, b] \),

\[ \|f(t)\|_p \equiv \left( \int_a^b |f(t)|^p dt \right)^{1/p} < \infty \]

\[ \mathcal{B} \] Borel \( \sigma \)-algebra

\[ N(\cdot) \] Standard normal distribution.
Integers:

\(d, i, j, k, l, m, n, p.\)

Various variables:

\(X(t), X_t, X\) \hspace{1cm} \text{Random variable}

\(W(t), W_t\) \hspace{1cm} \text{Standard Brownian motion (or standard Wiener process)}

\(L_t\) \hspace{1cm} \text{Likelihood process.}

Abbreviations:

PDE \hspace{1cm} \text{Partial Differential Equation}

SDE \hspace{1cm} \text{Stochastic Differential Equation}

SOR \hspace{1cm} \text{Successive Over Relaxation}

TSE \hspace{1cm} \text{Term Structure Equation}

CN \hspace{1cm} \text{Crank-Nicolson}

FDM \hspace{1cm} \text{Finite Difference Method}

\(Sup\) \hspace{1cm} \text{Supremum}

\(Inf\) \hspace{1cm} \text{Infimum}

UD \hspace{1cm} \text{Unbounded Domain}

\(a.s.\) \hspace{1cm} \text{Almost surely.}

Elements of the one-factor Short interest rate model:

\(r(t), r_t\) \hspace{1cm} \text{Strong solution of the one-factor short interest rate model}

\(S, T\) \hspace{1cm} \text{Maturity dates}

\(K\) \hspace{1cm} \text{Strike price}

\(\sigma(t)\) \hspace{1cm} \text{Volatility term of } \ r(t)
\( \mu(t) \)  \( r(t) \) Drift term of \( r(t) \)

\( \theta(t) \) Time-dependent drift for the Hull-White model

\( a(t) \) The drift of the Hull-White model

\( V(t, r, T) \) Price of the European option

\( p(t, S) \) Bond price maturing at the date \( S \)

\( p_c(t, S_i) \) Price of the coupon-bearing bond

\( R(t, T) \) Yield curve.

\( f(t, S, T) \) The forward rate for \( [S, T] \) contracted at date \( t \)
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Two sample paths for the GBM</td>
<td>23</td>
</tr>
<tr>
<td>1.2</td>
<td>Three sample paths for the OU process</td>
<td>23</td>
</tr>
<tr>
<td>2.1</td>
<td>Four different yield curves</td>
<td>35</td>
</tr>
<tr>
<td>3.1</td>
<td>Diagram for pricing interest rate derivatives</td>
<td>61</td>
</tr>
<tr>
<td>3.2</td>
<td>Prices of European call/put option for $a = 0.1, \sigma = 0.1$</td>
<td>71</td>
</tr>
<tr>
<td>4.1</td>
<td>Results of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Flat yield Curve</td>
<td>93</td>
</tr>
<tr>
<td>4.2</td>
<td>Results of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Upward yield Curve</td>
<td>93</td>
</tr>
<tr>
<td>4.3</td>
<td>Results of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Downward yield Curve</td>
<td>94</td>
</tr>
<tr>
<td>4.4</td>
<td>Results of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Humped yield Curve</td>
<td>94</td>
</tr>
<tr>
<td>4.5</td>
<td>Relative errors estimation of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for flat yield Curve</td>
<td>95</td>
</tr>
<tr>
<td>4.6</td>
<td>Relative errors estimation of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Upward yield Curve</td>
<td>95</td>
</tr>
<tr>
<td>4.7</td>
<td>Relative errors estimation of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Downward yield Curve</td>
<td>96</td>
</tr>
<tr>
<td>4.8</td>
<td>Relative errors estimation of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Humped yield Curve</td>
<td>96</td>
</tr>
</tbody>
</table>
List of Tables

Table 2.1: Summary of one-factor short interest rate models 45
Table 3.1: Results of European bond option 71
Table 4.1: Call option on American zero-coupon bond option under the CN method over an unbounded domain 90
Table 4.2: Relative errors in percentage for the CN method over an unbounded domain 90
Table 4.3: Call option on American zero-coupon bond option under the CN method 91
Table 4.4: Relative errors of explicit FDM 91
Table 4.5: Call option on American zero-coupon bond option under the explicit FDM. 92
Table 4.6: Relative errors in percentage for the explicit FDM. 92
Chapter 0:

Introduction

In the financial world, the short interest models play an important role in fixed-income security pricing; among them, the Hull-White model [36–40]. As an extension of Vasicek model [36], the Hull-White model assumes that the short rate follows the mean-reverting stochastic differential equation (SDE) and presents special features which are analytical tractability on liquidly traded derivatives [36], super calibration ability to the initial term structure [37] and elegant tree-building procedure [39]. These make the model very attractive as a practical tool.

On another hand, when we need to price interest rate derivatives such as bond option, interest rate swap, interest rate cap and interest rate swaption, we have to perform options on these derivatives. One attractive and simple option that offers us nice analytic results is the European option, see for e.g. [8, 23], through which the option is exercised only on the expiration date. For the case where there is early exercise of the option, we talk in terms of American option, see [13, 44]. Thus, an American option is a European one with the additional right to exercise it any time prior to expiration.

In the arbitrage free framework, pricing interest rate derivatives under the one-factor short interest rate model lead us to the parabolic partial differential equation (PDE) called term structure equation (TSE) [8] with the boundary condition given as the payoff function. The main problem for pricing options written on interest rate derivatives under interest rate models is how to solve...
these kinds of parabolic PDE associated to a given payoff option. This dissertation explores the bond option being considered as a standard interest rate derivative\(^1\).

Many papers have addressed the solution of the problem stated above, including Amin and Madsen [1], Brace and Musiela [11], and Madsen [49], who all worked within the Gaussian Heath-Jarrow-Morton framework. In addition, Jamshidian [43] derives the European bond option under the Vasicek model where the resulting pricing formula resembles the Black-Scholes formula [10] that has a similar interpretation. In this dissertation, by referring mainly to [43], we also derive the formula for pricing European bond option under the Hull-White extended Vasicek model. In this study, we introduce the forward price process and, from the risk neutral valuation formula, the formula used in [43, formula (8)] and by using probability computational skills, we arrive to derive a simple closed-form expression. This performance can be also considered as a special case of [1], [11] or [49].

The study of American bond option relies on numerical experiments due to its complexity. We first reduce the term structure equation which is a parabolic PDE to the diffusion equation with the help of some transformations and define a pricing boundary value problem under the diffusion equation which shall be discussed later. As finite difference methods (FDMs) are straightforward to implement and the resulting uniform rectangular grids are comfortable, we then first use these methods, especially explicit, implicit and Crank-Nicolson methods to solve the obtained pricing boundary value problem.

It is well-known that the explicit Finite Difference Method requires the condition of the type \(0 < \Delta t \leq \frac{\Delta z^2}{2}\) for stability, where \(\Delta t\) and \(\Delta z\) represent respectively the small time step and the step width of the scheme (see for e.g. Scott [53], Seydel [54] or Proposition 4.3.1). In practice, it is sometimes desirable to change the length of time step. In contrast, both the implicit FDM and Crank-Nicolson method can achieve unconditional stability [54]. Unfortunately, implicit schemes including both implicit FDM and CN method are constructed for a PDE with a bounded domain. Therefore, the implementation requires the truncation of

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\(^1\) Simply because from it we may derive other interest rate derivatives without any difficulties.
the infinite domain to the finite one which may deteriorate the computational efficiency of American bond option.

To circumvent the issue stated above, Kangro and Nicolaides [46] studied the boundary condition of the PDE of the Black–Scholes type. In their work, they stipulate that an alternative method to solve problems with unbounded domains is to impose an artificial boundary condition and then an exact boundary condition is derived on the artificial boundary based on the original problem. In the field of interest rate derivatives, Hun and Wu [41] extend the Kangro and Nicolaides results and propose an artificial Neumann boundary condition for pricing American bond option under Black–Scholes dynamics. Wong and Zhao [60] generalize the artificial boundary condition to the CEV model\(^2\) and show that the proposed artificial boundary condition is exact and the corresponding implicit scheme is unconditionally stable, efficient and accurate. In contrast, Tangman et al. [59] developed a high-order optimal compact scheme for pricing American options under the Black-Scholes dynamics without considering artificial boundary conditions as in [46].

To make more consistent these approaches listed above, Wong and Zhao [63] recently proposed an artificial boundary method based on the PDEs to price interest rate derivatives with early exercise feature. This approach is accurate, efficient and robust to the truncation. On the debit side of the balance sheet, the obtained result is very complex and very difficult to implement numerically. To remedy this issue we refer to above papers to study the Crank-Nicolson method over an unbounded domain in which we perform the CN method on the initial boundary value problem obtained from an exact artificial boundary condition. We then compare our performance with standard explicit, implicit FDMs and standard CN methods and draw the conclusion.

The rest of the dissertation is structured as follows. The **first chapter** introduces the general theory of stochastic process for the dynamics of one-factor short interest rates. In particular, two important SDEs are discussed, namely, the Geometric Brownian Motion and the Ornstein-Uhlenbeck model. We then make

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\(^2\) CEV is an acronym of Constant Elasticity of Variance widely used in stochastic volatility model and resembles Cox–Ingersoll–Ross short interest rate model.
the link between SDEs and parabolic PDEs and show how to solve them through some theorems. Since the Term Structure Equations are modelled in terms of bond price, we introduce this topic in the second chapter. Firstly, we derive the term structure equation and find the bond price process under both Vasicek and Hull-White extended Vasicek models which are very useful in the pricing methodology. The third chapter derives simple closed-form expression for pricing European option written on a zero-coupon and coupon-bearing bonds in terms of forward price under the Hull-White extended Vasicek model with the help of bond price process found in the previous chapter, risk neutral valuation formula and Jamshidian formula. Finally, the last chapter deals more with numerical methods for pricing American bond option. We show the complexity of analytic solutions of American bond option. To circumvent this difficulty, we first transform the American bond option problem into the diffusion (or heat) problem by making some transformations of the Hull-White TSE until we get the diffusion (or heat) equation. We then apply standard FDMs to the obtained pricing boundary value problem and we build the Cranck-Nicolson method over an unbounded domain. We close this work with a general conclusion and Matlab codes in the Appendix.
Chapter 1:

Stochastic processes for the dynamics of one factor short interest rates

Most of the research papers show that the dynamics of one-factor short interest rates in continuous time get the form (see [37])

\[ dr(t) = \mu(r, t) dt + \sigma(r, t) dW_t, \]  

(1.0)

where \( r = r(t) \) is the stochastic process which represents the one-factor short interest rate. In this dissertation, \( r(t) \) is the mean reverting Ornstein-Uhlenbeck (OU) process. \( \mu(r, t) \) and \( \sigma(r, t) \) are respectively the drift and the volatility terms of the stochastic process \( r(t) \) and finally \( W_t \) is another special case of stochastic process known as the standard Brownian motion. This is the reason we introduce this chapter as a mathematical background used for pricing interest rate derivatives.

The chapter begins with definitions and properties of stochastic processes in continuous time and we introduce the standard Brownian motion as a special case of stochastic processes in continuous time in Section 1.1. Afterward, we introduce some useful tools such as stochastic Ito integral and Ito formula that will help us to solve the equation (1.0) in general case in Section 1.2. In Section 1.3, we solve stochastic differential equations of the type (1.0). In particular, the OU equation, which is very useful for the Hull-White extended Vasicek model and the Geometric Brownian Motion (GBM) which models the bond price as discussed in the next chapter.
There is a link between stochastic differential equations and parabolic partial differential equations. We make that link by introducing the Feynman-Kac theorem and discuss the Girsanov theorem which performs transformations of measures in a Martingale framework through Radon-Nikodym derivative.

1.1. Stochastic processes and Brownian motion as a specific case

Roughly speaking, a stochastic differential equation is an equation from which the solution is a stochastic process. It seems appropriate to introduce this topic in this dissertation. There are several types of stochastic processes, including the Brownian motion, the Geometric Brownian motion and the mean-reverting Ornstein-Uhlenbeck process, which are very useful for this study. This section introduces the theory on Brownian motion and the two remaining stochastic processes should be the purpose of Section 1.3. First and foremost, definitions and some important properties of stochastic processes drawn from [4], [7] and [17], are discussed:

1.1.1. Stochastic process in continuous time

Definitions and some important properties

Let us first start with some preliminaries.

**Definition 1.1.1.** Let $\Omega$ be a non-empty set. The collection $\mathcal{F}$ of subsets of $\Omega$ is called a $\sigma$-algebra on $\Omega$ if

(i) The empty set $\emptyset$ and the set $\Omega$ belong to $\mathcal{F}$;

(ii) If $F \in \mathcal{F}$ then its complement $F^c = \Omega \setminus F \in \mathcal{F}$;

(iii) For all $n \in \mathbb{N}$, if $\{F_i\}_{i=1}^n$ is a finite sequence of subsets such that $F_i \in \mathcal{F}$ then the union $\bigcup_{i=1}^n F_i \in \mathcal{F}$. Likewise, if $\{F_i\}_{i=1}^\infty$ is an infinite countable sequence of subsets, each of which is in $\mathcal{F}$, then the union $\bigcup_{i=1}^\infty F_i$ is also in $\mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space, the set $\Omega$ is known as the sample space and the members of the collection $\mathcal{F}$ are called measurable sets. In addition, the measurable function is defined as a map from one measurable space to another. Concretely, we define a measurable function as follows:
Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. A function

$$X: (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$$

is said to be measurable if for every set $F \in \mathcal{F}_2$, $X^{-1}(F) \in \mathcal{F}_1$.

We note that for finite or countable infinite sample spaces, the set $\mathcal{F}$ is often defined as the set of all subsets of $\Omega$. For sample space consisting of the real numbers or intervals of real numbers, the set $\mathcal{F}$ is often defined as the Borel sigma algebra\(^3\) that is denoted by $\mathcal{B}(\cdot)$, the sample space will be usually denoted by $\mathcal{B} \subseteq \mathbb{R}$ and the measurable space under the Borel sigma algebra by $(\mathcal{B}, \mathcal{B}(\cdot))$.

This will enable us to introduce later the concept of random variable.

**Definition 1.1.2** Let $(\Omega, \mathcal{F})$ be measurable space. A probability measure $P$ is a function $P: \mathcal{F} \to [0,1]$ such that

(i) For all measurable sets $F \in \mathcal{F}$, $P(F) \geq 0$

(ii) $P(\emptyset) = 0$ and $P(\Omega) = 1$

(iii) For all collections $\{F_i\}_{i \in \mathbb{N}}$, of pairwise disjoint sets in $\mathcal{F}$,

$$P\left(\bigcup_{i \in I} F_i\right) = \sum_{i \in I} P(F_i).$$

For the case of infinite unions of measurable sets we have the following additional property

$$P\left(\bigcup_{i=1}^{\infty} F_i\right) = \lim_{i \to \infty} P(F_i),$$

where all measurable sets are disjoints in $\mathcal{F}$ and where $F_i \subseteq F_{i+1}$, $i = 1, 2, ..., +\infty$.

The triple $(\Omega, \mathcal{F}, P)$ is called probability space.

**Definition 1.1.3** Let $(\Omega, \mathcal{F})$ and $(B \subseteq \mathbb{R}, \mathcal{B})$ be two measurable spaces. A random variable is a function $X: \Omega \to B$ such as $X^{-1}(B) \in \mathcal{F}$, for all $B \in \mathcal{B}$. Therefore, a random variable is also a measurable function.

Using the definition above, we can now define the stochastic process in continuous time as follows:

\(^3\) A good explanation about Borel sigma algebra can be found in [67].
Definition 1.1.4 (Stochastic process). Let \((\Omega, \mathcal{F}, P)\) be a probability space, \(T\) an index set. A family of random variables \(X = \{X(t, \omega), t \in T \text{ and } \omega \in \Omega\}\) is called a stochastic process on \((\Omega, \mathcal{F}, P)\).

The mapping \(T \ni t \mapsto X(t, \omega)\) is the sample path of \(X(t, \omega)\). If \(T\) is an interval in \(\mathbb{R}\) then we shall say that \(\{X(t, \omega), t \in T \text{ and } \omega \in \Omega\}\) is a stochastic process in continuous time. A useful example of stochastic process in continuous time is the Brownian motion (or Wiener process) denoted by \(W(t)\) and discussed in the next subsection.

Definition 1.1.5 (Filtration). A filtration is a collection of sub-\(\sigma\)–algebras \(\{\mathcal{F}_t; t \geq 0\}\) of the \(\sigma\)–algebra \(\mathcal{F}\) such that if \(s \leq t\) then \(\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}\). For a given stochastic process \(X\), the notation \(\mathcal{F}_t^X\) for the filtration \(\sigma\{X_s; 0 \leq s \leq t\}\) is called the natural filtration associated to the process \(X\) or simply the filtration generated by the process \(X\). In particular, the filtration associated to a Brownian motion \(W\) is denoted by \(\mathcal{F}_t^W\).

We note that a filtration \(\mathcal{F}_t\) can also be viewed as all known information up to the time \(t\).

Definition 1.1.6. We say that the stochastic process \(X\) is adapted to the filtration \(\mathcal{F}_t\) if the random variable \(X_t\) is \(\mathcal{F}_t\)-measurable. We may then observe that a stochastic process is always adapted to its natural filtration.

We note that a random variable is \(\mathcal{F}_t\)-measurable if \(X_t^{-1}(B) \in \mathcal{F}_t\) for all measurable sets \(B\).

Now we give an important property of stochastic processes known as a martingale property in terms of a filtration.

Definition 1.1.7 (Martingale Property). The stochastic process \(X = \{X_t, t \in \mathbb{R}^+\}\) adapted to the filtration \(\mathcal{F}_t\) has a martingale property if the following conditions hold:

1. \(X_t\) is \(P\)-integrable for all \(t \in \mathbb{R}_+\) \hspace{1cm} (1.1)

---

\(^4\) For the simplicity reason, we will denote the random variable by \(X(t)\) instead of \(X(t, \omega)\) throughout.
2. For all \((s,t) \in \mathbb{R}_+^2, s < t\),
\[
E[X_t \mid \mathcal{F}_s] = X_s \text{ a.s.} \quad (1.2)
\]
The stochastic process \(X_t \mid \mathbb{R}_+^+\) is a submartingale if in addition to Condition 1
\[
E[X_t \mid \mathcal{F}_s] \geq X_s, \quad (1.3)
\]
If the inequality is reversed, then \(X_t \mid \mathbb{R}_+^+\) is a supermartingale.

The martingale Condition (1.2) is equivalent to
\[
E[X_t - X_s \mid \mathcal{F}_s] = 0. \quad (1.4)
\]
The Condition (1.4) means that a martingale is a real valued stochastic process defined by the property that the conditional mean of an increment of the process conditioned on past information is zero.

In addition, a stochastic process can possess the following properties:

- homogeneous or stationary increments
- independent increments
- Markovian Property

These properties are well explained with details in [4].

1.1.2. Brownian motion

The Brownian motion plays an important role in stochastic calculus involving integration (which is our next section) with respect to Brownian motion. This calculus is used to study dynamical systems modelled by stochastic differential equations.

**Definition 1.1.8.** The Brownian motion with drift is a stochastic process \(X = \{X(t), t \in \mathbb{R}^+\}\) with the following properties:
1. \(X(0) = 0\) a.s and the sample paths of \(X(t)\) are continuous almost surely,
2. \(X(t)\) has independent increments and for \(s < t\) the increment \(X(t) - X(s)\) has a normal distribution with mean \(\mu t\) and variance \(\sigma^2 t\), \(\mu\) and \(\sigma\) are fixed parameters.
3. For every $t_1 < t_2 < \cdots < t_n$, the increments
\[ X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1}) \]
are independent random variables with distributions given in assertion 2.

Since the second property of the definition stipulates that a Wiener Process $X_t$ has stationary, independent increments; it follows that $X_t$ is a Markov process with continuous sample paths and consequently (see for e.g. [7]) a Brownian motion is a diffusion process. Furthermore, the normal (or Gaussian) distribution of Brownian process with drift rate $\mu$ and variance rate $\sigma^2$ is given by
\[ N(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{ -\frac{(x - \mu t)^2}{2\sigma^2 t} \right\}. \]

For the particular case $\mu = 0$ and $\sigma^2 = 1$, the Brownian Process is called the standard Brownian motion (or Standard Wiener process) that we denote by $\{W_t, t \geq 0\}$. Therefore, the normal distribution is given by
\[ N(x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{x^2}{2t} \right\}, \]
and it is also called the standard normal distribution. In this study, we will deal with only the standard Wiener process $\{W_t, t \geq 0\}$. Useful properties are stated below:

**Proposition 1.1.9.** If $\{W_t, t \geq 0\}$ is the standard Wiener process and let $k \in \mathbb{N}$ a constant. Then
\[ E[W_t^{2k}] = \frac{(2k)!}{2^k k!} t^k \text{ and } E[W_t^{2k+1}] = 0. \]

**Proof:** We know that
\[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} dx = 1, \]
by scaling the variable $x$ to $\alpha x$ with $\alpha$ positive constant equal to $\frac{1}{2t}$ we get
\[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \alpha^{-1/2}, \]
and so now by shifting the variable $x$ to $x - \frac{i\beta}{2\alpha}$ we get
\[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\{-\alpha x^2 + i\beta x\} dx = \alpha^{-1/2} \exp\left\{ -\frac{\beta^2}{4\alpha} \right\}, \]
by expanding both sides of this equation in a power series in $\beta$ we obtain
\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^{2k+1} \exp(-ax^2) dx = 0
\]
and
\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} x^{2k} e^{-ax^2} dx = \frac{(2k)!}{2^{2k} k!} \alpha^{-k-1/2}.
\]
Thus from the two last equations we deduce
\[E[W_t^{2k+1}] = 0,
\]
and
\[E[W_t^{2k}] = \frac{1}{2\pi t} \int_{-\infty}^{+\infty} x^{2k} \exp\left\{-\frac{x^2}{2t}\right\} dx.
\]
which gives
\[E[W_t^{2k}] = \frac{1}{\sqrt{2\pi t}} \frac{(2k)!}{2^{2k} k!} \left(\frac{1}{2t}\right)^{-k} = \frac{(2k)!}{2^k k!} \sqrt{\pi t}.
\]

**Proposition 1.1.10.** Let \(\{W_t, \ t \geq 0\}\) be the family of standard Brownian motions. For any \(0 \leq s < t\), we have
1. \(E[W(t)W(s)] = \min(s, t),\)
2. \(E[|W_t - W_s|^2] = t - s.\)

**Proof:** To show the result in Assertion 1, we decompose the product \(W(t)W(s) = [W(t) - W(s)]W(s) + W^2(s)\) and we obtain
\[E[W(t)W(s)] = E[[W(t) - W(s)]W(s)] + E[W^2(s)]
\]
Since \(W(t) - W(s)\) and \(W(s)\) are independent and both \(W(t) - W(s)\) and \(W(s)\) have zero mean, so
\[E[W(t)W(s)] = E[W^2(s)] = s = \min(s, t).
\]
The proof of the Assertion 2 is a direct application of Definition 1.1.12.

**Proposition 1.1.11.** The stochastic processes \(W(t)\) and \(|W(t)|^2 - t\) are martingales with respect to the filtration \(\mathcal{F}_t\) generated by the Brownian motion

**Proof:** For any \(0 \leq s < t\)
\[E[W(t)|\mathcal{F}_s] = E[W(t) - W(s)|\mathcal{F}_s] + E[W(s)|\mathcal{F}_s]
\]
and
\[E[|W(t)|^2|\mathcal{F}_s] = E[|W(t) - W(s)|^2|\mathcal{F}_s] + E[2W(t)W(s)|\mathcal{F}_s] - E[W(s)^2|\mathcal{F}_s]
\]
Hence
\[ E[(W(t) - W(s))^2] + 2W(s)E[W(t)|F_s] - W(s)^2 = t - s + 2W(s)^2 - W(s)^2 = t - s + W(s)^2 \]

From this proposition, we give another one which characterizes the Wiener process by its martingale properties.

**Theorem 1.1.12 (Levy's martingale characterization).** The process \( W(t) \) is a Brownian motion if and only if the following conditions are satisfied:

1. \( W(0) = 0 \) a.s., and the sample paths \( t \to W(t) \) are continuous a.s.
2. \( W(t) \) is a martingale with respect to \( \mathcal{F}_t \)
3. \( |W(t)|^2 - t \) is a martingale with respect to \( \mathcal{F}_t \)

**Proof:** See, for e.g. [52].

We will use this theorem mainly for the proof of Girsanov theorem discussed in the next section.

### 1.2. The Stochastic integral and Ito formula

Stochastic integral and Ito formula are important topics in stochastic calculus. So, they are needed when we attempt to solve stochastic differential equations analytically. Here we follow [15] and [45].

#### 1.2.1. Definitions and properties

**Definition 1.2.1.** Let \( f \) be a stochastic process and \( W \) the standard Brownian motion both of them adapted to the same filtration \( \mathcal{F}_t \) generated by the Brownian motion \( W \). The (stochastic) Itô integral is a random variable given by

\[ X(t) = \int_0^t f(s) dW(s). \]  \hspace{1cm} (1.5)

We note that the stochastic process \( f \) has an additional property to be mean square integrable, that means

\[ E \left[ \int_0^t |f(s)|^2 \, ds \right] < \infty. \]
We further note that the mean square integrability implies that \( f(t) \in L^2(\Omega, F, P) \) that is
\[
\|f(t)\|_2 \equiv \left( \int_0^t |f(s)|^2 ds \right)^{1/2} < \infty.
\]

**Proposition 1.2.2.** Let \( f, g \) be two adapted processes as defined above and \( \alpha, \beta \) scalars then the following properties of Ito integral hold:

1. \( E \left[ \int_a^b f(t) dW(t) \right] = 0 \), \hspace{1cm} (1.6)

2. \( E \left[ \int_a^b f(t) dW(t) \int_a^b g(t) dW(t) \right] = \int_a^b E[f(t)g(t)] dt \), \hspace{1cm} (1.7)

3. \( E \left[ \left( \int_a^b f(t) dW(t) \right)^2 \right] = \int_a^b E[(f(t))^2] dt = \text{Var} \left[ \int_a^b f(t) dW(t) \right] \), \hspace{1cm} (1.8)

4. \( \int_a^b (\alpha f(t) + \beta g(t)) dW(t) = \alpha \int_a^b f(t) dW(t) + \beta \int_a^b g(t) dW(t) \). \hspace{1cm} (1.9)

**Proof:** See for e.g. [52].

**Proposition 1.2.3.** The stochastic Ito integral is a martingale with respect to the filtration \( F_t = \sigma \{ X_t; t \geq 0 \} \).

**Proof:** See for e.g. [55]. \( \square \)

1.2.2. **The stochastic differential and Ito formula**

**Definition 1.2.4.** Let \( X(t) \) be a stochastic process. If \( \alpha(t, X_t) \) is a continuous function and \( \sigma(t, X_t) \) is a mean square integrable function then the stochastic process defined by
\[
X(t) = X_0 + \int_0^t \alpha(t, X_t) ds + \int_0^t \sigma(t, X_t) dW_t, \hspace{1cm} (1.10)
\]
is called an Ito process. This is equivalently written by
\[
dX(t) = \alpha(t, X_t) dt + \sigma(t, X_t) dW_t, \hspace{1cm} (1.11)
\]
where \( dX(t) \) is the stochastic differential of \( X(t) \). The functions \( \alpha(t, X_t) \) and \( \sigma(t, X_t) \) are called respectively the drift and volatility term of the process \( X(t) \).
Now we introduce one of the most important topics on Brownian motion, the Ito formula which will be very useful in computation of stochastic integrals and stochastic differential equations.

**Theorem 1.2.5** (Ito formula). Suppose that \( g(t,x) \) is a real valued function with continual partial derivatives \( g_x, g_{xx} \) and \( g_t \) then the stochastic differential of the process \( g(t,X(t)) \) is given by

\[
    dg(t,X) = \left[ g_t(t,X)dt + g_x(t,X)dX(t) + \frac{1}{2} g_{xx}(t,X)(dX)^2 \right],
\]

where \( X = X(t) \) and \((dX)^2\) is computed using

\[
    dt \cdot dt = dt \cdot dW(t) = 0 \text{ and } \left(dW(t)\right)^2 = dt.
\]

**Proof:** see for e.g. [52].

**Proposition 1.2.6.** If the stochastic differential of \( \{X_i(t), \ 0 \leq t \leq T, \ i = 1,2\} \) is given by

\[
    dX_i(t) = \alpha_i(t)dt + \sigma_i(t)dW_i,
\]

then \( X_1(t)X_2(t) \) has the stochastic differential

\[
    d\left( X_1(t)X_2(t) \right) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + dX_1(t)dX_2(t).
\]

**Proof:** The proof is obtained by a direct application of Ito formula stated in Proposition 1.2.5.

This proposition will be very useful in Section 2.2.1 in order to find the dynamics of the portfolio in the bond market. We give an explicit way that makes it easy to compute stochastic integral using Ito formula.

**1.2.3. Evaluation of Ito integral via Ito formulae**

Let us consider a stochastic integral of the form

\[
    I = \int_0^t f(s,W(s))dW(s).
\]

We would like to propose a simple way for evaluation of stochastic integral of the form (1.13) within the following algorithm:
1. Before doing anything, we have to be sure that the integral \( I \) exists which means that \( f(t, x) \) is mean square integrable and where \( x \) is a variable with respect to \( W(t) \).

2. We let

\[
h(t, x) = \int_{x_0}^{x} f(t, y) \, dy,
\]

where \( x_0 \) is any constant taking values in \( \mathbb{R} \). Then we compute \( h(t, x) \) as a Riemann integral by the classic methods in calculus.

3. Apply the Itô’s formulae to \( h(t, x) \).

4. Find the result.

To illustrate what has been elaborated above, the following example is given.

**Example 1.2.7.** Let \( W_t^{} \) be a standard Brownian motion. Then for any \( n \in \mathbb{N} \), we have the following result

\[
\int_0^t W^n(s) dW(s) = \frac{1}{n-1} W^{n+1}(t) - \frac{n}{2} \int_0^t W^{n-1}(s) \, ds. \tag{1.14}
\]

**Proof:** We are following the four steps stated above for the proof.

1. In this example, our function \( f(t, x) \) is \( x^n \). Therefore, it is obvious to say that \( f(t, x) \) is a mean square integrable function.

2. Let us now define our \( h(t, x) \) as

\[
h(t, x) = \int_{x_0}^{x} y^n \, dy,
\]

where \( x_0 \) is any constant taking values in \( \mathbb{R} \). Then from calculus theory, we obtain

\[
h(t, x) = \frac{1}{n-1} (x^{n-1} - x_0^{n-1}),
\]

\( h(t, x) \in C^2([0, \infty]) \) (i.e. the first and second derivatives exist and are continuous in \([0, \infty[ \)). The partial derivatives are \( h_t(t, x) = 0, \ h_x(t, x) = x^n \) and \( h_{xx}(t, x) = nx^{n-1} \). By applying Itô’s formula, we obtain:

\[
h_t(t, W(t)) = W^n(t) dW(t) + \frac{n}{2} W^{n-1}(t) dt,
\]

it follows that

\[
d \left( \frac{1}{n-1} W^{n-1}(t) \right) = W^n(t) dW(t) + \frac{n}{2} W^{n-1}(t) dt,
\]

or removing the differential, we obtain
\[
\frac{1}{n-1}W^{n-1}(t) = \int_0^t W^n(s) dW(s) + \frac{n}{2} \int_0^t W^{n-1}(s) ds,
\]
and finally, we get
\[
\int_0^t W^n(s) dW(s) = \frac{1}{n-1}W^{n-1}(t) - \frac{n}{2} \int_0^t W^{n-1}(s) ds. \tag{1.16}
\]

1.3. **Stochastic differential equations for one-factor short interest rate model**

One of the keywords in our dissertation is the Hull-White model which is a stochastic differential equation. So, this topic is very important in this study. In fact, the stochastic differential equation is an unknown equation which is a stochastic process and taking the form
\[
dX(t) = \mu(t, X(t)) dt + \sigma(t, X(t)) dW(t). \tag{1.15}
\]
Later in third chapter, Section 3.1.1 we will state the financial meaning of the terms for the SDE that describe the Hull-White model. Here, we simply show how to solve it.

By solving a SDE (1.15), we mean determining a process \(X(t)\) such that the integral equation
\[
X(t) = X(0) + \int_0^t \mu(t, X(s)) ds + \int_0^t \sigma(t, X(s)) dW(s) \tag{1.16}
\]
is valid for all \(t\). In other words, the solution of \(X(t)\) starting from initial point \(X(0)\) is determined by the two integrals in the right hand side. That solution (1.16) is called an Ito process which is a diffusion process\(^5\), or the strong solution of the Equation (1.15).

1.3.1. **Existence and uniqueness result**

**Definition 1.3.1** Let \((\Omega, \mathcal{F}, P)\) be a probability space and
\[
\mu(s, x): [0, t] \times \mathbb{R}^n \rightarrow \mathbb{R}^n
\]
\[
\sigma(s, x): [0, t] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}
\]

\(^5\)Generally speaking, a diffusion process is an arbitrary strong Markov process with continuous sample paths. In our framework, a diffusion process is given as a strong solution of a stochastic differential equation driven by the underlying Brownian motion \(W\).
be measurable functions. A solution \( (1.16) \) is an \( \mathcal{F}_t \) - adapted stochastic process \( X(t) \) such that

- For any \( t \geq 0 \), the integrals
  \[
  \int_0^t \mu(t, X(s))ds \quad \text{and} \quad \int_0^t \sigma(t, X(s))dW(s),
  \]
  exist i.e.
  \[
  \int_0^t |\mu(t, X(s))|ds < +\infty \quad \text{and} \quad \int_0^t |\sigma(t, X(s))|^2ds < +\infty \text{ a.s.}
  \]

- \( X(t) \) satisfies the equation \( (1.16) \) i.e.
  \[
  X(t) = X(0) + \int_0^t \mu(t, X(s))ds + \int_0^t \sigma(t, X(s))dW(s).
  \]

**Theorem 1.3.2** (existence and uniqueness for SDE). If \( \mu(t, X(t)) \) and \( \sigma(t, X(t)) \) are measurable functions and if \( C \) and \( K \) are finite constants such that

1. \( |\mu(t, x)| + |\sigma(t, x)| \leq C(1 + |x|) \), \hspace{1cm} (1.17)
2. \( |\sigma(t, x) - \sigma(t, y)| + |\alpha(t, x) + \alpha(t, y)| \leq K|x - y| \), \hspace{1cm} (1.18)
3. \( E[X(0)^2] < +\infty \) \hspace{1cm} (1.19)

then for any \( T \geq 0 \), the equation \( (1.15) \) admits a unique solution in the interval \( [0, T] \). Moreover, this solution \( \{X(s), 0 \leq s \leq t\} \) satisfies

\[
E\left[\sup_{0 \leq s \leq t} |X(s)|^2\right] < +\infty.
\]

The uniqueness of the solution means that if \( X(s) \) and \( Y(s), 0 \leq s \leq t \) are two solutions of \( (1.15) \), then for all \( 0 \leq s \leq t, X(s) = Y(s) \) a.s.

**Proof**: See for e.g. [52].

### 1.3.2. Two important SDEs for our work

As introduced earlier, this study needs two important SDEs; the Geometric Brownian Motion also known as exponential Brownian motion which models the bond price process discussed later in the following chapter and the mean-reverting Ornstein Uhlenbeck process that is a variant of the Hull-White extended Vasicek model. Let us discuss them:
**Definition 1.3.3.** The stochastic process \( \{X(t), t \geq 0\} \) is a Geometric Wiener process if its differential has the form

\[
\begin{align*}
\{dX(t) &= \mu X(t)dt + \sigma X(t)dW(t) \\
X(0) &= X_0
\end{align*}
\]

(1.20)

where parameters \( \mu \) and \( \sigma \) are constant.

**Proposition 1.3.4** The solution to the equation (1.20) exists and is given by:

\[
X(t) = X_0 \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t - \sigma W(t) \right\}.
\]

(1.21)

Moreover, the expected value is given by

\[
E[X(t)] = X_0 e^{\mu t},
\]

(1.22)

and the variance by

\[
\text{Var}[X(t)] = \frac{\sigma^2}{2\alpha} (e^{2\mu t} - 1).
\]

(1.23)

**Proof:** first, we rewrite (1.20) as

\[
\frac{dX(t)}{X(t)} = \mu dt + \sigma dW(t).
\]

(1.24)

By integrating both sides of (1.24), the following is obtained:

\[
\int_0^t \frac{dX(s)}{X(s)} = \mu t + \sigma W(t)
\]

(1.25)

Let us now compute the stochastic integral which appears to the right side of equation (1.25) following the four steps stated in Section 1.2.3. By choosing the function \( g(t, x) \in C^2([0, t]) \) as

\[
h(t, x) = \int_{x_0}^x dy \text{ or } h(t, x) = \ln \left| \frac{x}{x_0} \right|
\]

with their partial differentials

\[
h_t(t, x) = 0, \quad h_x(t, x) = \frac{1}{x} \text{ and } h_{xx}(t, x) = -\frac{1}{x^2}.
\]

It follows by Ito formula that

\[
d \left( \ln \frac{X(t)}{X_0} \right) = -\frac{1}{2} \left( \frac{dX(t)}{X(t)} \right)^2 + \frac{dX(t)}{X(t)}.
\]

This implies that

\[
\ln \frac{X(t)}{X_0} = -\frac{1}{2} \int_0^t \left( \frac{dX(s)}{X(s)} \right)^2 + \int_0^t \frac{dX(s)}{X(s)}
\]
\[ \int_{0}^{t} (\mu ds + \sigma dW(s))^2 + \int_{0}^{t} \mu ds + \int_{0}^{t} \sigma dW(s) \]
\[ = -\frac{1}{2} \int_{0}^{t} \mu^2 (ds)^2 + 2\mu \sigma ds dW(s) + (\sigma dW(s))^2 + \int_{0}^{t} \mu ds + \int_{0}^{t} \sigma dW(s). \]

As given in Theorem 1.2.7, \( ds \cdot ds \), \( ds \cdot dW(s) \) are vanish and \( (dW(s))^2 = ds \), it follows that
\[ \ln \frac{X(t)}{X_0} = \left( \mu - \frac{1}{2} r^2 \right) t - \sigma W(t). \]

The final result is given by
\[ X(t) = X_0 \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t - \sigma W(t) \right). \]

To calculate the expectation, we write (1.21) under the form (1.17) as
\[ X(t) = X_0 + \int_{0}^{t} \mu X(s) ds + \int_{0}^{t} \sigma X(s) dW(s). \]

Taking expected values and defining \( E[X(t)] = m(t) \), we obtain
\[ m(t) = X_0 + \mu \int_{0}^{t} m(s) ds. \]

Since
\[ E \left[ \int_{0}^{t} \sigma X(s) dW(s) \right] = 0, \]

differentiating both sides with respect to \( t \), we obtain the following ordinary differential equation
\[ \begin{cases} \frac{dm(t)}{dt} = \mu m(t) \\ m(0) = X_0 \end{cases} \tag{1.26} \]
and solving the ordinary differential equation (1.26), we get the result
\[ m(t) = E[X(t)] = X_0 e^{\mu t}. \]

In order to compute the variance, we recall first from probability theory that
\[ \text{Var}[X(t)] = E[X^2(t)] - (E[X(t)])^2, \]
then we require \( E[X^2(t)] \). For this, we proceed in deriving the stochastic differential of \( X^2(t) \), differentiating twice with respect to \( t \), integrating and taking expectation, we obtain
\[ E[X^2(t)] = X_0^2 e^{\mu t} + \frac{\sigma^2}{2\alpha} (e^{\mu t} - 1). \]

Therefore, the variance is
**Definition 1.3.5.** The mean-reverting Ornstein-Uhlenbeck process is the solution $X(t)$ of the SDE

$$dX(t) = (a - bX(t))dt + \sigma dW(t); X(0) = X_0$$

where $a$, $b$ and $\sigma$ are real constant.

We note that the solution to the equation (1.27) is a Markov process with continuous sample paths and Gaussian increments. In fact, the equation (1.27) can be solved explicitly, as the following proposition shows.

**Proposition 1.3.6.** The solution to the equation (1.27) is given by

$$X(t) = \frac{a}{b} + X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dW(s)$$

Moreover, the expectation value is given by

$$E[X(t)] = \frac{a}{b} + X_0 e^{-bt}$$

and the variance by

$$Var[X(t)] = \frac{\sigma^2}{2b} [1 - e^{-2bt}]$$

**Proof:** To find the solution of the mean-reverting Ornstein-Uhlenbeck process we first define the process $Y_t$ as follows

$$Y_t = X_te^{bt}. \tag{1.30}$$

By choosing the function $g(t,x) \in C^2([0,t])$ as $g(t,x) = xe^{bt}$ with their partial differentials

$$g_t(t,x) = bx e^{bt}, \quad g_x(t,x) = e^{bt} \quad \text{and} \quad g_{xx}(t,x) = 0,$$

it follows by Ito formula that

$$dY(t) = bX(t)e^{bt} \ dt + e^{bt}dX(t)$$

$$= bX(t)e^{bt} \ dt + e^{bt}[a - bX(t)]dW(t)$$

$$= ae^{bt} dt + \sigma e^{bt} dW(t),$$

consequently

$$Y(t) = Y_0 + \frac{a}{b}e^{bt} + \sigma \int_0^t e^{bs} dW(s).$$

Using the expression of $Y(t)$ in (1.30), we have

$$Var[X(t)] = E[X^2(t)] - (E[X(t)])^2 = \frac{\sigma^2}{2\mu} (e^{2\mu t} - 1) \quad \blacksquare$$
\[ X(t)e^{bt} = X_0 + \frac{a}{b}e^{bt} + \sigma \int_0^t e^{bs} \, dW(s) \]

and finally, we obtain

\[ X(t) = \frac{a}{b} + X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)} \, dW(s) \] (1.31)

From (1.31) we can now find the expectation of \( X(t) \) as

\[ E[X(t)] = \frac{a}{b} + X_0 e^{-bt} \]

As in Proposition 1.3.4, the variance is given by

\[ Var[X(t)] = E[X^2(t)] - (E[X(t)])^2. \]

so that we need to find \( E[X^2(t)] \). Then squaring (1.31) and taking expectations yields

\[ E[X^2(t)] = \left( \frac{a}{b} + X_0 e^{-bt} \right)^2 + \left( \sigma \int_0^t e^{-b(t-s)} \, dW(s) \right)^2 \]

\[ = (E[X(t)])^2 + \sigma^2 \int_0^t e^{-2b(t-s)} \, ds. \]

The last equation is due to the Ito isometrics stated in the Proposition 1.2.2. Hence, the variance of the process \( X(t) \) given in the equation (1.35) is given by

\[ Var[X(t)] = (E[X(t)])^2 + \sigma^2 \int_0^t e^{-2b(t-s)} \, ds - (E[X(t)])^2, \]

or finally

\[ Var[X(t)] = \frac{\sigma^2}{2b} [1 - e^{-bt}]. \]

Let us now show how these two stochastic processes discussed above look like by the following figures:

**Figure 1.1.** Two sample paths for the GBM (Blue \( \mu = 1, \sigma = 0.2 \) and Green : \( \mu = 1, \sigma = 0.2 \))
1.4. Link between SDEs and parabolic PDEs

Let us create a link between SDEs and parabolic partial differential equations (PDEs) through some theorems and find the closed-form expression of the obtained PDE through Kolmogorov-Backward and Feynman-Kac theorems.

1.4.1. Formula for the generator of diffusion process

**Definition 1.4.1.** Let $X = \{X(t), t \geq 0\}$ be an Ito process in $\mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{F}, P)$. For a point $x \in \mathbb{R}^n$, let denote $P^x$ the probability law of...
Given the initial point \(X_0 = x\) and let \(E^x\) be the expectation with respect to \(P^x\). For a function \(f\) defined from \(\mathbb{R}^n\) to \(\mathbb{R}\), the generator \(A\) of \(X(t)\) is defined by

\[
Af(x) = \lim_{t \to 0} \frac{E^x[f(X(t))] - f(x)}{t}.
\]

The set of functions \(f : \mathbb{R}^n \to \mathbb{R}\) such that the limit exists at \(x\) is denoted by \(\mathcal{D}_A(x)\). \(A\) is also called Itô operator or Dynkin operator.

We recall that the probability law is the mapping \(P^X : \mathcal{B} \to \mathbb{R}\) where

\[
P^X(B) = P(X^{-1}(B)) = P([X \in B]) \quad \forall B \in \mathcal{B}
\]

where \(\mathcal{B}\) is the Borel sigma-algebra.

Let us give an explicit formula for the Ito operator

**Theorem 1.4.2.** Let consider the SDE given by

\[
dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t),
\]

then the (infinitesimal) generator \(A\) of \(X(t)\) is given by

\[
Af(t,x) = \sum_{i=1}^{n} \mu_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}
\]

for any function \(f(t,x) \in C^2(\mathbb{R}^n)\) and where \(\sigma^t\) is the transpose of \(\sigma\).

**Proof:** See for e.g. [52].

In terms of the infinitesimal generator, the Ito formula takes the form

\[
df(t,x) = \left(\frac{\partial f(t,x)}{\partial t} + Af(t,x)dt + [\mathcal{V}_x f(t,x)]\right) \sigma dW(t)
\]

where the gradient \(\mathcal{V}_x\) is defined for \(f(t, x) \in C^1(\mathbb{R}^n)\) as

\[
\mathcal{V}_x f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right].
\]

### 1.4.2. Kolmogorov’s backward equation

We start by giving the so-called Dynkin formula which follows from Definition 1.3.12 and Theorem 1.3.13. In addition, we start by an important definition about stopping times.

**Definition 1.4.3.** A random variable \(\tau\) with values in the set \(\mathbb{R}_+ \cup \{+\infty\}\) is called a stopping time with respect to the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) if for any \(t \geq 0\)
The σ-algebra associated with \( \tau \) is defined as follows
\[
\mathcal{F}_\tau = \{ A \in \mathcal{F}_t, \text{ for any } t \geq 0, \ A \cap \{ \tau \leq t \} \in \mathcal{F}_t \}.
\]
The all information available before the random time \( \tau \) is represented by \( \mathcal{F}_\tau \).

**Theorem 1.4.4** (Dynkin’s formula). Let \( f \in C^2(\mathbb{R}^n) \) and suppose \( \tau \) is a stopping time. Then
\[
E[f(X_\tau)] = f(x) + E^x \left[ \int_0^\tau A_f(x_s) ds \right] \tag{1.37}
\]
where \( E^x[\cdot] \) is the expectation with respect to the probability law \( P^x \) at the initial point \( X_0 = x \).

**Proof:** See for e.g. [50]. \( \square \)

Now, from (1.37) we let \( \tau = t \) we get
\[
u(t, x) = E^x[f(X_t)] = f(x) + E^x \left[ \int_0^t A_f(x_s) ds \right].
\]
In differential terms, we have
\[
\frac{\partial u(t, x)}{\partial t} = E^x[A_f(X_t)]. \tag{1.38}
\]
If we wish to express in terms of \( u \) the right hand side of (1.38), we obtain the following result:

**Theorem 1.4.5** (Kolmogorov’s Backward Equation). Let \( X_t \) be a Ito process as defined in equation (1.29) satisfying the conditions of Theorem 1.3.2 and let \( f \in C^2(\mathbb{R}^n) \). Then the function
\[u(t, x) = E^x[f(X_t)]\]
satisfies the equation
\[
\begin{cases}
\frac{\partial u(t, x)}{\partial t} + Au = 0, & t > 0, \ x \in \mathbb{R}^n \\
u(0, x) = f(x)
\end{cases} \tag{1.39}
\]
or more explicitly
\[
\frac{\partial u(t, x)}{\partial t} + \sum_{i=1}^n b_i(x) \frac{\partial u(t, x)}{\partial x_i} + \frac{1}{2} \sum_{i,j}^n (\sigma \sigma^T)_{ij}(x) \frac{\partial^2 u(t, x)}{\partial x_i \partial x_j} = 0 \tag{1.40}
\]
\[u(0, x) = f(x). \tag{1.41}\]
Note that in one dimension, the equation (1.40) can be written as
The equation (1.40) as well as equation (1.42) is called *Kolmogorov's backward differential equation.*

**Proof:** see for e.g. [17].

A useful generalization of Kolmogorov's backward equation differential equation is given by

**Theorem 1.4.6** (Feynman-Kac formula). *Under the assumptions of Theorem 1.4.3, let* \( f \in C^2(\mathbb{R}^n) \) *and* \( r \in C(\mathbb{R}^n) \). *Then the function*

\[
\nu(t, x) = E^x \left[ \exp \left\{ - \int_0^t r(X_s) \, ds \right\} f(X_t) \right]
\]

*satisfies the equation*

\[
\begin{align*}
\frac{\partial \nu(t, x)}{\partial t} &= A\nu - rv, \quad t > 0, x \in \mathbb{R}^n \\
\nu(0, x) &= f(x).
\end{align*}
\]

**Proof:** The proof can be obtained by combining Ito’s formula and Kolmogorov’s backward equation differential equation. (For more detail see [17]).

**Example 1.4.7.** Let us now compute the stochastic PDE given by

\[
\frac{\partial u(t, x)}{\partial t} + b(x) \frac{\partial u(t, x)}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2 u(t, x)}{\partial x^2} - r u(t, x) = 0
\]

\[
u(0, x) = (x - K)^+ = \max(x - K, 0); \quad x \in \mathbb{R}
\]

where \( r > 0, \ a, \ \sigma \neq 0 \) and \( K > 0 \) are constant.

Referring to Feynman – Kac Theorem, the solution of (1.47) has the form

\[
u(t, x) = \exp\{ -rt \} E^x [ (x - K)^+]\]

where \( X_t \) satisfies the stochastic differential equation

\[dX_t = \alpha X_t \, dt + \sigma X_t \, dW_t\]

which is exactly the geometric Brownian motion and its solution is already found in Proposition 1.3.4. It is
$X_t = X_0 \exp \left\{ \left( \alpha - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}.$ \hfill (1.47)

Let us now develop (1.47):

$$u(t, x) = \exp(-rt) E^x [ (X_t - K) 1_{\{X_t \geq K\}}].$$

This last equation can be written as

$$u(t, X_t) = \exp(-rt) \int_{-\infty}^{+\infty} \max (X_t - K, 0) f(t, X_t) dX_t.$$

From (1.47) we have,

$$\ln \frac{X_t}{X_0} = \left( \alpha - \frac{1}{2} \sigma^2 \right) t + \sigma W_t.$$  

So, $\ln \frac{X_t}{X_0}$ is normally distributed with mean $\left( \alpha - \frac{1}{2} \sigma^2 \right) t$ and variance $\sigma^2 t$. Then referring to Section 1.1.2, the probability density function is given by

$$f(t, X_t, X_0) = \frac{1}{X_t \sigma \sqrt{2\pi t}} \exp \left\{ -\frac{\ln \frac{X_t}{X_0} - \left( \alpha - \frac{1}{2} \sigma^2 \right) t}{2\sigma^2 t} \right\}.$$  

Let $\rho = \ln X_t$ and $\xi = \ln X_0$. Then $\ln \frac{X_t}{X_0} = \ln X_t - \ln X_0 = \rho - \xi$ and $d\rho = \frac{dX_t}{X_t}$.

Substituting the above transition density function into (1.48), we get

$$u(t, X_t) = e^{-rt} \int_{-\infty}^{+\infty} \max (X_t - K, 0) \frac{\exp \left\{ -\frac{\ln \frac{X_t}{X_0} - \left( \alpha - \frac{1}{2} \sigma^2 \right) t}{2\sigma^2 t} \right\}}{X_t \sigma \sqrt{2\pi t}} dX_t,$$

it follows that

$$u(t, \rho, \xi) = e^{-rt} \int_{-\infty}^{+\infty} \max (e^\rho - K, 0) \frac{\exp \left\{ -\frac{\xi + \left( \alpha - \frac{1}{2} \sigma^2 \right) t - \rho}{2\sigma^2 t} \right\}}{\sigma \sqrt{2\pi t}} d\rho.$$  

We know that our boundary condition can be written as

$$u(0, X_t) = \begin{cases} 0, & X_t \leq K \\ (X_t - K), & X_t > K \end{cases}$$

or in terms of parameter $\rho$ we have

$$u(0, \rho) = \begin{cases} 0, & e^\rho \leq K \\ (e^\rho - K), & e^\rho > K \text{ or } \rho > \ln K. \end{cases}$$

So, (1.62) becomes
The integral defined in $u(t, \rho, \xi)$ above can be set into two integrals that we name $u_1(t, \rho, \xi)$ and $u_2(t, \rho, \xi)$:

$$u_1(t, \rho, \xi) = \int_{\ln K}^{+\infty} \frac{e^\rho}{\sigma \sqrt{2\pi t}} \exp \left\{ -\frac{\left[ \xi + \left( \alpha - \frac{\sigma^2}{2} \right) t - \rho \right]^2}{2\sigma^2 t} \right\} d\rho$$

$$= e^{at} e^t \int_{\ln K}^{+\infty} \frac{1}{\sigma \sqrt{2\pi t}} \exp \left\{ -\frac{\left[ \xi + \left( \alpha + \frac{\sigma^2}{2} \right) t - \rho \right]^2}{2\sigma^2 t} \right\} d\rho$$

$$= e^{at} e^t N \left( \frac{\xi + \left( \alpha + \frac{\sigma^2}{2} \right) t - \ln K}{\sigma \sqrt{t}} \right) = e^{at} X_0 N \left( \frac{\ln \frac{X_0}{K} + \left( \alpha + \frac{\sigma^2}{2} \right) t - \ln K}{\sigma \sqrt{t}} \right).$$

It follows that

$$u_1(t) = e^{at} X_0 N \left( \frac{\ln \frac{X_0}{K} + \left( \alpha + \frac{\sigma^2}{2} \right) t}{\sigma \sqrt{t}} \right).$$

The other integral $u_2(t, \rho, \xi)$ can be expressed as

$$u_2(t, \rho, \xi) = \int_{\ln K}^{+\infty} \frac{K}{\sigma \sqrt{2\pi t}} \exp \left\{ -\frac{\left[ \xi + \left( \alpha - \frac{\sigma^2}{2} \right) t - \rho \right]^2}{2\sigma^2 t} \right\} d\rho$$

$$= KN \left( \frac{\xi + \left( \alpha - \frac{\sigma^2}{2} \right) t - \ln K}{\sigma \sqrt{t}} \right) = KN \left( \frac{\ln \frac{X_0}{K} + \left( \alpha - \frac{\sigma^2}{2} \right) t}{\sigma \sqrt{t}} \right).$$

Hence the solution is given by

$$u(t) = e^{(\alpha-r)t} X_0 N(d_1) - e^{-rt} KN(d_2),$$

where

$$d_1 = \frac{\ln \frac{X_0}{K} + \left( \alpha + \frac{\sigma^2}{2} \right) t}{\sigma \sqrt{t}}, \quad d_2 = d_1 - \sigma \sqrt{t},$$

and where $N(\cdot)$ is the standard normal distribution.
Note that in financial language if we set $\alpha = r$ (in order to avoid arbitrage opportunity [7]) in the equation (1.49), we get the well-known Black–Scholes equation for European call option with exercise price $K$ and the result is known as Risk Neutral Valuation formula associated to the Black–Scholes equation or shortly the Black–Scholes formula.

### 1.4.3. Change of measure: The Girsanov Theorem

The Girsanov Theorem states that changing to an equivalent measure changes the drift of a stochastic process but nothing else. Let us make some mathematical comments.

**Proposition 1.4.8.** Let $(\mathcal{F}_t)_{t \in [0,T]}, T > 0$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$ and let $(L_t)_{t \in [0,T]}$ be a strict positive $\mathcal{F}_t$ martingale with respect to the probability measure $P$ with $L_0 = 1$. A sufficient condition for an adapted stochastic process $W^Q_t$ to be an $\mathcal{F}_t$–martingale with respect to the measure $dQ = L_T dP$ is that the process $(L_t W^Q_t)_{t \in [0,T]}$ is an $\mathcal{F}_t$–martingale with respect to the probability measure $P$.

**Proof:** Since $L_t W^Q_t$ is an $\mathcal{F}_t$–martingale with respect to $P$, thus for $s \leq t \leq T$ we have

$$ E\left[L_T W^Q_t | \mathcal{F}_s\right] = L_s W^Q_s, \quad (1.50) $$

as a consequence we have that

$$ E^Q\left[W^Q_t | \mathcal{F}_s\right] = \frac{E\left[L_T W^Q_t | \mathcal{F}_s\right]}{E\left[L_T | \mathcal{F}_s\right]} = \frac{L_s W^Q_s}{L_s} = W^Q_s. $$

We note that the process $(L_t)_{t \in [0,T]}$ is called Likelihood process and is defined by

$$ L_t = \frac{dQ}{dP}. \quad (1.51) $$

Since $L_t$ is a strictly positive $\mathcal{F}_t$–martingale with respect to the probability measure $P$, it is natural to define $L$ as the solution of stochastic differential equation

$$ dL_t = \varphi_t L_t dW^P_t, \quad L_0 = 1 \quad (1.52) $$

for some choice of the process $\varphi_t$ that we call Girsanov Kernel of the measure transformation. Let us choose a certain adapted process $h_t$ such that $W^Q_t$ be a martingale. We do it by giving the following proposition:
Proposition 1.4.9.

1) Let \( h \in L^2(\Omega,\mathcal{F},Q) \) be a \( Q \)-deterministic function, \( W_t \) a Brownian motion with respect to \( P \), and define
\[
W_t^Q = \exp \left\{ \int_0^t h_s \, dW_s^p - \frac{1}{2} \int_0^t h_s^2 \, ds \right\}; \quad t \in [0,T],
\]
then by Ito formula
\[
dW_t^Q = W_t^Q h_t \, dW_t^p.
\]

2) Let \( \varphi \in \mathcal{C}([0,T]) \) with \( T \leq +\infty \) and define
\[
L_t = \exp \left\{ \int_0^t \varphi_s \, dW_s^p - \frac{1}{2} \int_0^t \varphi_s^2 \, ds \right\}; \quad t \in [0,T],
\]
then by Ito formula
\[
dL_t = \varphi_t L_t \, dW_t^p.
\]

**Proof:** We give the proof only for point 1 and for the point 2 we have to follow the same way. By Ito formula, we have
\[
dW_t^Q = W_t^Q \left( h(t) \, dW_t^p - \frac{1}{2} h(t)^2 \, dt \right) + \frac{1}{2} W_t^Q h(t)^2 \, dt = W_t^Q h_t \, dW_t^p.
\]

Lemma 1.4.10. (Novikov condition). Under the assumptions of point 2 of Proposition 1.2.21, if
\[
E \left[ \exp \left\{ \frac{1}{2} \int_0^t |\varphi_s|^2 \, ds \right\} \right] < +\infty
\]
then \((L_t)_{t \in [0,T]}\) is a martingale and \( E^P[L_T] = E^P[L_0] = 1 \).

**Proof:** See for e.g. [46].

Thus, the following theorem is achieved:

**Theorem 1.4.11** (Girsanov Theorem). Let \((L_t)_{t \in [0,T]}\) be a \( P \)-martingale Likelihood process and \( \varphi_t \) the Girsanov Kernel process satisfying the novikov condition mentioned above. Then the process
\[
W_t^Q = W_t^p - \int_0^t \varphi_s \, ds,
\]
is a Wiener process with respect to the measure \( dQ = L_t \, dP \).

**Proof:** To prove this theorem, we will use Levy’s martingale characterization (Theorem 1.1.13) by verifying the three conditions:
a. It is obvious that $W^Q(0) = 0$ and the sample paths $t \rightarrow W^Q(t)$ are continuous a.s.

b. To verify the Condition 2 of Theorem 1.1.13 (Which stipulates that $W^Q(t)$ must be a martingale), let $M(t) = L(t)W^Q(t)$ as defined in Proposition 1.4.8, to prove that $W^Q(t)$ is a Q-martingale and it is sufficient to show that $M(t), t \in [0,T]$ is a P-martingale. So, it is assumed that $L(t), t \in [0,T]$ satisfies the Novikov condition and that $L(t), t \in [0,T]$ is a martingale with $E[L(t)] = 1$.

By the Ito's formula, the following result is obtained:

$$
dM(t) = L(t)dW^Q(t) + W^Q(t)dL(t) + L(t)\varphi(t)dt
$$

$$
= L(t)(dW^P(t) - \varphi(t)dt) + W^Q(t)L(t)\varphi(t)dW^P(t) + L(t)\varphi(t)dt
$$

$$
= L(t)(1 + \varphi(t)W^Q(t))dW^P(t).
$$

Hence $M(t), t \in [0,T]$ is a martingale, it follows that $W^Q(t)$ is a Q-martingale.

c. Now let us prove that $|W^Q(t)|^2 - t$ is a martingale. So given

$$
W_t^Q = W_t^P - \int_0^t \varphi_s ds,
$$

then it follows that

$$
(W_t^Q)^2 = (W_t^P)^2 - 2W_t^P \int_0^t \varphi_s ds + \left( \int_0^t \varphi_s ds \right)^2.
$$

For any $0 \leq s \leq t$, we have

$$
E \left[ (W_t^Q)^2 | \mathcal{F}_s \right] = E[(W_t^P)^2 | \mathcal{F}_s] - 2E \left[ W_t^P \int_0^t \varphi_s ds | \mathcal{F}_s \right] + E \left[ \left( \int_0^t \varphi_s ds \right)^2 | \mathcal{F}_s \right]
$$

$$
= (W_s^P)^2 + t - s - 2W_s^P \int_0^t \varphi_s ds + \left( \int_0^t \varphi_s ds \right)^2
$$

$$
= \left( W_s^P - \int_0^t \varphi_s ds \right)^2 - s + t = (W_s^Q)^2 - s + t.
$$

Thus,

$$
E \left[ (W_t^Q)^2 - t | \mathcal{F}_s \right] = (W_s^Q)^2 - s,
$$

and then it follows that $(W_t^Q)^2 - t$ is a Q-martingale.

We conclude by Levy's martingale characterization that $W_t^Q$ is a Wiener process with respect to the probability measure $Q$. □
The equation (1.53) can be written in differential form as

$$dW^Q(t) = dW^P(t) - \varphi(t)dt.$$ 

1.5. Conclusion

The dynamics of one-factor short interest rate are modelled by mean of stochastic differential equations (SDEs) where their solutions are stochastic processes. Therefore, important ways for solving SDEs, in particular, the Geometric Brownian Motion and the mean-reverting Ornstein-Uhlenbeck process have been stated. As there is a link between SDEs and parabolic PDEs, different tools have been introduced, namely, Ito operator, Kolmogorov's backward equation and Feynman-Kac theorem that can allow us to make that link effective. The Girsanov theorem and Radom-Nikodym derivative are introduced when probability measures need to be changed. These theorems will be very useful in the construction of the term structure equation in arbitrage free framework which is a parabolic partial differential equation and in the building of analytic formula for pricing European bond option discussed in the two next chapters.
Chapter 2:

**Derivation of the term structure equation and the bond price process under the Hull-White extended Vasicek model**

Loosely speaking, a bond option is a financial contract which gives the right to buy or to sell a bond at a certain price called strike price on or before the maturity date. In this chapter, the term structure equation, which is a parabolic PDE in the arbitrage free framework, is derived and also deduce the affine term structure. We then consider the bond price as a function of interest rate getting the exponential form and we look for the dynamics of bond prices under both Vasicek and Hull-White models. This pricing has an important impact for pricing European bond option as we can see it in our third chapter.

### 2.1. Bond prices and interest rate models

The zero coupon bond of maturity date \( T \) is a financial derivative paying to its holder one unit of currency at the date \( T \) in the future. This means that the principal bond (known also as face or nominal value) is one unit of currency. The price at time \( t \) of a bond of maturity \( T \) is denoted by \( p(t, T), t \leq T \) and it is thus obvious that \( p(T, T) = 1 \). The study will assume that the bond price \( p(t, T), t \in \)

---

\(^6\)more details in [7, pages 302 and 303].
$[0,T]$ is a strictly positive and adapted process on a filtered probability space $(\Omega, \mathcal{F}, P)$ defined in our first chapter.

Thus, in a zero-coupon bond, there is only one bond. If we have more than one bond, we are talking about coupon-bearing bond which is a financial derivative that pays to its holder the amounts $c_1, c_2, \ldots, c_d$ at the dates $T_1, T_2, \ldots, T_d$. The price of coupon bonds is given by

$$p_c(t, T_i) = \sum_{i=1}^{d} c_i p(t, T_i).$$  \hfill (2.1)

The relationship between discount and coupon bond prices is rather crucial because in real world, zero-coupon bond markets are rarely available. Instead, we are unfortunately stuck with extracting our necessary information from market prices of coupon-bearing bonds, which might be additionally equipped with special features that make the analysis even worse.

Investors can borrow or lend over different periods at different interest rates. If we plot out these interest rates, they form the term structure of interest rates or yield curve. So, let us consider the price $p(t, T)$ of a zero-coupon bond with a fixed maturity date $T$ (or simply $T$-bond price). The term structure $R$ is the solution of

$$p(t, T) = e^{-(T-t)R(t,T)}. \mathbf{1}$$

where the factor $\mathbf{1}$ indicates the face value of the bond. We are thus led to the following concepts:

2.1.1. Bond market model

Let us define the basic building blocks of the bond market, consisting of the prices of a zero-coupon bonds, coupon bearing bonds, money account and the interest rates, as follows:

**Definition 2.1.1.** Given the price $p(t, T)$ of a zero-coupon bond, an adapted process $R(t, T)$ defined by the formula

$$R(t, T) = -\frac{\ln p(t, T)}{T - t},$$  \hfill (2.2)

is called the yield to maturity on a zero-coupon bond maturing at time $T$.

The function $t \mapsto R(t, T)$ for a fixed $T$ is called the yield curve.
Throughout this study, we will consider four yield curves, namely, the flat yield curve $R_f(t, T)$, upward yield curve $R_u(t, T)$, downward yield curve $R_d(t, T)$ and the humped yield curve $R_h(t, T)$ defined by:

$$\begin{align*}
R_f(t, T) &= 0.03, \\
R_u(t, T) &= 0.03 + 0.003(T - t)^{0.5}, \\
R_d(t, T) &= 0.03 - 0.003(T - t)^{0.5}, \\
R_h(t, T) &= 0.06e^{-0.01(T-t)} - 0.03e^{-0.3(T-t)}
\end{align*}$$

(2.2')

**Figure 2.1:** Four different yield curves

**Definition 2.1.2.** If the bond pays the positive cash flows $c_1, c_2, ..., c_m$ at the dates $T_1, T_2, ..., T_m$, then its continuously compounded yield – to – maturity

$$R(t, c, T) = R(t; c_1, c_2, ..., c_m; T_1, T_2, ..., T_m)$$

is uniquely determined by the following relationship

$$p_c(t, T_i) = \sum_{T_j \geq t} c_i e^{-R(t,c,T)(T_j-t)}$$

(2.3)

Here, $p_c(t, T_i)$ denotes the coupon – bearing bond price at time $t < T_m$ and $c = (c_1, c_2, ..., c_d)$ and $T = (T_1, T_2, ..., T_d)$

Let us now introduce a helpful topic for the derivation of the European bond option "the forward price". In fact, the forward price is the price of a forward contract between two parties to buy or to sell a bond at a specified future date at a price agreed upon today. In what follow, we give a formal definition.
**Definition 2.1.3.** Let $p(t, S)$ and $p(t, T)$ be the bond prices maturing respectively at the dates $S$ and $T$. The forward rate for $[S, T]$ contracted at date $t$ is defined as

$$f(t, S, T) = \frac{\ln p(t, S) - \ln p(t, T)}{T - S}$$

or equivalently

$$\frac{p(t, T)}{p(t, S)} = e^{-f(t, S, T)(T - S)}.$$  

By taking the limit of $T$ converges to $S$, we get

$$f(t, T) = -\frac{\partial \ln p(t, T)}{\partial T},$$  

which constitutes the rates for instantaneous for borrowing and lending in $T$. Hence, we remark that

$$f(t, T) = \lim_{S \to t} f(t, S, T),$$

From equation (2.4), we can solve the price $p(t, T)$ as

$$p(t, T) = \exp \left( - \int_t^T f(t, s) ds \right).$$

A special case of instantaneous forward rate is the instantaneous short rate defined by

$$r(t) = f(t, t) = \lim_{T \to t} \left( -\frac{\ln p(t, T)}{T - t} \right).$$

In a stochastic set-up, the short-term interest rate is modelled by means of an Ito process, or more specifically, as a one dimensional diffusion process.

**Assumption 2.1.4.** We assume that the dynamics of the bond price and the short – interest rate have respectively the form

$$dp(t, T) = p(t, T)[a(t, T)dt + b(t, T)dW_t],$$

and

$$dr(t) = \mu(r, t)dt + \sigma(r, t)dW_t,$$

where $W_t$ is the standard Brownian motion with respect to a certain probability measure $P$. Here $a(t, T)$ and $b(t, T)$ are adapted process parametrized by $T$ which are respectively the drift and volatility term of the bond price process $p(t, T)$, and $\mu(r, t)$ and $\sigma(r, t)$ are smooth functions representing the drift and volatility terms of the stochastic process $r(t)$. 
2.1.2. The money account process

We may then introduce an adapted process $B(t)$ of finite variation and with continuous sample paths, given by the formula

$$B(t) = \exp\left(\int_0^t r(s)ds\right)$$

or equivalently

$$\begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1 \end{cases}$$

In financial interpretation, $B(t)$ represents the price process of a risk-free security. The process $B(t)$ is referred to as a money market.\(^7\)

2.2. Derivation of the term structure equation

Assume that the short interest rate $r(t)$ follows the diffusion process as described by the following stochastic differential equation

$$dr(t) = \mu(r,t)dt + \sigma(r,t)d\tilde{W}_t,$$

where $\tilde{W}_t$ is the standard Brownian motion with respect to an objective probability measure $P$, $\mu(r,t)$ and $\sigma(r,t)$ are respectively the drift and volatility term of the process $r(t)$. Let us derive the governing partial differential equation for the bond price under the no-arbitrage conditions.

2.2.1. Term Structure Equation

Let us assume that the bond price $p(t,T)$ is a strictly positive and adapted process on a probability space $(\Omega, \mathcal{F}, P)$ and can be modelled by a function of the short rate $r = r(t)$, this means that

$$p(t,T) = p(t,r;T),$$

with the boundary condition

$$p(T, T) = 1 = p(T, r; T).$$

\(^7\) A money market is also known as accumulator or saving account. For more details about price process of a risk-free security see, for example, [50].
In order to find the arbitrage free price of zero-coupon bonds, we need to use the idea of an investor seeking for arbitrage opportunities. He has the possibility to hold three different assets: the saving account and two zero-coupon bond prices.

We can develop the stochastic differential equation for \( p(t, r; T_i) \). Here \( T_i, i = 1, 2 \) are two fixed maturities. For simplicity, we will write \( p_i \) instead of \( p(t, r; T_i) \). So by Ito formula we get as follows:

\[
dp_i = p_i \alpha_i dt + p_i \sigma_i d\tilde{W} \quad (2.7)
\]

with

\[
\alpha_i = \frac{1}{p_i} \left( \frac{\partial p_i}{\partial t} + \mu \frac{\partial p_i}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 p_i}{\partial r^2} \right), \quad (2.8)
\]

and

\[
\sigma_i = \frac{\sigma}{p_i} \frac{\partial p_i}{\partial r} \quad (2.9)
\]

Denoting \( n_j = n_j(t), j = 0, 1, 2 \) the weight of the saving account and two zero-coupon bonds \( p_1 \) and \( p_2 \) respectively then our portfolio is built as

\[
V(t) = n_0 B(t) + n_1 p_1 + n_2 p_2,
\]

where \( B(t) \) is the money account process. Then referring to Proposition 1.2.9, the dynamics of the portfolio is given by the following differential equation

\[
dV(t) = n_0 dB(t) + n_1 dp_1 + n_2 dp_2 + B(t) dn_0 + p_1 dn_1 + p_2 dn_2 \\
+ dn_0 dB(t) + dn_1 dp_1 + dn_2 dp_2. \quad (2.10)
\]

To ensure that the formed portfolio is self-financing, it needs to satisfy the following conditions [7, chapters 6 and 7]:

\[
dn_0 (B(t) + dB(t)) + dn_1 (p_1 + dp_1) + dn_2 (p_2 + dp_2) = 0.
\]

This reduces the equation (2.10) to

\[
dV(t) = n_0 dB(t) + n_1 dp_1 + n_2 dp_2.
\]

By inserting expression for \( dp_1 \) and \( dp_2 \) of equation (2.7) into the equation above we get

\[
dV(t) = [n_0 r(t) B(t) + n_1 \alpha_1 p_1 + n_2 \alpha_2 p_2] dt + [n_1 \sigma_1 p_1 + n_2 \sigma_2 p_2] d\tilde{W}.
\]
The no-arbitrage possibilities lead us to the following conditions (see for example [7, Proposition 7.6 and Chapter 10]):

a. The portfolio is self-financing at the beginning which means that

\[ V(0) = n_0 B(t) + n_1 p(0, T_1) + n_2 p(0, T_2) \]

b. No market risk which requires

\[ n_1 \sigma_1 p_1 + n_2 \sigma_2 p_2 = 0 \]

c. Finally,

\[ n_0 r(t) B(t) + n_1 \alpha_1 p_1 + n_2 \alpha_2 p_2 = 0 \]

Let \( w_0 = n_0 B(t) \), \( w_1 = n_1 p_1 \) and \( w_2 = n_2 p_2 \) the relative portfolio weights, the conditions of no-arbitrage reduce to

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & \sigma_1 & \sigma_2 \\
r & \alpha_1 & \alpha_2
\end{bmatrix}
\begin{bmatrix}
w_0 \\
w_1 \\
w_2
\end{bmatrix} = 0
\](2.11)

The matrix in equation (2.11) needs to be singular. With one degree of freedom, we choose, for example, the last row being a linear combination of the remaining rows. Hence

\[ \alpha_1 = r + \lambda \sigma_1 \text{ and } \alpha_2 = r + \lambda \sigma_2 \]

These equations reduce the result for the bond market to

\[ \lambda = \frac{\alpha_1(t) - r(t)}{\sigma_1} = \frac{\alpha_2(t) - r(t)}{\sigma_2} = \frac{\alpha(t) - r(t)}{\sigma} \]

(2.12)

The quantity \( \lambda \) is called the market price of interest rate risk and one can declare: “In a no-arbitrage market, all bonds have the same market price of risk, regardless of maturity time”.

Now inserting the equations (2.8) and (2.9) into (2.12), we obtain that

\[
\frac{1}{p_i} \left( \frac{\partial p_i}{\partial t} + \mu \frac{\partial p_i}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 p_i}{\partial r^2} \right) - r(t) = \lambda(t) \frac{\sigma \frac{\partial p_i}{p_i}}{\partial r},
\]

or equivalently

\[
\frac{\partial p_i}{\partial t} + (\mu - \lambda \sigma) \frac{\partial p_i}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 p_i}{\partial r^2} - rp_i = 0.
\]

(2.13)

The parabolic partial differential equation (2.13) above is called “term structure equation” that we reformulate as follows:
Proposition 2.2.1. In an arbitrage free bond market, the price $p(t, T) = p(t, r; T)$ satisfies the term structure equation

$$\frac{\partial p}{\partial t} + (\mu - \lambda \sigma) \frac{\partial p}{\partial r} + 1 \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial r^2} - rp = 0$$  \hspace{1cm} (2.14)

with the terminal condition given by

$$p(T, r; T) = p(T, T) = 1.$$  

In general, the term structure equation is given by [7, page 322]

$$\frac{\partial V}{\partial t} + (\mu - \lambda \sigma) \frac{\partial V}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = 0$$  \hspace{1cm} (2.15)

while the terminal condition is given by

$$V(T, r; T) = \Phi(r(T)).$$  \hspace{1cm} (2.16)

Where the function $\Phi(r(T))$ is called payoff, $V(T, r; T)$ is the price of any option and $r = r(t)$ is the one-factor short interest rate model defined by

$$dr(t) = \mu(r, t)dt + \sigma(r, t)d\tilde{W}_t.$$  

2.2.2. The risk neutral valuation formula

The solution of the bond price can be formally represented in an integral form as an expectation under the physical measure $P$ by (see for e.g. [50]).

$$V(t, r; T) = E^P \left[ \exp \left[ - \int_t^T \left( r(s) - \frac{\lambda^2(r, s)}{2} \right) ds + \int_t^T \lambda(r, s) d\tilde{W}_s \right] \right],$$

where $E^P$ denotes the expectation under the physical probability measure $P$ conditional on the filtration $\mathcal{F}_t$. To show the result, we define the following auxiliary function

$$V(t, r; T) = \exp \left[ - \int_t^S \left( r(s) - \frac{\lambda^2(r, s)}{2} \right) ds + \int_t^S \lambda(r, s) d\tilde{W}_s \right], t \leq S$$

and when applying Ito formula to $p(t, S; T)V(t, r; S)$, we find that

$$\frac{dV(t, r; S)}{V(t, r; S)} = \left( -r(S) - \frac{\lambda^2(r, S)}{2} \right) dS + \lambda(r, S) d\tilde{W}_s + \frac{\lambda^2(r, S)}{2} dS$$

$$= -r(S) dS + \lambda(r, S) d\tilde{W}_s.$$  

Therefore,
\[ dp(t, S; T) V(t, r; S) = -\lambda(r, S) V(t, r; S) \frac{\partial p}{\partial r} dS. \]

It follows, with simplifications \( p = p(t, S; T) \) and \( V = V(t, r; S) \) that

\[
\begin{align*}
    d(pV) &= V dp + pdV + dpdV \\
    &= V \left( \frac{\partial p}{\partial t} + \mu \frac{\partial p}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial r^2} \right) dS + V \sigma \frac{\partial p}{\partial r} d\bar{W}_t + pV (-rdS + \lambda d\bar{W}_t) \\
    &\quad - \lambda V \sigma \frac{\partial p}{\partial r} dS \\
    &= V \left( \frac{\partial p}{\partial t} + (\mu - \lambda \sigma) \frac{\partial p}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial r^2} - rp \right) dS + pV \lambda d\bar{W}_t + V \sigma \frac{\partial p}{\partial r} d\bar{W}_t \\
    &= pV \lambda d\bar{W}_t + V \sigma \frac{\partial p}{\partial r} d\bar{W}_t.
\end{align*}
\]

Next, we integrate the above equation from \( t \) to \( T \) and take expectation with respect to the probability measure \( P \). Since the expectation of a stochastic integral is zero, we have:

\[
E^P [p(T, r; T) V(t, r; T) - p(t, r; T) V(r, t; t)] = 0.
\]

Applying the terminal conditions \( p(T, r; T) = 1 \) and \( V(r, t; t) = 0 \), we finally obtain

\[ p(t, r; T) = E^P [V(r, t; T)]. \]

Now, we would like to apply the change of measure from the physical measure \( P \) to the risk neutral measure \( Q \) such that the bond price is a martingale under \( Q \). Let us assume that the market price \( \lambda(r, t) \) satisfies the Novikov condition, that is,

\[
E^Q \left[ \exp \left( \int_0^T \frac{\lambda^2(r, s)}{2} ds \right) \right] < \infty,
\]

and we define the Likelihood process as in section 1.3.7 by

\[ L_t = \frac{dQ}{dP}, \]

as the process \( L_t \) satisfies the stochastic differential equation

\[ dL(t) = L(t) \lambda(r, t) dW^p. \]

Then, it can be written as

\[ L(t) = \exp \left\{ \int_0^t \lambda(r, s) dW(s) - \frac{1}{2} \int_0^t |\lambda(r, s)|^2 ds \right\}. \]
By the virtue of Girsanov theorem, it follows that

\[ dW^Q = dW^P + \lambda(r, t)dt, \]

where \( W^Q \) is martingale under the probability measure \( Q \). Therefore, the dynamics of the short rate \( r \) under \( Q \) becomes

\[ dr(t) = [\mu(r, t) - \lambda(r, t)\sigma(r, t)]dt + \sigma(r, t)dW^Q(t). \quad (2.17) \]

Suppose the bond price \( p(t, r; T) \) satisfies the term structure equation (2, 13) and the dynamics of \( r(t) \) are governed by (2.17). Then by Feynman-Kac theorem, \( p(t, r; T) \) admits the expectation representation given by

\[ p(t, r; T) = E^Q \left[ \exp \left\{ - \int_t^T r(s)ds \right\} \right], \]

where \( r(t) \) is a solution to a stochastic differential equation given by

\[ dr(t) = \mu(r, t)dt + \sigma(r, t)dW_t. \quad (2.18) \]

In general, the pricing boundary value problem given by

\[ \begin{cases} \frac{\partial V}{\partial t} + (\mu - \lambda \sigma) \frac{\partial V}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = 0 \\
V(T, r; T) = \Phi(r) \end{cases} \]

(2.19)

admits a solution taking expectation form and given by

\[ V(t, r; T) = E^Q \left[ \left( \exp \left\{ - \int_t^T r(s)ds \right\} \times \Phi(r(T)) \right| \mathcal{F}_t \right]. \quad (2.20) \]

Note the formula (2.20) is called risk-neutral valuation formula. Recall that in what we have developed above, we started by a \( P \) -- dynamics of the short interest rate given by

\[ dr(t) = \mu(t, r)dt + \sigma(t, r)d\tilde{W}_t, \]

where \( \tilde{W} \) is the standard Brownian motion under the objective probability measure \( P \), \( \mu \) and \( \sigma \) are respectively the drift and diffusion term. On the other hand, the obtained stochastic differential equation (2.18) is under a new probability measure \( Q \) called risk neutral martingale measure. So, instead of working with the new drift \( \mu - \lambda \sigma \), we consider only the old drift \( \mu \) but under risk neutral martingale probability measure \( Q \) and then we can proceed by calibration. This procedure is known as martingale modelling. Let us describe shortly some popular papers for one-factor short interest rate.
2.2.3. A brief literature review for the dynamics of one-factor short interest rate.

Mean-reverting Ornstein-Uhlenbeck process was one of the first models used to describe short interest rates where it is called the Vasicek (1977) model given by

$$dr_t = (b - ar)dt + \sigma dW_t,$$

where $a$ (speed of reversion), $b$ and $\sigma$ are positive constants. Applying the Proposition 1.3.6 above, we may find that

$$r_t = r_s e^{-b(t-s)} + \frac{a}{b} \left(1 - e^{-b(t-s)}\right) + \sigma \int_s^t e^{-b(t-s)}dW_s.$$

Due to the Brownian term in the stochastic integral, it is possible that the short interest rate may become negative. To rectify the problem, Cox, Ingersoll and Ross (1985) proposed the following so-called square root diffusion process for the short rate

$$dr_t = (b - ar)dt + \sqrt{r} \sigma dW_t$$

where with an initially non-negative interest rate, the process $r_t$ will never be negative.

In 1990, John Hull and Alain White show that the Vasicek and Cox–Ingersoll–Ross (CIR) models can be extended as follows: (see [36])

$$dr_t = (\theta(t) - a(t)r_t)dt + \sigma(t)r_t \xi dW_t$$

where $\theta(t)$ is called time-independent drift to the process for $r_t$, $a(t)$ is the reversion rate function and $\sigma(t)$ the volatility function. $\xi$ is a constant taking values zero and $1/2$; zero for the so-called Hull–White extended Vasicek model and $1/2$ for the Hull–White extended CIR model.

It is deduced that Hull–White extended both Vasicek and CIR models to present advantages that have been discussed in [36]. In this dissertation, we are focusing on the Hull–White extended Vasicek model and sometimes we will talk about Vasicek model since they are associated. We note moreover that if $a(t)$ and $\xi$ get values zero, we obtain the so-called Black-Derman-Toy (BDT) model.

On the other hand, Geometric Brownian motion can model not only the price process of an asset as we can see it in the Black–Scholes model [10] but also the dynamics of the short interest rate; it is the case of Dothan model (1978) given by
where \( a \) and \( \sigma \) are positive constants. The solution \( r_t \), the expectation and variance of the process \( r_t \) can be found easily by applying the Proposition 1.3.4.

The list of the dynamics of one-factor short interest rate models is so long that we cannot dissect all of them. However, Hull-White [37] recently proposed the generalized Hull-White model containing many popular dynamics of one-factor short interest rate models. The equation is given by

\[
df(r_t) = [\theta(t) - a(t)f(r_t)]dt + \sigma(t)dW_t.
\]

As special cases, when for

- \( f(r_t) = r_t \) and \( a(t) = a \), \( \sigma(t) = \sigma \) and \( \theta(t) = b \) are positive constants, we get the Vasicek model,
- \( f(r_t) = r_t, a(t) = 0 \) and \( \sigma(t) = \sigma \) is a constant, it is the Ho-Lee (1986) model,
- \( f(r_t) = r_t, a(t) \) is not zero, it is the original Hull-White [36] model,
- \( f(r_t) = \sqrt{r_t} \), it is a model developed by Pelsser (1996),
- \( f(r_t) = ln r_t \), it is the Black-Karasinski (1991) model, and so on.

We summarize popular one-factor short interest rate models by the following table.
Table 2.1 Summary of one – factor short interest rate models

<table>
<thead>
<tr>
<th>Models</th>
<th>Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Vasicek</td>
<td>( dr_t = (b - ar_t) dt + \sigma dW_t )</td>
</tr>
<tr>
<td>2 Exponential Vasicek</td>
<td>( dr_t = r_t (b - a \ln r_t) dt + \sigma r_t dW_t )</td>
</tr>
<tr>
<td>3 Cox – Ingersoll – Ross (CIR)</td>
<td>( dr_t = a(b - r_t) dt + \sigma \sqrt{r_t} dW_t )</td>
</tr>
<tr>
<td>4 Dothan</td>
<td>( dr_t = ar_t dt + \sigma r_t dW_t )</td>
</tr>
<tr>
<td>5 Blac – Derman – Toy</td>
<td>( dr_t = \theta(t) r_t dt + \sigma r_t dW_t )</td>
</tr>
<tr>
<td>6 Ho – Lee</td>
<td>( dr_t = \theta(t) dt + \sigma r_t dW_t )</td>
</tr>
<tr>
<td>7 Hull – White (extended Vasicek)</td>
<td>( dr_t = (\theta(t) - a(t) r_t) dt + \sigma(t) dW_t )</td>
</tr>
<tr>
<td>8 Hull – White (extended CIR)</td>
<td>( dr_t = (\theta(t) - a(t) r_t) dt + \sigma \sqrt{r_t} dW_t )</td>
</tr>
<tr>
<td>9 Black – Karasinski</td>
<td>( dr_t = r_t (b_t - a \ln r_t) dt + \sigma r_t dW_t )</td>
</tr>
<tr>
<td>10 Mercurio – Moraleda</td>
<td>( dr_t = r_t \left( b_t - \left( a \frac{1}{1 + at} \right) \ln r_t \right) dt + \sigma r_t dW_t )</td>
</tr>
<tr>
<td>11 CIR++</td>
<td>( r_t = x_t + \alpha_t, ) ( dx_t = a(b - x_t) dt + \sigma \sqrt{x_t} dW_t )</td>
</tr>
<tr>
<td>12 Extended Exponential Vasicek model (EEV)</td>
<td>( r_t = x_t + \alpha_t, ) ( dx_t = x_t (b - a \ln x_t) dt + \sigma x_t dW_t )</td>
</tr>
</tbody>
</table>

2.2.4. Affine term structures

Now we are going to investigate the case where the bond price process gets an exponential form. Here we follow [8].

Let first recall the definition of affine function.

**Definition 2.2.2 (Affine function).** Assume that we are given a function \( L: \mathbb{R}^m \to \mathbb{R}^n \). We say that \( L \) is linear if for any vectors \( X \) and \( Y \) in \( \mathbb{R}^m \)

\[
L(\alpha X + \sigma Y) = \alpha L(X) + \sigma L(Y),
\]

where \( \alpha \) and \( \sigma \) are scalar. The function \( A: \mathbb{R}^m \to \mathbb{R}^n \) is affine if there is a linear function \( L \) and a vector \( \beta \) in \( \mathbb{R}^n \) such that

\[
A(X) = L(X) + \beta,
\]
for all $X$ in $\mathbb{R}^m$. The formula above can be written as

$$A(X) = MX + \beta$$

where $M$ is the $n \times m$ matrix. In particular, if $n = m = 1$ then $f: \mathbb{R} \to \mathbb{R}$ is affine if there are real numbers $\alpha$ and $\beta$ such that

$$f(x) = \alpha x + \beta.$$ 

Thus an affine function is just a linear function plus a translation.

**Definition 2.2.3.** Let us consider the bond price processes $p(t, T) = p(t, r(t); T)$ getting the following form

$$p(t, r, T) = e^{m(t,T) - n(t,T)r}$$

(2.21)

where $m(t, T)$ and $n(t, T)$ are deterministic functions. Then, the model is said to possess an affine term structure (ATS).

Let us now discuss about existence of ATS. We start by investigating some of the implications of an affine term structure. Let the process $r_t$ be a solution of

$$dr(t) = \mu(t)dt + \sigma(t)dW_t$$

(2.22)

and assuming that the bond price process gets the form (2.21), we compute the various partial derivatives of $p(t, r; T)$ as

$$\frac{\partial p(t, r; T)}{\partial t} = \left[ \frac{\partial m(t, T)}{\partial t} - \frac{\partial n(t, T)}{\partial t}r \right] p(t, r; T);$$

$$\frac{\partial p(t, r; T)}{\partial r} = -n(t, T) p(t, r; T);$$

$$\frac{\partial^2 p(t, r; T)}{\partial r^2} = n^2(t, T) p(t, r; T).$$

Substituting these three partial derivatives above into the governing partial differential equation (2.14), we get the following result:

$$\frac{\partial m(t, T)}{\partial t} - \frac{\partial n(t, T)}{\partial t} - \mu n(t, T)r + \frac{1}{2} \sigma^2 n^2(t, T) - r = 0,$$

or equivalently

$$\frac{\partial m(t, T)}{\partial t} - \{1 + \mu n(t, T)\}r + \frac{1}{2} \sigma^2 n^2(t, T) = 0.$$  

(2.23)

and the boundary condition $p(T, r; T) = 1$ implies $m(T, T) = n(T, T) = 0$. 
We observe that the equation (2.23) gives us the relations which must hold between \( m, \mu, n \) and \( \sigma \) in order for an ATS to exist, and for a certain choice of \( \mu \) and \( \sigma \) there may or may not exist functions \( m(t, T) \) and \( n(t, T) \) such that (2.23) is satisfied. We observe again that if \( \mu \) and \( \sigma^2 \) are both affine functions of \( r_t \) according to Definition 2.1.1 above, then equation (2.23) becomes a separable differential equation for the unknown function \( m(t, T) \) and \( n(t, T) \). Assume thus that \( \mu \) and \( \sigma^2 \) have the form

\[
\mu(t) = \mu_1(t)r + \mu_2(t),
\]

\[
\sigma^2(t) = \sigma_1(t)r + \sigma_2(t),
\]

then

\[
\frac{\partial m(t, T)}{\partial t} - \mu_2 n(t, T) + \frac{1}{2} \sigma_2 n^2(t, T)
\]

\[
- \left\{ 1 + \frac{\partial n(t, T)}{\partial t} \mu_1(n, T) - \frac{1}{2} \sigma_1(n^2(t, T) \right\} r = 0.
\]

(2.24)

Since the equation holds for all values of \( r = r(t) \), therefore, the coefficient of \( r \) in the equation above must be equal to zero. Thus, we have

\[
1 + \frac{\partial n(t, T)}{\partial t} \mu_1(n, T) - \frac{1}{2} \sigma_1(n^2(t, T) = 0
\]

or equivalently we get the following Riccatti equation

\[
\frac{\partial n(t, T)}{\partial t} \mu_1(n, T) - \frac{1}{2} \sigma_1(n^2(t, T) = -1.
\]

Since the equation (2.24) must hold, then the other term in (2.24) must also vanish, so we have

\[
\frac{\partial m(t, T)}{\partial t} - \mu_2 n(t, T) + \frac{1}{2} \sigma_2 n^2(t, T) = 0,
\]

or equivalently

\[
\frac{\partial m(t, T)}{\partial t} = \mu_2 n(t, T) - \frac{1}{2} \sigma_2 n^2(t, T).
\]

Let us now formulate this result properly as the following proposition:

**Proposition 2.2.4** (Affine term structure formula). Assume that \( \mu \) and \( \sigma^2 \) are given by

\[
\begin{cases}
\mu(t) = \mu_1(t)r + \mu_2(t) \\
\sigma^2(t) = \sigma_1(t)r + \sigma_2(t)
\end{cases}
\]

(2.25)
then the model admits an ATS of the form (2.25) where \( m(t, T) \) and \( n(t, T) \) satisfy the system

\[
\begin{align*}
\frac{\partial n(t, T)}{\partial t} &- \mu_1(t)n(t, T) - \frac{1}{2}\sigma_1(t)n^2(t, T) = 1 \\
\frac{\partial m(t, T)}{\partial t} &- \mu_2(t)n(t, T) - \frac{1}{2}\sigma_2(t)n^2(t, T) = 0
\end{align*}
\]  

(2.26) (2.27)

with \( m(T, T) = n(T, T) = 0. \)

**Corollary 2.2.5.** Both the Vasicek model and the Hull-White extended Vasicek model admit an ATS.

**Proof:** From Hull–White model given by

\[
dr_t = (\theta(t) - a(t)r_t)dt + \sigma(t)dB_t
\]

we may identify

\[
\begin{align*}
\mu(t, r) &= -a(t)r + \theta(t) \\
\sigma^2(t, r) &= \sigma^2(t)
\end{align*}
\]

where \( \mu(t, r) \) and \( \sigma^2(t, r) \) are affine functions. Thus, from Proposition 2.2.4, the model admits an ATS and it is the same with Vasicek model where the term \( \theta(t) \) is a constant.

2.3. **Bond prices under Vasicek and Hull–White models**

2.3.1. **Bond price under Vasicek model**

The diffusion process proposed by Vasicek is a mean-reverting version of the Ornstein-Uhlenbeck process. The short-term interest rate \( r \) is defined as the unique strong solution of the SDE

\[
dr_t = (b - ar)dt + \sigma dB_t,
\]

(2.28)

where \( b, a \) and \( \sigma \) are strictly positive constants. It is well-known from our first chapter that the solution of the SDE (2.28) is a Markov process with continuous sample paths and Gaussian increments.

**Proposition 2.3.1.** The unique solution to the stochastic differential equation (2.28) is given by the formula
Moreover, for any \( s < t \) the conditional law of \( r_t \) with respect to the filtration \( \mathcal{F}_s = \sigma \{ r(s), 0 \leq s \leq t \} \), is Gaussian with the conditional expected value
\[
E^Q \left( r_t \mid \mathcal{F}_s \right) = r_s e^{-b(t-s)} + \frac{a}{b} \left(1 - e^{-b(t-s)}\right),
\]
and the conditional variance
\[
\text{Var}(r_t \mid \mathcal{F}_s) = \frac{\sigma^2}{2b} \left(1 - e^{-2b(t-s)}\right).
\]

Furthermore, the limits of \( E^Q \left( r_t \mid \mathcal{F}_s \right) \) and \( \text{Var}(r_t \mid \mathcal{F}_s) \) when \( t \) tends to infinity are:
\[
\lim_{t \to \infty} E^Q \left( r_t \mid \mathcal{F}_s \right) = \frac{a}{b},
\]
and
\[
\lim_{t \to \infty} \text{Var}(r_t \mid \mathcal{F}_s) = \frac{\sigma^2}{2b}.
\]

**Proof:** This proposition is already proved in our first chapter, Proposition 1.3.6. However, we will make some more comments. For \( s > 0 \) fixed, let us consider the process \( Y_t = r_t e^{b(t-s)} \). By Ito's lemma and by using the expression of \( dr_t \) in (2.28), we obtain the following result:
\[
dY_t = e^{b(t-s)} \, dr_t + b e^{b(t-s)} r_t \, dt
\]
\[
= e^{b(t-s)} \left(\frac{a}{b} - b r_t \, dt + \sigma dW_t + b r_t \, dt\right)
\]
\[
= e^{b(t-s)} \left(\frac{a}{b} + \sigma dW_t\right).
\]

Thus, we have
\[
\int_s^t e^{b(t-s)} \, dY_u = \int_s^t e^{b(u-s)} \, du + \sigma \int_s^t e^{b(u-s)} \, dW_u
\]
and consequently
\[
r_t = e^{-b(t-s)} r_s + e^{-b(t-s)} \int_s^t e^{b(u-s)} \, du + \sigma e^{-b(t-s)} \int_s^t e^{b(u-s)} \, dW_u.
\]

Since
\[
\int_s^t e^{b(u-s)} \, du = \frac{1}{b} \left[ e^{b(t-s)} - 1 \right],
\]
we finally obtain
It is well-known again from our first chapter that for any function $g(u)$, the Ito integral

$$
\int_s^t g(u) \, dW_u,
$$
is a random variable independent of the filtration $\mathcal{F}_s$, $s \geq 0$ and has the Gaussian law

$$N\left(0, \int_s^t g^2(u) \, du\right).
$$

In our case

$$\int_s^t g^2(u) \, du = \sigma^2 \int_s^t e^{-2b(t-u)} \, du = \frac{\sigma^2}{2b} \left(1 - e^{-2b(t-s)}\right).
$$

We conclude that for any $s < t$ the conditional law of $r_t$ with respect to the filtration $\mathcal{F}_s = \sigma(r(s), 0 \leq s < t)$ is Gaussian, with the conditional expected value

$$E_Q(r_t | \mathcal{F}_s) = r_s e^{-b(t-s)} + \frac{a}{b} \left(1 - e^{-b(t-s)}\right),
$$

and the conditional variance

$$Var(r_t | \mathcal{F}_s) = \frac{\sigma^2}{2b} \left(1 - e^{-2b(t-s)}\right).
$$

Finally, it is easy to obtain their limits:

$$\lim_{t \to \infty} E_Q(r_t | \mathcal{F}_s) = \lim_{t \to \infty} \left[r_s e^{-b(t-s)} + \frac{a}{b} \left(1 - e^{-b(t-s)}\right)\right] = \frac{a}{b},
$$

and

$$\lim_{t \to \infty} Var(r_t | \mathcal{F}_s) = \lim_{t \to \infty} \left[\frac{\sigma^2}{2b} \left(1 - e^{-2b(t-s)}\right)\right] = \frac{\sigma^2}{2b}.
$$

The lemma is thus proved.

Let us now find the price of a bond in the Vasicek model framework.

**Proposition 2.3.2.** The price at time $t$ of a zero coupon bond under the Vasicek model equals to

$$p(t, T) = e^{m(t,T)-n(t,T)r},$$

where $n(t, T)$ and $m(t, T)$ are functions given by

$$n(t, T) = \frac{1}{a} \left(1 - e^{-a(T-t)}\right)$$

and

$$m(t, T) = nt.$$
Furthermore, the dynamics of the bond price under the martingale probability measure $Q$ are given by

$$dp(t,T) = p(t,T)[r(t)dt - \sigma n(t,T)dW_t].$$

\textbf{Proof:} From Proposition 2.2.4, we may identify the functions $\alpha(t), \beta(t), \gamma(t)$ and $\delta(t)$ as follows:

$$\mu_1(t) = -a, \quad \mu_2(t) = b, \quad \sigma_1(t) = 0 \text{ and } \sigma_2(t) = \sigma^2,$$

and from the same Proposition 2.2.4, we arrive at the following system of differential equations:

$$\begin{cases}
\frac{\partial m(t,T)}{\partial t} = bn(t,T) - \frac{1}{2} \sigma^2 n^2(t,T) \\
\frac{\partial n(t,T)}{\partial t} = an(t,T) - 1
\end{cases}$$

with $m(T,T) = n(T,T) = 0$.

It is easy to find the solution to the equation (2.34) since it is a simple linear ordinary differential equation in $t$-variable. Thus, for each fixed $T$, we obtain

$$n(t,T) = \frac{1}{a} \left(1 - e^{-a(T-t)}\right).$$

On the other hand, integrating equation (2.33) we obtain

$$m(t,T) = \frac{1}{2} \sigma^2 \int_t^T n^2(s,T)\,ds - b \int_t^T n(s,T)\,ds.$$  \hspace{1cm} (2.36)

Now, inserting the expression (2.35) into the equation (2.36), we obtain what follows:

$$m(t,T) = \frac{1}{2} \sigma^2 \int_t^T \left[\frac{1}{a} \left(1 - e^{-a(T-s)}\right)\right]^2 \,ds - b \int_t^T \frac{1}{a} \left(1 - e^{-a(T-s)}\right) \,ds.$$

We set the expression of $m(t,T)$ into two integrals named $m_1(t,T)$ and $m_2(t,T)$ that we define as follows:

$$m_1(t,T) = \frac{\sigma^2}{2a^2} \int_t^T \left[(1 - e^{-a(T-s)})\right]^2 \,ds \quad \text{and} \quad m_2 = \frac{b}{a} \int_t^T \left(1 - e^{-a(T-s)}\right) \,ds$$

So, the integral $m_1(t,T)$ gives
\[
m_1(t,T) = \frac{\sigma^2}{2a^2} \int_t^T \left( (1 - 2e^{-a(T-s)} + e^{-2a(T-s)}) \right) ds
\]
\[
= \frac{\sigma^2}{2a^2} \left[ (T - t) - \frac{2}{a} \left( 1 - e^{-a(T-t)} \right) + \frac{1}{2a} \left( 1 - e^{-2a(T-t)} \right) \right]
\]
\[
= \frac{\sigma^2}{2a^2} \left[ (T - t) - \frac{3}{2a} + \frac{2}{a} e^{-a(T-t)} - \frac{1}{2a} e^{-2a(T-t)} \right]
\]
\[
= \frac{\sigma^2}{2a^2} \left[ (T - t) - \frac{1}{2a} \left( 3 - 4e^{-a(T-t)} - e^{-2a(T-t)} \right) \right]
\]
\[
= \frac{\sigma^2}{2a^2} \left[ (T - t) - \frac{1}{2a} \left( 2(1 - e^{-a(T-t)}) + (1 - e^{-a(T-t)})^2 \right) \right]
\]
and the integral \( m_2(t,T) \) gives
\[
m_2(t,T) = \frac{b}{a} \int_t^T (1 - e^{-a(T-s)}) ds = \frac{b}{a} \left[ (T - t) - \frac{1}{a} (1 - e^{-a(T-t)}) \right].
\]
Finally we get:
\[
m(t,T) = m_1(t,T) - m_2(t,T)
\]
\[
= \left( \frac{b}{a} - \frac{\sigma^2}{2a^2} \right) \left[ \frac{1}{a} (1 - e^{-a(T-t)}) - (T - t) \right] - \frac{\sigma^2}{4a^3} \left( (1 - e^{-a(T-t)})^2 \right)
\]
\[
= \left( \frac{b}{a} - \frac{\sigma^2}{2a^2} \right) \left[ \frac{1}{a} n(t,T) - (T - t) \right] - \frac{\sigma^2}{4a} n^2(t,T).
\]
\[
= \frac{[n(t,T) - (T - t)] (ab - \frac{1}{2} \sigma^2)}{a^2} - \frac{\sigma^2 n^2(t,T)}{4a}.
\]

The general valuation formula (2.29) is an easy consequence of the Markov property or \( r \). Finally, to establish the relation (2.32), it suffices to apply Ito formula to
\[
p(t,T) = e^{m(t,T) - n(t,T)r}
\]
in order to obtain
\[
dp(t,T) = p(t,T) \left[ r(t) dt - \sigma n(t,T) dW_t \right]
\]
or equivalently
\[
dp(t,T) = p(t,T) \left[ r(t) dt - \frac{\sigma}{a} \left( 1 - e^{-a(T-t)} \right) dW_t \right]
\]
which finishes our proof.

Note that from Assumption 2.1.4, we may identify functions \( a(t,T) \) and \( b(t,T) \) under Vasicek model as
\[
a(t,T) = r(t),
\]
and

\[ b(t, T) = -\frac{\sigma}{a} \left( 1 - e^{-a(T-t)} \right). \]

### 2.3.2. Bond price under the Hull-White model

As introduced in our first chapter, the one-factor short interest rate model proposed by Hull and White (1990) is one which follows the mean-reverting diffusion process defined by

\[ dr_t = (\theta(t) - a(t)r)dt + \sigma(t)r^\beta dW_t, \]  \hspace{1cm} (2.37)

where \( 0 \leq \beta \leq 1 \) is a constant, \( W_t \) is a one-dimensional Brownian motion under the risk neutral probability measure \( Q \) and \( \theta(t), a(t), \sigma(t) : \mathbb{R}_+ \to \mathbb{R} \) are locally bounded functions. \( \theta(t) \) is called the time-dependent drift to the process \( r_t \). Note that both Vasicek and CIR models are special cases of the Hull–White model. As a result, by setting \( \beta = 1/2 \) and \( \beta = 0 \), we get the extended CIR and Vasicek model respectively. In this study, we will make a fairly detailed exploration of the Hull-White extension of the Vasicek model, in which the dynamics of \( r \) are

\[ dr_t = (\theta(t) - a(t)r)dt + \sigma(t)dW_t. \]  \hspace{1cm} (2.38)

To solve this equation explicitly, let us denote

\[ y(t) = \int_0^t a(s)ds. \]

Then, we have

\[ d\left( e^{y(t)}r_t \right) = e^{y(t)}(\theta(t)dt + \sigma(t)dW_t), \]

which lead us to the solution given by

\[ r_t = e^{-y(t)} \left( r_0 + \int_0^t e^{y(s)}a(s)ds + \int_0^t e^{y(t)}\sigma(s)dW_s \right). \]  \hspace{1cm} (2.39)

Unlike the Vasicek model, the Hull-White model gives the possibility of the choice of the time-dependent drift \( \theta(t) \) for the fixed parameters \( a(t) \) and \( \sigma(t) \). This gives rise of a perfect fitting the theoretical to the observed bond price at initial date as discussed in their original papers [37, 40]. To illustrate the point, we derive \( \theta(t) \) in terms of initial observed forward rate \( \tilde{f}(0, T) \) and initial yield curve to maturity

\[ A \text{ special case of such a model, with } b=0, \text{ was considered by Merton (1973).} \]
Let then consider the governing equation for the bond price $p = p(r_t, t; T)$ under the standard Hull–White model given by
\[
\frac{\partial p}{\partial t} + (\theta(t) - a(t)r) \frac{\partial p}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial r^2} - rp = 0.
\]
As in Proposition 2.3.2, the price at time $t$ of a zero-coupon bond in the Hull–White model equals to
\[
p(r_t, t; T) = e^{m(t,T) - n(t,T)r},
\]
or equivalently,
\[
\ln p(r_t, t; T) = m(t, T) - n(t, T)r,
\]
this implies that
\[
m(t, T) = \ln p(r_t, t; T) + n(t, T)r,
\]
where $m(t, T)$ and $n(t, T)$ are solution of the following Riccatti equations
\[
\begin{align*}
\frac{\partial m(t, T)}{\partial t} &= \theta(t)n(t, T) - \frac{1}{2} \sigma^2(t)n^2(t, T); \\
\frac{\partial n(t, T)}{\partial t} &= a(t)n(t, T) - 1.
\end{align*}
\]
with $m(T, T) = n(T, T) = 0$.

Thus, $m(t, T)$ and $n(t, T)$ are given by
\[
\begin{align*}
m(t, T) &= \frac{\sigma^2}{2} \int_t^T n^2(s, T)ds - \int_t^T \theta(s)n(s, T) ds \quad (2.42) \\
n(t, T) &= \frac{1}{a} \left( 1 - e^{-a(T-t)} \right). \quad (2.43)
\end{align*}
\]
Let us now determine $\theta(T)$ in terms of the current term structure of bond price $p(r_t, t; T)$. From (2.41) and (2.42), we have
\[
m(t, T) = \ln p(r_t, t; T) + n(t, T)r
\]
\[
= \frac{\sigma^2}{2} \int_t^T n^2(s, T)ds - \int_t^T \theta(s)n(s, T) ds,
\]
Using only one equality, we may write
\[
\ln p(r_t, t; T) + n(t, T)r = \frac{\sigma^2}{2} \int_t^T n^2(s, T)ds - \int_t^T \theta(s)n(s, T) ds,
\]
or equivalently
\[
\int_t^T \theta(s)n(s, T) ds = \frac{\sigma^2}{2} \int_t^T n^2(s, T)ds - \ln p(r_t, t; T) - n(t, T)r. \quad (2.44)
\]
To solve for $\theta(s)$, the first step is to obtain an explicit expression for
This can be achieved by differentiating the integral
\[ \int_t^T \theta(s) ds. \]
with respect to \( T \) and subtracting the terms involving
\[ \int_t^T \theta(s) e^{-a(T-t)} ds. \]
The derivative of the left-hand side of (2.44) with respect to \( T \) gives
\[ \frac{\partial}{\partial T} \left( \int_t^T \theta(s)n(s,T) ds \right) = \theta(s)b(s,T)|_{s=T} + \int_t^T \theta(s) \frac{\partial}{\partial T} b(s,T) ds \]
\[ = \int_t^T \theta(s)e^{-a(T-t)} ds. \] (2.45)
Next, we equate the derivatives on both sides to obtain
\[ \int_t^T \theta(s)e^{-a(T-t)} ds \]
\[ = \frac{\sigma^2}{a} \int_t^T (1 - e^{-a(T-s)}) e^{-a(T-s)} ds - \frac{\partial}{\partial T} \ln p(r_t,t;T) - re^{-a(T-t)} \] (2.46)
Now, we multiply (2.44) by \( a \) and add it to (2.46), which gives us the following result:
\[ \int_t^T \theta(s) ds = \frac{\sigma^2}{2a} \int_t^T (1 - e^{-a(T-s)}) ds - \frac{\partial}{\partial T} \ln p(r_t,t;T) - a \ln p(r_t,t;T) - r. \]
Finally, by differentiating the above equation with respect to \( T \) again, we obtain \( \theta(T) \) in terms of the current terms structure of bond prices \( p(r_t,t;T) \) as follows:
\[ \theta(T) = \frac{\sigma^2}{2a} \left(1 - e^{-a(T-s)}\right) - \frac{\partial^2[\ln p(r_t,t;T)]}{\partial T^2} - a \frac{\partial[\ln p(r_t,t;T)]}{\partial T} \]
Alternatively, one may express \( \theta(T) \) in terms of the current term structure of forward rate \( f(t,T) \). Recall from Definition 2.1.1 that
\[ f(t,T) = -\frac{\partial[\ln p(r_t,t;T)]}{\partial T}, \]
so that we may rewrite \( \theta(T) \) in the form
\[ \theta(T) = \frac{\sigma^2}{2a} \left(1 - e^{-a(T-s)}\right) + \frac{\partial f(t,T)}{\partial T} + af(t,T). \] (2.47)
If we start from initial time \( t = 0 \), then equation (2.47) above can be written as
\[
\theta(T) = \frac{\sigma^2}{2a} (1 - e^{-aT}) + \frac{\partial f(0,T)}{\partial T} + af(0,T).
\] (2.48)

Suppose that \( \theta(T) \) get the following form:

\[
\theta(T) = \frac{d\alpha(T)}{dT} + a\alpha(T),
\]

and define the observed forward rate \( \bar{f}(0,T) \) as

\[
\bar{f}(0,T) = \alpha(T) - \beta(T),
\]

where \( \beta(T) \) is defined by \( \beta(T) = \frac{\sigma^2}{2} n(0,T)^2 \). Then it follows that

\[
\theta(T) = \frac{d\alpha(T)}{dT} + a\alpha(T),
\]

\[
= \frac{d\bar{f}(0,T)}{dT} + a\alpha(T) + \frac{d\beta(T)}{dT}
\]

\[
= \frac{d\bar{f}(0,T)}{dT} + a \left( \frac{d\bar{f}(0,T)}{dT} + \beta(T) \right) + \frac{d\beta(T)}{dT}.
\] (2.49)

Moreover, in term of yield curve \( R(0,T) \) starting at the initial time \( t = 0 \), we have

\[
\alpha(T) = R(0,T) + T \frac{dR(0,T)}{dT} + \frac{d\beta(T)}{dT},
\]

and then it follows that

\[
\theta(T) = \frac{d\bar{f}(0,T)}{dT} + aR(0,T) + aT \frac{dR(0,T)}{dT} + \frac{d\beta(T)}{dT}.
\] (2.50)

and thus, the function \( \theta \) is indeed uniquely determined in terms of current term structure in equation (2.47), initial observed forward rate in equation (2.47) and initial yield curve in equation (2.50). This terminates the fitting procedure. As for Vasicek model, we have to apply Ito formula to (2.29) to obtain the bond price process as

\[
dp(t,T) = p(t,T) \left[ r(t)dt - \frac{\sigma}{a} \left( 1 - e^{-a(T-t)} \right) dW_t \right].
\]

As the bond volatility is independent of \( r \), so the distribution of the bond price at any given time conditional on its price at an earlier time is lognormal. We note that from Assumption 2.1.4, we may identify functions \( a(t,T) \) and \( b(t,T) \) under Hull – White model as

\[
a(t,T) = r(t),
\]

and
So we can write shortly the dynamics of bond prices as

\[ b(t, T) = -\frac{\sigma}{\alpha} \left( 1 - e^{-\alpha(T-t)} \right), \]

which is a Geometric Brownian Motion with respect to the probability measure \( Q \) introduced in Subsection 1.3.2 of our first chapter.

2.4. Conclusion

In this chapter, we have derived the term structure equation for pricing one-factor short interest rate in the arbitrage free framework; this equation is a parabolic partial differential equation. We have also derived the price process under the Vasicek and Hull-White extended Vasicek models and the result shows that the volatility term of the two price processes are equivalent. However, the Hull-White model presents special features than the Vasicek model which are best calibration to the current term structure, to the observed forward rate and to the initial yield curve. These features make the model more attractive and more interesting as a practical tool. Notably, these results will be very useful in our next chapters.
In this chapter, we find a simple closed-form expression for pricing European option written on a zero-coupon and coupon-bearing bond option under the Hull-White extended Vasicek model by solving the Hull-White term structure problem. Our derivation procedure starts from the risk neutral valuation formula and our ideas mainly come from [43], [1], [11] and [49]. Before we enter into the heart of the chapter, let us start with some preparations.

3.1. Preparations

3.1.1. Statement of the problem

Let \( V(t, r; T) \) be the price of the European option which is function of the time \( t \), the short interest rate process \( r \) with the terms \( \mu(t, r) \) and \( \sigma(t, r) \) as parameters and the maturity date \( T \). As introduced in our second chapter, pricing interest rate derivatives requires solving the term structure problem given by

\[
\begin{align*}
\frac{\partial V(t, r; T)}{\partial t} + \mu(t, r) \frac{\partial V(t, r; T)}{\partial r} + \frac{1}{2} \sigma^2(t, r) \frac{\partial^2 V(t, r; T)}{\partial r^2} - rV &= 0 \\
V(T, r; T) &= \Phi(r(T)),
\end{align*}
\]

where the equation (3.1) represents the term structure equation, the term \( r = r(t) \) is the one factor short interest rate model with their dynamics given by

\[
dr_t = \mu(t, r)dt + \sigma(t, r)dW_t^Q. \tag{3.3}
\]
Here $\mu(t,r)$ and $\sigma(t,r)$ are respectively the drift and the volatility term of the process $r(t)$. The boundary condition (3.2) is called payoff. The solution to Problem (3.1) - (3.2) is given under the form

$$V(t,r;T) = E^Q \left[ \exp \left( - \int_t^T r(s)ds \right) \times V(T,r;T) \bigg| \mathcal{F}_t \right],$$

(3.4)

where $\mathcal{F}_t$ represents the filtration generated by the standard Brownian motion $W_t^Q$ on the probability space $(\Omega, \mathcal{F}, Q)$.

In this chapter, we are going to solve the Hull-White term structure problem with the boundary condition or the payoff given as the European option written on the zero-coupon and coupon bearing bonds. This means that we need to solve the following problem for the call option:

$$\begin{cases}
\frac{\partial V}{\partial t} + (\theta(t) - a(t)r) \frac{\partial V}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = 0 \\
V(T,r;T) = \Phi(r(T)) = \max(p_c(T, S_t) - K, 0),
\end{cases}$$

(3.5)

(3.5’)

where $r = r(t)$ is the solution to the Hull-White extended Vasicek model given by

$$dr_t = (\theta(t) - a(t)r)dt + \sigma(t)dW_t.$$

Here $\theta(t)$ has the time-dependence which is an unknown parameter that can be calculated from the initial yield curve as done in Section 2.3.2, $a(t)$ represents the rate at which the average of interest changes. Most of the time, it is considered as a constant and is left as a user input. The $\sigma(t)$ is the volatility of the process $r(t)$ which represents the pace at which interest rates move higher or lower and can be determined via calibration. As these two parameters are determined statistically then in this dissertation we will not discuss how to find them rather we will pick up their values to test the illustration of our result.

On the other hand, the process $p_c(t, S_t)$ is the price of a coupon bearing bond given by

$$p_c(t, T_i) = \sum_{i=1}^{d} c_i p(t, S_i),$$

where $p(t, S_i)$ is the price of each bond as the dynamics given in the (2.5). If $d = 1$ and $c_1 = 1$, then we say that the option is written on a zero-coupon bond.
Furthermore, we note that the put option is a slight modification of the call option and it is given by the following problem:

\[
\begin{align*}
\frac{\partial V}{\partial t} + (\theta(t) - a(t)r) \frac{\partial V}{\partial r} + \frac{1}{2} \sigma^2 r^2 \frac{\partial^2 V}{\partial r^2} - rV &= 0 \\
V(T, r; T) &= \Phi(r(T)) = \max(K - p_c(T, S), 0).
\end{align*}
\]

As the solution for put option goes in the same lines with the call option, we will only give the final results for the put option. Before we solve the problems (3.4) and (3.5), let us explain briefly the theory of pricing interest rate derivatives in the general case. For more details see [50].

### 3.1.2. Valuation of interest rate derivatives in general

There are two main ways to price interest rate derivatives as shown by the first two branches in Figure 3.1. If we want to price interest rate derivatives, we could either model the bond price directly or model the equivalent representations in terms of interest rates. The most common way for practitioners to price basic interest rate derivatives (e.g. bonds, caps, floors and swaptions) is to adapt the Black-Scholes model as introduced in our first chapter, Example 1.3.19 and implement the Black's (1976) model. However, common practices involve making a number of different inconsistent assumptions for each instrument.

We note that the Black's (1976) model is very versatile. When a cap is valued, the underlying variable that is assumed to be lognormal is the interest rates that are being capped. When a bond option is valued, it is the bond price at the maturity of the option. When a swap option is valued, it is the swap rate at the maturity of the option. However, the main disadvantage of Black's model and its extensions is that it can be used only when a derivative depends on an interest rate observed at a single time. As a result, the model provides no linkages between different interest rates and their volatilities. Thus, the model cannot be used for valuing American options and other more complex interest rate derivatives. For these types of instruments, a no – arbitrage model of the term structure is essential as explained in next paragraph.

---

*The Black’(1976) model assumes that, at the maturity of the option, the variable underlying the option (typically an interest rate or a bond price) is lognormally distributed.*
Another consistent way to price interest rate derivatives is by modelling the underlying interest rate directly. One way of constructing an arbitrage free model for interest rates is in terms of the process followed by the instantaneous short rate \( r(t) \). We have shown that the process for the short rate in a risk–neutral measure determines the current term structure and how it can evolve. It is the case of the bond price formula (2.22). The term structure consistent models set out to model the dynamics of entire term structure in a way that is automatically consistent with the initial (observed) market data.

We can further subdivide models in this approach into those that fit the term structure of interest rate only and those that fit both the term structure of rates and term structure of rate volatilities. Models that do not fit the volatility structure have them determined by the parameters of the model.

There are two basic ways of achieving this goal. One is to specify the process for short interest rate and then increase the parameterization of the model by using time–independent factors until all initial data can be returned. The second starts by specifying the initial yield curve and its volatility structure and to determine a drift structure that makes the model arbitrage free.
The Hull-White model, as presented in this dissertation, is a widely used one-factor interest rate model because it fits both the term structure of rates and term structure of rate volatilities. Moreover, the model presents analytical tractability on traded derivatives, super-calibration ability to the term structure and elegant tree-building procedure [36 - 39].
3.2. European bond option under the Hull-White model

In this section, we do our own calculations to derive the price formula for European option written on a bond under the Hull-White extended Vasicek model. We start by the zero-coupon bond which should be generalized to the coupon-bearing bond. Once done, we state an illustration with results which will lead us to a concluding chapter.

3.2.1. European zero-coupon bond option

As introduced previously, we can price interest rate derivatives either through bond price model directly or interest rate model. Here, we change the turn; we put a link between model of interest rates and model of bond prices. We then subsequently apply the risk neutral valuation formula to bond price models and make some calculations. Our derivation can be considered as a special case of [1], [11] and [49]. Let us reconsider the dynamics of bond price under the Hull-White model as given in equation (2.51) by

\[ dp(t, T) = p(t, T)[r(t)dt + b(t, T)dW_t], \]

where

\[ b(t, T) = -\frac{\sigma}{a}(1 - e^{-a(T-t)}) \]

and where \( W_t \) is the standard Brownian motion with respect to the probability measure \( Q \). Now we introduce the forward price \( f^p(t, S_i, T) \) defined by

\[ f^p(t, S_i, T) = \frac{p(t, S)}{p(t, T)}, \]

with \( f^p(T, S, T) = p(T, S) \). By virtue of Ito formula introduced in the first chapter, the dynamics of \( f^p(t, S_i, T) \) noted by \( df^p(t, S_i, T) \) is given by

\[ df^p(t, S, T) = d \left( \frac{p(t, S)}{p(t, T)} \right) \]

\[ = f^p(t, S, T)((b(t, S) - b(t, T))(dW_t - b(t, T))). \]

By letting \( dW^p_t = dW_t - b(t, T) \), or equivalently

\[ W^p_t = W_t - \int_0^t b(s, T)ds, \]
where $W_t^P$ is, by Girsanov theorem, a standard Brownian motion with respect to a new probability measure $P$, we arrive to

$$df^P(t,S,T) = f^P(t,S,T) \varphi(s,S,T)dW^P_t,$$

which is a variant of geometric Brownian motion and where

$$\varphi(s,S,T) = -\frac{\sigma}{\alpha}(e^{-\alpha(T-t)} - e^{-\alpha(S-t)}).$$

From (3.7) and by considering the terminal condition $f^P(T,S,T)$, we get

$$f^P(T,S,T) = f^P(t,S,T)\left(\int_t^T \varphi(s,S,T)dW^P_s - \frac{1}{2}\int_t^T |\varphi(s,S,T)|^2 ds\right).$$

Writing shortly, we have

$$f^P(T,S,T) = f^P(t,S,T)\exp\left(\xi(t,T) - \frac{1}{2}v^2(t,T)\right),$$

where

$$\xi(t,T) = \int_t^T \varphi(s,S,T)dW^Q_s$$

and where

$$v^2(t,T) = \int_t^T |\varphi(s,S,T)|^2 ds$$

$$= \left(\frac{\sigma}{\alpha}\right)^2 \int_t^T \left[e^{-\alpha(T-t)} - e^{-\alpha(S-t)}\right]^2 ds$$

$$= \frac{\sigma^2}{2\alpha^3} \left(1 - e^{-\alpha(S-t)}\right)^2 \left(1 - e^{-2\alpha(T-t)}\right).$$

On the other hand, the boundary condition (3.5') becomes

$$V(S,r;T) = \max(p(S,T) - K, 0)$$

$$= \max(f^P(T,S,T) - K, 0) = (f^P(T,S,T) - K)1_D$$

Where $D = \{f^P(T,S,T) \geq K\}$ and where $1_D$ is the indicator function, that is,

$$1_D = \begin{cases} 
 1 & \text{if } f^P(T,S,T) \geq K \\
 0 & \text{otherwise}
\end{cases}$$

Before we continue with our calculations, let state a useful result which is an issue of the risk neutral valuation formula.
Proposition 3.2.1. Let \( p(t, T) \) be the price of a zero-coupon bond maturing at time \( T \). Then the solution to problem (3.1)-(3.2) is given by the formula

\[
V(t, r; T) = p(t, T)E_P(V(T, r; T) | \mathcal{F}_t).
\]

Where \( E_P(\cdot) \) is the expectation taken under the probability measure \( P \)

**Proof.** The proof is the straightforward application of the risk neutral valuation formula. See [43].

From the Proposition 3.2.1 above, we have

\[
V(t, r; T) = p(t, T)E_P(V(T, r; T)|\mathcal{F}_t)
= p(t, T)E_P\left((f^P(t, S, T) - K)1_D|\mathcal{F}_t\right)
= p(t, T)E_P(f^P(t, S, T)1_D|\mathcal{F}_t) - p(t, T)K P(D|\mathcal{F}_t).
\]

To make calculations easier let

\[
V_1(t, r; T) = p(t, T)E_P(f^P(t, S, T)1_D|\mathcal{F}_t),
\]

and

\[
V_2(t, r; T) = p(t, T)K P(D|\mathcal{F}_t).
\]

Then the function price \( V(t, r; T) \) can be written as follows

\[
V(t, r; T) = V_1(t, r; T) - V_2(t, r; T).
\]

Now, we first find the expression for \( V_1(t, r; T) \). From the property of Ito integrals proved in Proposition 1.2.4, it is declared that \( \xi(t, T) \) is, under the probability measure \( P \), a Gaussian process, independent of the \( \sigma - algebra \) \( \mathcal{F}_t \) with expected value 0 and variance \( Var[\xi(t, T)] = v^2(t, T) \). Using the properties of conditional expectation (See for e.g. [7]), we obtain the following results:

\[
Q(D|\mathcal{F}_t) = P(\{ f^P(t, S, T) \geq K \}|\mathcal{F}_t)
= P\left(\left\{ f^P(t, S, T) \exp\left(\xi(t, T) - \frac{1}{2}v^2(t, T)\right) \geq K \right\} |\mathcal{F}_t\right)
= P\left(\xi(t, T) - \frac{1}{2}v^2(t, T) \geq -\ln\frac{f^P(t, S, T)}{K}\right).
\]
Consequently, we find the expression of $V_2(t, r; T)$ which is

$$V_2(t, r; T) = K p(t, T) N \left( \frac{\ln \left( \frac{f^P(t, s, T)}{K} \right) - \frac{1}{2} v^2(t, T)}{v(t, T)} \right),$$

where $N(\cdot)$ is the standard normal distribution. To evaluate $V_1(t, r; T)$, we introduce an auxiliary probability measure $\hat{P}$ equivalent to $P$ on $(\Omega, \mathcal{F}_T)$ by setting

$$\frac{d\hat{P}}{dp} = \exp \left( \int_0^T \varphi(s, s, T) dW^p_s - \frac{1}{2} \int_0^T |\varphi(s, s, T)|^2 ds \right) = \tilde{L}_T.$$ 

Here, $\tilde{L}_T$ represents the Likelihood process. By Girsanov Theorem 1.4.11, the process $\hat{W}^\xi_t$, given by

$$\hat{W}^\xi_t = W^p_t - \int_0^t \varphi(s, s, T) ds,$$

is a standard Brownian motion under the probability measure $\hat{P}$. It follows that, under the probability measure $\hat{P}$, the forward price process $f^P(T, s, T)$ is

$$f^P(T, s, T) = f^P(t, s, T) \exp \left( \int_t^T \varphi(s, s, T) d\hat{W}^\xi_s + \frac{1}{2} \int_t^T |\varphi(s, s, T)|^2 ds \right).$$

We can then write the equation (3.23a) shortly as

$$f^P(T, s, T) = f^P(t, s, T) \varphi(t, s, T) \exp \left( \xi(t, T) + \frac{1}{2} v^2(t, T) \right),$$

where

$$\xi(t, T) = \int_t^T \varphi(s, s, T) d\hat{W}^\xi_s.$$  

The process $\xi(t, T)$ is a Gaussian under the probability measure $\hat{P}$ with expected value 0 and variance $v^2(t, T)$, it is also independent $\sigma$-algebra $\mathcal{F}_t$. So, we have

$$V_1(t, r; T) = p(t, S) E_{\hat{P}} \left( f^P(T, s, T) 1_\mathcal{F}_t \right)$$

$$= p(t, S) E_{\hat{P}} \left\{ 1_\mathcal{F}_t \left( \exp \left( \int_t^T \varphi(s, s, T) d\hat{W}^\xi_s + \frac{1}{2} \int_t^T |\varphi(s, s, T)|^2 ds \right) \right) \right\},$$

10 Recall that sometimes we write $W_t$ instead of $W_t^Q$ denoting the standard Brownian motion under the measure $Q$. 
and thus
\[
V_1(t,r;T) = p(t,S)E_\bar{\rho} \left[ 1_D \left( \frac{\bar{L}_T}{\bar{L}_t} \bigg| F_t \right) \right] = p(t,S)\bar{P}(D|F_t) = p(t,S)\bar{P} \left( \frac{\xi(t,T)}{v(t,T)} \leq \ln \left( \frac{f^P(t,S,T)}{K} \right) + \frac{1}{2} v^2(t,T) \right).
\]

Note that the second equality above is due to the Bayes Rule which stipulates that
\[
E_\bar{\rho}(X|F_t) = \frac{E_\rho(L_T X|F_t)}{E_\rho(L_T|F_t)} = E_\rho \left( \frac{L_T}{L_t} X \bigg| F_t \right); \quad \forall t \in [0,T],
\]
where \(X\) is any stochastic process and \(L_T\) is the Likelihood process.

Finally, we find the expression for \(V_1(t,r;T)\) as
\[
V_1(t,r;T) = p(t,S)N \left( \frac{\ln \left( \frac{f^P(t,S,T)}{K} \right) + \frac{1}{2} v^2(t,T)}{v(t,T)} \right).
\]

Therefore, the price of an arbitrage free bond market at time \(t \in [0,T]\) of an European call option with expiry date \(T\) and exercise price \(K\) written on a zero-coupon bond maturing at time \(S \geq T\) equals to
\[
V(t,r;T) = p(t,S)N(d_1) - Kp(t,T)N(d_2), \quad (3.23'')
\]
where
\[
d_{1,2}(t,S,T) = \frac{\ln \left( \frac{f^P(t,S,T)}{K} \right) \pm \sigma^2 (1 - e^{-\alpha(S-t)})^2 (1 - e^{-2\alpha(T-t)})}{2\sigma a\sqrt{2\alpha(1 - e^{-\alpha(S-t)})\sqrt{(1 - e^{-2\alpha(T-t)})}}} \quad (3.23''')
\]
The formula that gives the price of the put option can be established along the same lines, it is given by the following formula:
\[
V(t,r;T) = Kp(t,T)N(-d_2) - p(t,S)N(-d_1),
\]
where \(d_{1,2}\) are defined in (3.23'''). This terminates our derivation procedure.

3.2.2. European coupon-Bearing bond option

This subsection is just a general case of the previous one; here instead of using zero-coupon bond, we consider a coupon-bearing bond option. So, let us consider the boundary condition (3.5'). Then, it follows that
\[ V(T, r, T) = \Phi_T = \max(p_c(T, S_i) - K, 0) \]
\[ = \max\left( \sum_{i=1}^{d} c_i \ p(T, S_i) - K, 0 \right) \]
\[ = \left[ \sum_{i=1}^{d} c_i \ p(T, S_i) - K \right] 1_D - K 1_D, \tag{3.24} \]
where \( D \) is the exercise set of the option and is given by
\[ D = \left\{ \sum_{i=1}^{d} c_i \ p(T, S_i) > K \right\}. \]
In term of forward price process, we have
\[ p(T, S_i) = f^P(T, S_i, T). \]
So, the equation (3.24) becomes
\[ V(T, r, T) = \left[ \sum_{i=1}^{d} c_i \ f^P(T, S_i, T) - K \right] 1_D - K 1_D. \]
Here, \( D \) has the same meaning as above
\[ D = \left\{ \sum_{i=1}^{d} c_i \ f^P(T, S_i, T) > K \right\}. \]
From the Proposition 3.1.1, we have
\[ V(t, r, T) = p(t, T) E_Q (\Phi_T | F_t) \]
\[ = p(t, T) E_Q \left( \left[ \sum_{i=1}^{d} c_i \ f^P(T, S_i, T) - K, 0 \right] 1_D - K 1_D \right| F_t \]
\[ = p(t, T) \left( \sum_{i=1}^{d} c_i E_Q \left( f^P(T, S_i, T) \right) 1_D \right| F_t \) - Kp(t, T) P(D | F_t). \]
As done previously, let define functions \( V_1(t, r, T) \) and \( V_2(t, r, T) \) as
\[ V_1(t, r, T) = p(t, T) \left( \sum_{i=1}^{d} c_i E_p \left( f^P(T, T_i, T) \right) 1_D \right| F_t \), \tag{3.25} \]
and
\[ V_2(T, r, T) = Kp(t, T) P(D | F_t). \tag{3.26} \]
The function price \( V(t, r, T) \) is then given by
\[ V(t, r, T) = V_1(t, r, T) - V_2(t, r, T) \]
As before, we know that the dynamics of the process \( f^P(t, T_i, T) \) is given by
\[
f^P(T, S_t, T) = f^P(t, S_t, T) \left( \int_t^T \varphi(s, S_t, T) dW^P_s - \frac{1}{2} \int_t^T |\varphi(s, S_t, T)|^2 ds \right). \quad (3.27)
\]

Here

\[
\varphi(s, S_t, T) = b(s, S_t) - b(s, T) = -\frac{\sigma}{\alpha} \left( e^{-\alpha(T-t)} - e^{-\alpha(S_t-t)} \right).
\]

And if we define

\[
\xi_i(t, T) = \int_t^T \varphi(s, S_t, T) dW^P_s,
\]

then by the Ito integral properties, we have

\[
E[\xi_i(t, T)] = 0,
\]

and

\[
Var[\xi_i(t, T)] = v_{ii} = \int_t^T |\varphi(s, S_t, T)|^2 ds = \left( \frac{\sigma}{\alpha} \right)^2 \int_t^T e^{-\alpha(S_t-t)} - e^{-\alpha(T-t)} \right|^2 ds.
\]

So, the reduced form for equation (3.27) is given by

\[
f^P(T, S_t, T) = f^P(t, S_t, T) \exp \left( \xi_i - \frac{1}{2} v_{ii} \right).
\]

The process \( \xi_i(t, T) \) is, under the probability measure \( P \), a Gaussian process, independent of the \( \sigma \)-field \( \mathcal{F}_t \) with expected value zero and variance \( v_{ii} \) and the Gaussian law is given by \( N(0, v_{ii}) \). Hence

\[
P(D|\mathcal{F}_t) = P \left( \sum_{i=1}^d c_i f^P(T, S_t, T) > K \right)
\]

\[
= P \left( \sum_{i=1}^d c_i p(t, S_t) \exp \left( \xi_i - \frac{1}{2} v_{ii} \right) > K > Kp(t, T) \right) ; 0 < p(t, T) < 1,
\]

\[
= P \left( \sum_{i=1}^d c_i p(t, S_t) \exp \left( \xi_i - \frac{1}{2} v_{ii} \right) > Kp(t, T) \right).
\]

Therefore, we should express \( V_2(t, r, T) \) as follows:

\[
V_2(t, r, T) = Kp(t, T) P(D|\mathcal{F}_t) = Kp(t, T) J_2,
\]

where \( J_2 = J_2(p(t, S_t), p(t, T)) \), \( i = 1, \ldots, d \) is given by

\[
J_2 = P \left( \sum_{i=1}^d c_i p(t, S_t) \exp \left( \xi_i - \frac{1}{2} v_{ii} \right) > Kp(t, T) \right).
\]

To evaluate the expression (3.25), we proceed in the same way as in the previous section. So we introduce an auxiliary probability measure \( P_i \) by setting
\[
\frac{dP_i}{dP} = \exp\left( \int_t^T \varphi(s, S_i, T)dW_s^P - \frac{1}{2} \int_t^T |\varphi(s, S_i, T)|^2 ds \right) \\
+ \int_t^T \varphi(s, S_i, T) . \varphi(s, S_j, T) ds = L_i,
\]

where \(i, j = 1, ..., d; i \neq j\). Here \(L_t\) can be considered as the likelihood process in \(d\)-dimension. Then, under probability measures \(P_i\), the forward price process \(f_p(T, S_i, T)\) is given by

\[
f_p(T, S_i, T) = \exp\left( \int_t^T \varphi(s, S_i, T)dW_s^{P_i} - \frac{1}{2} \int_t^T |\varphi(s, S_i, T)|^2 ds \right) \\
+ \int_t^T \varphi(s, S_i, T) . \varphi(s, S_j, T) ds.
\]

We can write the expression above as

\[
f_p(T, S_i, T) = f_p(t, S_i, T)\exp\left( \xi_i - \frac{1}{2} v_{ii} + v_{ij} \right),
\]

where

\[
\xi_i(t, T) = \int_t^T \varphi(s, S_i, T)dW_s^{P_i}, \quad v_{ii} = \int_t^T |\varphi(s, S_i, T)|^2 ds
\]

and

\[
v_{ij} = \int_t^T \varphi(s, S_i, T) . \varphi(s, S_j, T) ds = \int_t^T [b(t, S_i) - b(t, T)] [b(t, S_j) - b(t, T)] ds,
\]

The random variable \((\xi_1, \xi_2, ..., \xi_d)\) is independent of \(\mathcal{F}_t\), with Gaussian law under \(P_i\). In addition, \(E_{P_i} [\xi_i] = 0\), \(Var [\xi_i] = v_{ii}\) and \(Cov [\xi_i, \xi_j] = v_{ij}, i \neq j\). Then, we get the following result:

\[
p(t, T)E_P \left( \left( f_p(T, S_i, T) \right) 1_D | \mathcal{F}_t \right) = p(t, S_i) P_i(D | \mathcal{F}_t),
\]

where

\[
P_i(D | \mathcal{F}_t) = P_i \left( \sum_{j=1}^d c_j p(t, S_j) \exp\left( \xi_i - \frac{1}{2} v_{ii} + v_{ij} \right) > Kp(t, T) \right).
\]

Finally, we have

\[
V_1(t, r, T) = \sum_{j=1}^d c_i p(t, S_i) f_j^1(t).
\]

where
\[ J_i^1 = P_i \left( \sum_{j=1}^{d} c_j p(t, S_j) \exp \left( \xi_i - \frac{1}{2} v_{ii} + v_{ij} \right) > Kp(t, T) \right). \]

Therefore, the arbitrage of a European call option on a coupon-bearing bond is given by the following result:

\[ V(t, r, T) = V_1(t, r, T) - V_2(t, r, T) = \sum_{j=1}^{d} c_i p(t, S_i) J_1^i - Kp(t, T) J_2, \quad (3.29) \]

where processes \( J_1^i \) and \( J_2 \) are given by

\[ J_1^i = P_i \left( \sum_{j=1}^{d} c_j p(t, S_j) \exp \left( \xi_i - \frac{1}{2} v_{ii} + v_{ij} \right) > Kp(t, T) \right), \quad (3.30) \]

\[ J_2 = P \left( \sum_{i=1}^{d} c_i p(t, S_i) \exp \left( \xi_i - \frac{1}{2} v_{ii} \right) > Kp(t, T) \right), \quad (3.31) \]

and where \( \xi_i, i = 1, ..., d \) are random variables whose distribution under \( P_i \) is Gaussian, with zero expected value and covariance \( v_{ij} \).

As a remark, the price \( V(t, r; T) \) resembles the Black-Scholes formula [10] (or see Example 1.4.7) and has the same interpretation.

### 3.2.3. Illustration and results

Let us consider the flat, upward, downward and humped yield curves as defined in (2.2'). Let us consider again parameters of the Hull–White model given by Speed reversion \( a = 0.1 \) and Volatility term \( \sigma = 0.1 \) with strike price given by 0.8. The price of a two year maturity bond in one year's time call option for these four yield curves are given with the help of the following calculations:

**Calculations**

For \( a = 0.1, \sigma = 0.1, K = 0.8, t = 0, T = 1 \) and \( S = 2 \) we have

\[ R_f(0,1) = 0.03, \]

\[ p(0,1) = e^{-(1-0)\times 0.03} = 0.970445, \]

\[ p(0,2) = e^{-(2-0)\times 0.03} = 0.941764, \]

\[ f^P(0,1,2) = \frac{p(0,2)}{p(0,1)} = 0.970445 \]

Let us find now the value of \( v(t, T) \) and \( d_{1,2}(t, S, T) \)
The remaining prices for the yield curves \( R_u(t,T) \), \( R_d(t,T) \) and \( R_h(t,T) \) are computed in the similar way.\(^{11}\)

### Table 3.1. Results of European bond option for these four different yield curves.

<table>
<thead>
<tr>
<th>Yield curves</th>
<th>( R_p(t,T) )</th>
<th>( R_u(t,T) )</th>
<th>( R_d(t,T) )</th>
<th>( R_h(t,T) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prices (call option)</td>
<td>0.175131</td>
<td>0.172118</td>
<td>0.178177</td>
<td>0.167961</td>
</tr>
<tr>
<td>Prices (put option)</td>
<td>0.009723</td>
<td>0.010018</td>
<td>0.009434</td>
<td>0.010445</td>
</tr>
</tbody>
</table>

\(^{11}\) We provide Matlab codes in Appendix A.1 for a fast computation.
Figure 3.2. Prices of European call/put option for $a = 0.1, \sigma = 0.1$

In the figure above, we have four values of the yield curves computed by using the formula (2.2'):

$$R_f(0,1) = 0.03, \quad R_u(0,1) = 0.033, \quad R_d(0,1) = 0.027 \text{ and } R_h(t, T) = 0.0372.$$  

Note that these results are also valid for the Vasicek model. Moreover, we observe that for the same fixed parameters for both Hull-White model and Vasicek model, the downward yield curve has the highest price and the humped yield curve has the lowest price for a European call bond option. In contrast, for a European put option, the humped yield curve has the highest price and the downward yield curve has the lowest price.
3.3. Conclusion

In this chapter, we have derived a simple closed-form expression for pricing European bond option, specifically the European option written on zero-coupon and coupon-bearing bond option under the Hull-White extended Vasicek model which can be considered as a special case of [1], [11] and [49]. By introducing the forward price process, we have found the price in terms of forward price with the help of the risk neutral valuation formula which is very useful and practical. Furthermore, we have observed that the price under the Hull-White extended Vasicek model is equivalent to the Jamshidian formula [43], which is a formula for pricing European bond option under the Vasicek model; this is due to the same bond price process volatility of both models obtained in Chapter 2. We further noted that the price for both models resembles the Black-Scholes formula [10] and, consequently, has the same interpretation.

As a European option can be exercised only on the maturity date $T$, let us examine the case of early exercise opportunity in the forthcoming chapter.
Chapter 4:

Pricing American bond options under the Hull-White extended Vasicek model

In the previous chapter, we have examined the European bond option where the option is exercised only on the expiration date $T$. In this chapter, we develop the case where the option is exercised prematurely commonly called American option which is written on the zero-coupon and coupon bearing bonds under the Hull-White extended Vasicek model. We then may write the pricing boundary value problem for American bond option by a slight modification of the problem (3.3) as:

$$\begin{cases}
\frac{\partial V}{\partial t} + (\theta(t) - a(t)r) \frac{\partial V}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = 0 \\
V(t_{st}, r; T) = \Phi(r, t_{st}) = (p(S, t_{st}) - K)^+,
\end{cases} \tag{4.1}$$

where $t_{st} \in [t, T]$ is the exercised time for the American option and where $r$ is the solution to the Hull-White model defined by

$$dr_t = (\theta(t) - a(t)r)dt + \sigma(t)dW_t. \tag{4.1'}$$

We observe that if $t_{st} = T$, we find the European option discussed in the third chapter. Let us first make a brief discussion about the analytical representation of the American bond option.

4.1. Analytic representation of American bond option

In this section, we consider two pricing formulations, namely, the linear complementarity formulation and the optimal stopping problem. First, we develop the variational inequalities that satisfied by the American bond option
price from which we derive the linear complementarity formulation. Alternatively, the American bond option price can be seen as the supremum of the expectation of the discounted exercise payoff among the possible stopping times. To the end of the section, we state analytic representations of American bond option. Here we follow [47] and [48].

4.1.1. Formulation

We know that the value of an American option cannot be smaller than the value of European option, so that we may write

\[ V(t_{st}, r; T) \geq V(T, r; T). \]

It is followed from the boundary condition of the problem (4.1) that

\[ V(t, r; T) \geq (p(S, T) - K)^+. \]  \hspace{1cm} (4.2)

The condition (4.2) above means that there exists a function \( p_f(t) \), with \( 0 < p_f(t) < K \), for which the payoff is reached so that the valuation of an American option can be formulated as the free boundary value problem, where the free boundary is the optimal exercise boundary which separates the continuation and stopping regions. The function \( p_f(t) \) is characterized by

\[
\begin{cases}
V(t, r; T) = p - K & \text{for } p \geq p_f(t) \\
V(t, r; T) > (p - K)^+ & \text{for } p < p_f(t).
\end{cases}
\]  \hspace{1cm} (4.3)

When we are in the stopping regions, we may write:

\[ V(t, r; T) = p - K \quad \text{for } p \geq p_f(t). \]

As the exercise payoff function, \( V(t, r; T) = p - K \) does not satisfy the Hull-White TSE, it follows that (see for e.g. [42]):

\[
\frac{\partial V}{\partial t} + (\theta(t) - a(t)r) \frac{\partial V}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} - rV > 0 \quad \text{for } p \geq p_f(t). \]  \hspace{1cm} (4.4)

Relations (4.3) and (4.4) lead us to

\[
\frac{\partial V}{\partial t} + (\theta(t) - a(t)r) \frac{\partial V}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} - rV \geq 0, \quad \text{for } p > 0 \text{ and } t > 0. \]  \hspace{1cm} (4.5)

In the continuation region, we have \( V(t, r; T) > (p - K)^+ \) for \( p < p_f(t) \), and so, the first inequality in (4.5) changes into the equality. Relations (4.3) and (4.5) constitute the linear complementarity formulation. The pricing of an American option can be also formulated as an optimal stopping problem.
Let \( t_{st} \) be a stopping time as defined in Definition 1.4.3 and \( \bar{t} \in [0, t_{st}] \). The price of an American put option is given by \(^{12}\)

\[
E^Q \left[ \exp \left( - \int_{\bar{t}}^{t_{st}} r_v \, dv \right) \max \left( K - p_f(t_{st}), 0 \right) \right],
\]

where \( E^Q \) denotes the expectation taken under the risk neutral probability measure \( Q \) and where \( r_v \) satisfies the Hull-White model (4.2). Following arguments given in [48, Chap. 5], we may write

\[
V(r, \bar{t}, T) = \Phi(r, \bar{t}) = \sup_{t \leq t_{st} \leq T} E^Q \left[ \exp \left( - \int_{\bar{t}}^{t_{st}} r_v \, dv \right) \max \left( K - p_f(t_{st}), 0 \right) \right] \quad (4.6)
\]

where the supremum is taken over all possible stopping times. The optimal stopping time is then given by

\[
t_{ost} = \inf \left\{ u \in [t_{st}, T] \mid \max \left( K - p_f(t_{st}) \right) \right\}.
\] (4.7)

To verify that the solution to the linear complementarity formulation gives the American put value (4.6), where the optimal stopping time is determined by (4.7), we refer to [66, Chapter 2]. By Ito formula applied to the formula (4.6), we obtain

\[
\exp \left( - \int_{\bar{t}}^{t_{st}} r_v \, dv \right) V(r, t_{st}, T)
\]

\[
= V(r, \bar{t}, T) \int_{\bar{t}}^{t_{st}} \left\{ \int_{\bar{t}}^{t_{st}} r_\xi \, d\xi \left[ \frac{\partial}{\partial t} + (\mu - \lambda \sigma) \frac{\partial}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial r^2} - r \right] V(r, \bar{t}, T) \right\} \, dv
\]

\[
= \int_{\bar{t}}^{t_{st}} \int_{\bar{t}}^{t_{st}} r_\xi \, d\xi \left[ \sigma \frac{\partial V(r, \bar{t}, T)}{\partial r} \right] dW_r.
\]

Similarly to Proposition 1.2.2 first assertion,

\[
E^Q \left[ \int_{\bar{t}}^{t_{st}} \left\{ \int_{\bar{t}}^{t_{st}} r_\xi \, d\xi \left[ \sigma \frac{\partial V(r, \bar{t}, T)}{\partial r} \right] \right\} dW_r \right] = 0,
\]

it follows that

\[
V(r, \bar{t}, T) \geq E^Q \left[ \exp \left( - \int_{\bar{t}}^{t_{st}} r_v \, dv \right) V(r, t_{st}, T) \right].
\]

\(^{12}\) In this section we talk about put option since it is more understandable. However the idea behind both put and call option remains the same.
Since the above result is valid for any stopping time and

$$V(r, t_{st}, T) \geq \max(K - p_f(t_{st}), 0),$$

it follows that

$$V(r, \tilde{t}, T) = \Phi(r, \tilde{t}) \geq \sup_{t_{st} \leq T} E^Q \left[ \exp \left( - \int_{\tilde{t}}^{t_{st}} r_v dv \right) \max(K - p_f(t_{st}), 0) \right].$$

This can be extended to the optimal stopping time and by the virtue of [46], we may find that

$$V(r, \bar{t}, T) = \Phi(r, \bar{t}) = \sup_{t_{ost} \leq T} E^Q \left[ \exp \left( - \int_{t_{ost}}^{t_{ost}} r_v dv \right) \max(K - p_f(t_{ost}), 0) \right].$$

This proves that the solution to the optimal stopping formulation satisfies the linear complementarity problem.

### 4.1.2. Analytic representations of the American bond option.

The American put price is given by (See for e.g. [43, 44])

$$V(r, \tilde{t}, T) = E^Q \left[ \exp \left( - \int_{\tilde{t}}^{t_{st}} r_v dv \right) \max(K - p(r, \tilde{t}, S), 0) \right]$$

$$+ E^Q \left( T \int_{\tilde{t}}^{T} \exp \left( - \int_{\tilde{t}}^{u} r_v dv \right) r_u K_1^{(r_u \geq r_u^*)} du \right) r_t,$$

where $r_u^*$ is a certain adapted stochastic process representing the critical level of the short interest rate and where $S$ represents the maturity date of the bond. In the equation above, the term represents the usual European bond put option while the second term represents the early exercise premium. Equivalently, the American call bond option is given by

$$V(r, \bar{t}, T) = E^Q \left[ \exp \left( - \int_{\bar{t}}^{t_{ost}} r_v dv \right) \max(p(r, \bar{t}, S) - K, 0) \right]$$

$$+ E^Q \left( T \int_{\bar{t}}^{T} \exp \left( - \int_{\bar{t}}^{u} r_v dv \right) r_u K_1^{(r_u \geq r_u^*)} du \right) r_t.$$

We observe that the formulas above are very difficult to handle due to the complexity of the early exercise premium. Therefore, we opt for one strategy; we
transform the Hull-White term structure problem into the diffusion problem and apply the finite difference methods for numerical solutions.

4.2. From the Hull-White TSE to the diffusion equation

In this section, we make transformations of the parabolic PDE that we have called Hull-White term structure equation (TSE) in the Problem (4.1a) until we get the simplest diffusion or heat equation. As in the Hull-White model the drift term $a(t)$ and the volatility term $\sigma(t)$ are given or determined statistically, then the main purposes of these transformations are to eliminate the time-independent drift $\theta(t)$ which is unknown; and to deal with only the obtained diffusion equation for pricing simply because it presents an easy algebraic and algorithmic derivation. Another special feature is under these transformations; the terminal condition given initially as the payoff function is changed into the initial condition which makes computations easier. The main idea of these transformations comes from [34] and [63]. Let us rewrite the Hull-White term structure equation from problem (4.1) as

$$\frac{\partial V}{\partial t} + (\theta(t) - a(t)r)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} - rV = 0 \quad (4.8)$$

where $\theta(t)$ is the time-dependent drift to the process $r$, $a(t)$ is the speed reversion, $\sigma(t)$ is the volatility term of the process $r$, and $W_t$ is a standard Brownian motion with respect to a certain probability measure $Q$.

We note that throughout the following sections, we will consider only the case where parameters $a(t) = a$ and $\sigma(t) = \sigma$ are constants as mentioned in our abstract.

4.2.1. Elimination of the time independent drift in the Hull-White TSE

Let us eliminate the unknown function $\theta(t)$ in the governing equation (4.8). In order to do so, we define a deterministic variable $\alpha$ such that

$$d\alpha(t) = (\theta(t) - a(t)\alpha(t))dt.$$

We then define a new variable $x(t)$ given by

$$x(t) = r(t) - \alpha(t).$$

It follows that
\[ dx(t) = dr(t) - d\alpha(t) \]
\[ = (\theta(t) - a(t)r)dt + \sigma(t)dW_t - (\theta(t) - a(t)\alpha(t))dt \]
\[ = -a(t)[r(t) - \alpha(t)]dt + \sigma(t)dW_t \]
\[ = -a(t)x(t)dt + \sigma(t)dW_t. \]  
(4.9)

where \( a(t) \) and \( \sigma(t) \) are deterministic functions defined in the Hull-White model (4.1b). We may adapt the governing PDE (4.8) to the stochastic process (4.9) by replacing respectively the expression \( \theta(t) - a(t)r \) and the short interest rate process \( r(t) \) of the governing equation (4.8) by expressions \(-a(t)x(t)\) and \(x(t) + \alpha(t)\). We then get the following term structure equation under the auxiliary variable:

\[ \frac{\partial V}{\partial t} - a(t)x \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2(t) \frac{\partial^2 V}{\partial x^2} - (x + \alpha(t))V = 0. \]  
(4.10)

Now we observe that \( \theta(t) \) disappears because it is absorbed by the deterministic stochastic process \( \alpha(t) \). For the case where the parameters \( a(t) = a \) and \( \sigma(t) = \sigma \) are strictly positive constant, we thus rewrite the governing equation (4.10) above as

\[ \frac{\partial V}{\partial t} - ax \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} - (x + \alpha(t))V = 0. \]  
(4.11)

### 4.2.2. Further transformations to the diffusion equation

We start this part of transformations by reverting the time by \( t^* = T - t \) in governing equation (4.11), we get:

\[ \frac{\partial \tilde{V}}{\partial t^*} - ax \frac{\partial \tilde{V}}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \tilde{V}}{\partial x^2} - x\tilde{V} = 0, \]  
(4.12)

with \( \tilde{V} = \tilde{V}(t^*, x) \).

Let us define the function \( \gamma(t^*) \) as

\[ \gamma(t^*) = \frac{1}{a}(\psi(t^*) - 1), \]

where \( \psi(t^*) \) is an exponential function defined by

\[ \psi(t^*) = e^{-at^*}. \]

The derivative of the function \( \gamma(t^*) \) with respect to \( t^* \) is then given by

\[ \frac{d\gamma(t^*)}{dt^*} = -\psi(t^*) = -[a\gamma(t^*) + 1]. \]

Assume that the function \( \tilde{V} \) can be written under the form:
\[ \tilde{V} = e^{xy(t^*)} \tilde{V}_1 \]

with their partial differential operators given by

\[
\begin{align*}
\frac{\partial \tilde{V}}{\partial t^*} &= x \frac{d\gamma(t^*)}{dt^*} \tilde{V}_1 + e^{xy(t^*)} \frac{\partial \tilde{V}_1}{\partial t^*} = \left[ -x(\alpha \gamma(t^*) + 1) \tilde{V}_1 + \frac{\partial \tilde{V}_1}{\partial t^*} \right] e^{xy(t^*)}, \\
\frac{\partial \tilde{V}}{\partial x} &= \gamma(t^*) \tilde{V}_1 + \frac{\partial \tilde{V}_1}{\partial x} e^{xy(t^*)}, \\
\frac{\partial^2 \tilde{V}}{\partial x^2} &= \gamma^2(t^*) \tilde{V}_1 + 2\gamma(t^*) \frac{\partial \tilde{V}_1}{\partial x} + \frac{\partial^2 \tilde{V}_1}{\partial x^2} e^{xy(t^*)},
\end{align*}
\]

plugged into the governing equation (4.12), lead us to

\[
\frac{\partial \tilde{V}_1}{\partial t^*} = -(ax - \sigma^2 \gamma(t^*)) \frac{\partial \tilde{V}_1}{\partial x} + \sigma^2 \frac{\partial^2 \tilde{V}_1}{\partial x^2} + \left( \frac{\sigma^2 \gamma^2(t^*)}{2} \right) \tilde{V}_1. \tag{4.13}
\]

Let us once again

\[ x_1 = x \psi(t^*), \]

then partial differential operators

\[
\begin{align*}
\frac{\partial \tilde{V}_1}{\partial x} &= \frac{\partial \tilde{V}_1}{\partial x_1} \frac{dx_1}{dx} = \psi(t^*) \frac{\partial \tilde{V}_1}{\partial x_1}, \\
\frac{\partial^2 \tilde{V}_1}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial \tilde{V}_1}{\partial x_1} \right) = \psi(t^*) \frac{\partial^2 \tilde{V}_1}{\partial x_1 \partial x_1} = \psi^2(t^*) \frac{\partial^2 \tilde{V}_1}{\partial x_1^2},
\end{align*}
\]

plugged into the equation (4.13) lead us to

\[
\frac{\partial \tilde{V}_1}{\partial t^*} = \frac{\sigma^2}{2} \psi^2(t^*) \frac{\partial^2 \tilde{V}_1}{\partial x_1^2} + \sigma^2 \gamma(t^*) \psi(t^*) \frac{\partial \tilde{V}_1}{\partial x_1} + \frac{\sigma^2 \gamma^2(t^*)}{2} \tilde{V}_1. \tag{4.14}
\]

Once more, assume that the function \( \tilde{V}_1 \) get the following form

\[ \tilde{V}_1 = e^{\beta(t^*)} \tilde{V}_2, \]

where the function \( \beta(t^*) \) is defined by

\[ \beta(t^*) = \frac{\sigma^2}{4 \alpha^3} \left[ 1 - 2 \ln(1 + \alpha \gamma(t^*)) - (1 - \alpha \gamma(t^*))^2 \right], \]

with its derivative with respect to \( t^* \) given by
then the partial differential operators of the function $\tilde{V}_1$ are given by

\[
\begin{align*}
\frac{\partial \tilde{V}_1}{\partial t^*} &= \frac{d\beta(t^*)}{dt^*}e^{\beta(t^*)}\tilde{V}_2 + e^{\beta(t^*)}\frac{\partial \tilde{V}_2}{\partial t^*} = \frac{\sigma^2}{2}y^2(t^*)e^{\beta(t^*)}\tilde{V}_2 + e^{\beta(t^*)}\frac{\partial \tilde{V}_2}{\partial t^*}, \\
\frac{\partial \tilde{V}_1}{\partial x_1'} &= e^{\beta(t^*)}\frac{\partial \tilde{V}_2}{\partial x_1'}, \\
\frac{\partial^2 \tilde{V}_1}{\partial x_1'^2} &= e^{\beta(t^*)}\frac{\partial \tilde{V}_2}{\partial x_1'^2}.
\end{align*}
\]

Plugging into the equation (4.14), we get what follows

\[
\frac{\partial \tilde{V}_2}{\partial t^*} = \frac{\sigma^2}{2}y^2(t^*)\frac{\partial^2 \tilde{V}_2}{\partial x_1'^2} + \sigma^2y(t^*)\psi(t^*)\frac{\partial \tilde{V}_2}{\partial x_1'}.
\] (4.15)

Let us introduce two other variables; the variable $\tau$ which is a function of $t^*$ (i.e. $\tau = \varphi(t^*)$); when $t = 0, t^* = T$ thus $\tau = \varphi(T)$ and defined as

\[
\tau = \varphi(t^*) = \frac{\sigma^2}{4\alpha}(1 - \psi^2(t^*))
\]

with

\[
\frac{d\tau}{dt^*} = \frac{\sigma^2}{2}\psi^2(t^*)
\]

and the variable

\[
z = x\psi(t^*) - \frac{\sigma^2}{4\alpha}(1 - \psi(t^*))^2.
\]

Setting the function $\tilde{V}_2 = u(z, \tau)$, then the governing equation (4.15) is further reduced to the so called diffusion or heat equation and it is finally given by:

\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2}.
\] (4.16)

with the overall changes summarized as follows:

\[
\begin{align*}
\tau &= \varphi(t^*) = \frac{\sigma^2}{4\alpha}(1 - \psi^2(t^*)) \quad (4.17') \\
z &= x\psi(t^*) - \frac{\sigma^2}{4\alpha}(1 - \psi(t^*))^2. \quad (4.17'') \\
u(z, \tau) &= \varphi(x, t)exp(-xy(t^*) - \beta(t^*)), \quad (4.17''')
\end{align*}
\]

where
4.2.3. A new formulation of a boundary value problem

Let us reconsider a boundary value condition (3.5') given by

\[
\sum_{i=1}^{d} c_i p(T, S_i) - K, 0
\]

As discussed in [48] or [49], a European option can have a value smaller than the payoff but it cannot happen with American options. Thus, the boundary condition (4.18) under the American bond option is then given by the following inequality

\[
u(z, \tau) \geq \exp(-xy(t^*) - \beta(t^*)) \max \left( \sum_{i=1}^{d} c_i p(T, S_i) - K, 0 \right)
\]

This can be written as

\[
u(z, \tau) \geq g(t^*, x)
\]

where

\[
g(t^*, x) = \exp(-xy(t^*) - \beta(t^*)) \max \left( \sum_{i=1}^{d} c_i p(T, S_i) - K, 0 \right),
\]

or equivalently

\[
g(t^*, z) = \exp \left( -\frac{4az + \sigma^2 (1 - \psi)^2}{4a\psi} \gamma - \beta \right) \max \left( \sum_{i=1}^{d} c_i p(T, S_i) - K, 0 \right).
\]

With the initial conditions given by

\[
\begin{align*}
\left\{ 
\begin{array}{l}
u(z, 0) = g(0, x) \\
p(x, t, S_i) \to 0, \quad \text{as} \quad x \to +\infty \\
V(x, t, S_i) \to 0, \quad \text{as} \quad x \to +\infty,
\end{array}
\right.
\end{align*}
\]

then we have

\[
\lim_{z \to +\infty} \nu(z, \tau) = \lim_{x \to +\infty} g(t^*, x) = 0.
\]

Therefore, the Hull-White term structure problem is reduced to the following boundary value problem:
where

\[
g(t^*,x) = \exp\left(-x\gamma(t^*) - \beta(t^*)\right) \max\left(\sum_{i=1}^{d} c_i p(T, S_i) - K, 0\right).
\]

**Remark 4.2.1.** The price process \( p(t, S_t) \) can be determined by two ways; either by using Proposition 2.3.2 or by expressing in terms of yield curve formula (2.2) which is an easy way. In our case, we consider the flat, upward, downward and humped yield curves as defined in (2, 2') and plotted in Figure 2.1.

The problem (4.18') above can be considered as the free boundary value problem of American option written on the bond under the Hull-White extended Vasicek model. Due to the complexity of the problem which is very hard to resolve analytically, we rely on numerical experiments in our last section. Our choice falls to finite difference methods because they are straightforward to implement and the resulting uniform rectangular grids are comfortable.

### 4.3. Solution of the obtained diffusion equation through FDMs

In the construction of finite difference schemes, we approximate the differential operators in the governing differential equation of the option model by appropriate finite difference operators, hence the name of this approach. In this section, we develop three cases of FDM which are explicit, implicit and Crank-Nicolson (CN) difference scheme and at the end; we derive another special case the Crank-Nicolson method over an unbounded domain.

#### 4.3.1. Explicit, Implicit and Crank-Nicolson scheme

For these all three types of schemes, we need first to transform the domain of the continuous problem \( \{(z, \tau) : -\infty < z < +\infty, \tau \geq 0\} \) into a discretized domain which must be approximated by a finite truncated interval \([-M, M]\) where to achieve a given level of accuracy requires \( M \) to be large enough.
We begin to build finite difference schemes by defining a grid of points in the $(\tau, z)$ plane. For any arbitrary integer $n$ and $m$, we denote $u(\tau_n, z_m)$ the value of $u$ at the grid point $(\tau_n, x_m)$ that can be shortly written as $u_{nm}^n$. The grid is then constructed for considering values of $u$ when the time is equal to

$$\tau_0, \tau_1, \ldots, \tau_n, \ldots, \tau_N = \tau_{max}$$

and when the variable $z$ is equal to

$$z_0, z_1, \ldots, z_m, \ldots, z_M = z_{max}.$$ 

Further considerations are given by

$$z_m = m\Delta z, \text{ for } m = 0, 1, \ldots, M$$

$$\tau_n = n\Delta\tau, \text{ for } n = 0, 1, \ldots, N$$

where $\Delta \tau$ and $\Delta z$ are called respectively the time step and step width and are given by the following formula:

$$\Delta \tau = \tau_{n+1} - \tau_n = \frac{\tau_N - \tau_0}{N}$$

and

$$\Delta z = z_{m+1} - z_m = \frac{z_M - z_0}{M}.$$ 

The general finite difference scheme is given by the following approximations (see for e.g. [15 or 53]):

$$\begin{align*}
\frac{\partial u}{\partial \tau} &\approx \frac{u_{m+1}^{n+1} - u_{m}^{n}}{\Delta \tau} \\
\frac{\partial u}{\partial z} &\approx \theta \frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta z} + (1 - \theta) \frac{u_{m+1}^{n} - u_{m-1}^{n}}{2\Delta z} \\
\frac{\partial^2 u}{\partial z^2} &\approx \theta \frac{u_{m+1}^{n+1} - 2u_{m}^{n+1} + u_{m-1}^{n+1}}{\Delta z^2} + (1 - \theta) \frac{u_{m+1}^{n} - 2u_{m}^{n} + u_{m-1}^{n}}{\Delta z^2},
\end{align*}$$

where $\theta$ is a constant taking values in the set $\{0, 1, 1/2\}$. According to whether $\theta$ get value 0, 1 or $1/2$, we have explicit, implicit and Crank-Nicolson method respectively. These approximations inserted into the heat equation (4.16) give

$$\begin{align*}
\frac{u_{m+1}^{n+1} - u_{m}^{n}}{\Delta \tau} & = \theta \frac{u_{m+1}^{n+1} - 2u_{m}^{n+1} + u_{m-1}^{n+1}}{\Delta z^2} + (1 - \theta) \frac{u_{m+1}^{n} - 2u_{m}^{n} + u_{m-1}^{n}}{\Delta z^2}, \quad (4.19)
\end{align*}$$

where $u_{m}^{n} = u(z_m, \tau_n), m = 0, \ldots, M$ and $n = 0, \ldots, N$.

Equation (4.19) is equivalent to
By letting
\begin{equation}
\eta = \frac{\Delta \tau}{\Delta z^2}
\end{equation}
we get the following result
\begin{equation}
u_m^{n+1} - \eta \vartheta (u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) - u_m^n - \eta (1 - \vartheta) (u_{m+1}^n - 2u_m^n + u_{m-1}^n) = 0.
\end{equation}

Let us define the following vectors as
\begin{equation}
b^{(n)} = \begin{bmatrix} b_1^n \\ b_2^n \\ b_3^n \\ \vdots \\ b_{M-1}^n \\ b_M^n \end{bmatrix}, \quad u^{(n)} = \begin{bmatrix} u_1^n \\ u_2^n \\ u_3^n \\ \vdots \\ u_{M-1}^n \\ u_M^n \end{bmatrix}, \quad g^{(n)} = \begin{bmatrix} g_1^n \\ g_2^n \\ g_3^n \\ \vdots \\ g_{M-1}^n \\ g_M^n \end{bmatrix}
\end{equation}

where
\begin{align}
\{ b_m^n &= u_m^n + \eta (1 - \vartheta) (u_{m+1}^n - 2u_m^n + u_{m-1}^n), \\
u_m^n &= u_m^n + \eta (1 - \vartheta) (u_{m+1}^n - 2u_m^n + u_{m-1}^n),
\end{align}

and where \( g_m^n \) approximate the value of the function \( g(t^*, z) \) at the grid point \((t_n, z_m)\). The initial and terminal conditions in the vector \( b^{(n)} \) are given by
\begin{align}
b_1^n &= u_0^n + \eta (1 - \vartheta) (u_2^n - 2u_1^n + g_0^n) + \eta \vartheta b_1^{(0)}, \\
b_{M-1}^n &= u_{M-2}^n + \eta (1 - \vartheta) (g_M^n - 2u_{M-1}^n + u_{M-2}^n) + \eta \vartheta b_{M-1}^{(0)}
\end{align}

We may formulate the American boundary value problem as
\begin{equation}
\begin{cases}
Au^{(n+1)} - b^{(n)} = 0 \\
u^{(n)} \geq g^{(n)} \\
u^{(0)} = g^{(0)} \\
u_0^n = g_0^n, \quad u_M^n = g_M^n
\end{cases}
\end{equation}

where \( A = (a_{mn})_{m,n=1,...,M} \) is a square tridiagonal matrix with
\begin{align}
a_{mn} &= 1 + 2\vartheta \eta \quad \text{and} \quad a_{n,n+1} = a_{n-1,n} = -\vartheta \eta,
\end{align}

**Proposition 4.3.1.** The explicit FDM is stable for
\begin{align}
0 < \eta \leq \frac{1}{2}
\end{align}

or equivalently
\begin{align}
0 < \Delta \tau \leq \frac{\Delta z^2}{2}.
\end{align}
In contrast, both Implicit and CN methods are unconditionally stable, which means that their stability holds for all time step $\Delta t$. Moreover, the Crank-Nicolson method has the highest order of convergence among standard FDMs.

**Proof:** See Seydel [54, pages 117, 118, and 121].

**Remark 4.3.2.** The time step $\Delta t$ can be found from (4.17') and it is given by

$$\Delta t = \frac{\sigma^2}{4a} \left(1 - e^{-2a\Delta t^*}\right), \quad \text{with} \quad \Delta t^* = -\Delta t = \frac{-T}{N},$$

and in our Matlab codes, referring to Proposition 4.2.1 above, we will choose for both methods

$$\Delta t = \frac{\Delta z^2}{4}$$

or equivalently

$$\Delta z = 2\sqrt{\Delta t}$$

**4.3.2. Crank-Nicolson method over an unbounded domain**

The three cases of finite difference method discussed above are constructed for a partial differential equation with a bounded domain. Therefore, their implementation requires the truncation of the infinite domain into the finite one which may deteriorate the computation efficiently. As the CN method is a well-known highest order of convergence and efficiency method among standard finite difference methods, we thus derive the CN method over an unbounded domain.

The main idea behind this is to perform the CN method on the initial boundary value problem obtained from an exact artificial boundary condition. In the derivation of the exact boundary value problem, we subdivide the domain into two: the interior domain denoted by $\Omega_{\text{int}}$ which contains the initial condition and the exterior domain denoted by $\Omega_{\text{ext}}$. These two domains are separated by the so-called artificial boundary $\Gamma_M$.

Here we build the unbounded domain according to [46] and [63]. So, let us consider the problem (4.18') defined on an unbounded domain $\Omega(\tau)$ given by

$$\Omega = \{(z, \tau) \in \mathbb{R}^2 | z < z^*, 0 \leq \tau \leq T^*\},$$

where

$$T^* = \frac{\sigma^2}{4a} \left(1 - \psi^2(T)\right).$$
We then define the artificial boundary $\Gamma_M$ as

$$\Gamma_M = \{(z, \tau)|z = z_M, 0 \leq \tau \leq T^*\}$$

which divides the unbounded domain $\Omega$ into the following domains:

$$\Omega_{int} = \{(z, \tau) \in \mathbb{R}^2|z < z_M, 0 \leq \tau \leq T^*\}$$

and

$$\Omega_{ext} = \{(z, \tau) \in \mathbb{R}^2|z > z_M, 0 \leq \tau \leq T^*\}$$

where

$$z_M = z_{max} = z_0 + \Delta z M.$$ 

As the initial condition is zero in the exterior domain. Then, the derivation of the exact artificial boundary condition will be based on the interior problem defined by:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2}, \quad u|_{\tau=0} = 0, \quad z > z_M, \quad u_{z=z_M} = u(z_M, \tau), 0 \leq \tau \leq T^*, u(z_M, 0) = 0, \quad u \to 0 \text{ when } z \to +\infty; \quad 0 \leq \tau \leq T^*. \quad (4.23)$$

By using Laplace Transform [30] and by the Duhamel Theorem (see [18, pp 31]), we may find:

$$\frac{\partial u(z, \lambda)}{\partial z} \bigg|_{z=z_M} = -\frac{1}{\sqrt{\pi}} \int_0^\tau \frac{1}{\sqrt{(\tau - \lambda)}} \frac{\partial u(z_M, \lambda)}{\partial \lambda} \, d\lambda \quad (4.24)$$

**Proposition 4.3.3.** The solution of the original problem (4.18') over an unbounded domain satisfies the following partial differential equation over a bounded domain

$$\frac{\partial u(z, \lambda)}{\partial \lambda} \bigg|_{z=z_M} = -\frac{1}{\sqrt{\pi}} \int_0^\tau \frac{1}{\sqrt{(\tau - \lambda)}} \frac{\partial u(z_M, \lambda)}{\partial \lambda} \, d\lambda \quad (4.25)$$

where $g(z, \tau) = \exp(-\lambda \Phi_1(x, t, T))$. Moreover the problem above admits a unique solution.

**Proof:** Assume that $u_1(z, \lambda)$ and $u_2(z, \lambda)$ are two solutions to problem (4.25). We define their difference to be $\zeta(z, \lambda) = u_1(z, \lambda) - u_2(z, \lambda)$. In $\Omega_{int}$ $\zeta(z, \lambda)$ satisfies:
By multiplying \( \zeta \) by both sides of the PDE (4.25) and performing integrations over \( \Omega_{int} \), we obtain:

\[
\int_0^T \int_{z_0}^{z_M} \left( \frac{\partial \zeta}{\partial z} \right)^2 \, dz \, d\lambda + \frac{1}{2} \int_{z_0}^{z_M} \zeta^2 \big|_{t=T} \, dz - \int_0^T \zeta \frac{d\zeta}{dz} \big|_{z=z_M} \, d\tau = 0
\] \hspace{1cm} \text{(4.27)}

We then consider the following problem on the unbounded domain \( \Omega_{ext} \):

\[
\begin{cases}
\frac{\partial^2 \zeta^*}{\partial z^2} = \frac{\partial^2 \zeta^*}{\partial z^2}, & 0 \leq \tau \leq T^* \\
\zeta^* \big|_{t=0} = 0, & z > z_M \\
\zeta^* \big|_{z=z_M} = \zeta \big|_{z=z_M}, & 0 \leq \tau \leq T^*, \\
\zeta^* \to 0 \text{ when } z \to +\infty; & 0 \leq \tau \leq T^*.
\end{cases}
\] \hspace{1cm} \text{(4.28)}

Given \( \zeta (z, \tau) \), the problem above has a unique solution \( \zeta^* (z, \tau) \). Moreover,

\[
\left. \frac{\partial \zeta^* (z, \lambda)}{\partial z} \right|_{z=z_M} = -\frac{1}{\sqrt{\pi}} \int_0^T \frac{1}{\sqrt{(\tau - \lambda)}} \frac{\partial u(z_M, \lambda)}{\partial \lambda} \, d\lambda = \left. \frac{\partial \zeta (z, \lambda)}{\partial z} \right|_{z=z_M}
\] \hspace{1cm} \text{(4.28')}  

By multiplying \( \zeta^* \) by both sides of (4.28') and integrating over \( \Omega_{ext} \), we obtain

\[
\int_0^T \int_{z_0}^{\infty} \left( \frac{\partial \zeta^*}{\partial z} \right)^2 \, dz \, d\lambda + \frac{1}{2} \int_{z_0}^{\infty} (\zeta^*)^2 \big|_{t=T} \, dz = \int_0^T \zeta^* \frac{d\zeta^*}{dz} \big|_{z=z_M} \, d\tau \geq 0
\] \hspace{1cm} \text{(4.29)}

According to the Equation (4.28'), we obtain

\[
-\int_0^T \zeta^* \frac{d\zeta^*}{dz} \big|_{z=z_M} \, d\tau = -\int_0^T \zeta \frac{d\zeta}{dz} \big|_{z=z_M} \, d\tau \leq 0
\] \hspace{1cm} \text{(4.30)}

Finally, combining (4.28'), (4.29) and (4.30) we find \( \zeta (z, \lambda) = 0 \). This means that \( u_1(z, \lambda) = u_2(z, \lambda) \).

After getting an initial boundary problem on a finite domain enclosed by the artificial boundary which is equivalent to the original problem, our little contribution is mainly based by building the Crank-Nicolson scheme over the obtained unbounded domain and to solve the obtained numerical problem. So let us first approximate the third boundary condition for the problem (4.25). We
know from the theory of approximation [47] that the integral in that boundary condition can be approximated as

$$\frac{1}{\sqrt{\pi}} \int_0^\tau \frac{1}{\sqrt{(\tau - \lambda)}} \frac{\partial u(z_M, \lambda)}{\partial \lambda} d\lambda \approx \sum_{l=0}^{n-1} \frac{(u_{M+1}^L - u_M^L)}{\Delta\tau} \Lambda_n,$$  \hspace{0.5cm} (4.31)

where

$$\Lambda_n = 2 \left( \sqrt{\frac{\tau_n - \tau_{l+1}}{\pi}} - \sqrt{\frac{\tau_n - \tau_{l}}{\pi}} \right) = 2 \sqrt{\frac{\Delta\tau}{\pi}} \left( \sqrt{n-l-1} - \sqrt{n-l} \right).$$

From the Crank-Nicolson scheme, we have

$$\frac{\partial u(z, \lambda)}{\partial z} \bigg|_{z=z_M} \approx \frac{u_{M+1}^{n+1} - u_{M-1}^{n+1}}{4\Delta z} + \frac{u_{M+1}^n - u_{M-1}^n}{4\Delta z}.$$  \hspace{0.5cm} (4.32)

Approximations (4.31) and (4.32) into the third boundary condition for the problem (4.25) lead to

$$\frac{u_{M+1}^{n+1} - u_{M-1}^{n+1}}{4\Delta z} + \frac{u_{M+1}^n - u_{M-1}^n}{4\Delta z} = \sum_{l=0}^{n-1} \frac{(u_{M+1}^L - u_M^L)}{\Delta\tau} \Lambda_n,$$

which can be rewritten as

$$u_{M+1}^{n+1} - u_{M-1}^{n+1} = (u_{M}^{n+1} - u_{M-1}^{n+1}) + 4\Delta z \sum_{l=0}^{n-1} \frac{(u_{M+1}^L - u_M^L)}{\Delta\tau} \Lambda_n,$$  \hspace{0.5cm} (4.33)

Keeping in our mind of linear complementarity, the Problem (4.25) under the Crank-Nicolson method over the unbounded domain is then given by

\[
\begin{cases}
\frac{u_{m+1}^{n+1} - u_m^n}{\Delta\tau} = \frac{u_{m+1}^{n+1} - 2u_m^n + u_{m-1}^{n+1}}{2\Delta z^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2\Delta z^2} \\
u_m^n > g_m^n, \ u_{M}^n = g_M^n \\
u_{M+1}^{n+1} - u_{M-1}^{n+1} = (u_{M+1}^{n+1} - u_{M-1}^{n+1}) + 4\Delta z \sum_{l=0}^{n-1} \frac{(u_{M+1}^L - u_M^L)}{\Delta\tau} \Lambda_n. 
\end{cases}
\]  \hspace{0.5cm} (4.34)

We observe that terms $u_{M+1}^{n+1}$ and $u_{M+1}^n$ in equation (4.37) are unknown; so, we need to eliminate them. In order to do so, we combine the equation (4.34) for $m = M$ with the equation (4.37) in order to obtain the following result:
As we have done in Section 4.3.1, let us look at this equation

\[
\begin{align*}
\frac{u_{m+1}^{n+1} - \frac{\eta}{2} (u_{m+1}^{n+1} - 2u_{m}^{n+1} + u_{m-1}^{n+1}) - u_{m}^{n} - \frac{\eta}{2} (u_{m+1}^{n} - 2u_{m}^{n} + u_{m-1}^{n})}{\Delta z} & = 0 \\
u_{m+1}^{n+1} - \left(1 + \frac{1}{\eta}\right) u_{m+1}^{n+1} - \frac{1 - \eta}{1 + \eta} g_{M}^{n} - \left(1 + \frac{1}{\eta}\right) u_{M-1}^{n} - 2 \left(1 + \frac{1}{\eta}\right) Dz \sum_{i=0}^{n-1} \frac{u_{i+1}^{n} - u_{i}^{n}}{\Delta \tau} & = 0 \\
u_{m}^{n} & > g_{m}^{n} \\
u_{0}^{n} & = g_{0}^{n}.
\end{align*}
\]

Finally, we arrive at the following problem\(^{13}\)

\[
\begin{align*}
Au^{(n+1)} - b^{(n)} & = 0 \\
\begin{bmatrix} u_{1}^{n} \\ u_{2}^{n} \\ \vdots \\ u_{M-1}^{n} \\ u_{M}^{n} \end{bmatrix} & \geq \begin{bmatrix} b_{1}^{n} \\ b_{2}^{n} \\ \vdots \\ b_{M-1}^{n} \\ b_{M}^{n} \end{bmatrix} \\
u_{0}^{n} & = g_{0}^{n}.
\end{align*}
\]

where

\[
b_{m}^{n} = u_{m}^{n} + \eta (1 - \theta) (u_{m+1}^{n} - 2u_{m}^{n} + u_{m-1}^{n}),
\]

with the initial and terminal conditions given by

\[
b_{1}^{n} = u_{1}^{n} + \eta (1 - \theta) (u_{2}^{n} - 2u_{1}^{n} + g_{0}^{n}) + \eta \theta g^{(0)}
\]

and

\[
b_{M}^{n} = u_{M}^{n} - \left(1 + \frac{1}{\eta}\right) u_{M}^{n-1} - \frac{1 - \eta}{1 + \eta} g_{M}^{n} - \left(1 + \frac{1}{\eta}\right) u_{M-1}^{n} - 2 \left(1 + \frac{1}{\eta}\right) Dz \sum_{i=0}^{n-1} \frac{u_{i+1}^{n} - u_{i}^{n}}{\Delta \tau} A_{n} = 0.
\]

The solution to problems (4.22) and (4.38) above is done iteratively. To find their numerical solutions, we prefer to use the Successive Over Relaxation (SOR) method because of its high speed of convergence (see for e.g. [15]). Since our problems are more complex and the standard SOR method cannot support this kind of problem, we provide a SOR method for our problems (4.22) and (4.38) in

\(^{13}\) We provide in Appendix A.2.1 Matlab codes which compute explicit FDM, Crank – Nicolson method and Crank – Nicolson method over an unbounded domain.
Appendix A.2.1 which is a slight adaptation of the standard SOR method. Let us now illustrate our performance.

4.3.3. Illustration and results

We consider a one-year call option on a zero coupon bond of strike price 0.8 with early exercise feature on a two-year with face value equals to unity. The model parameters are given as $a = 0.1$ and $\sigma = 0.1$. Comparisons are made by using the explicit FDM, Crank-Nicolson method and the Crank-Nicolson method over an unbounded domain. We are doing our essay with six numbers $N = M$ of steps: $N = 120, 240, 360, 480, 600$ and we regard the results of the explicit FDM with $N = 1200$ as the true value.\(^{14}\)

With the help of the Matlab 7.1 codes provided in Appendix A.2.2, we arrive at the following summarized results:

---

\(^{14}\) We acknowledge that in doing this, a discretization or programming error could affect what we take to be a true value
Table 4.1: Call option on American zero-coupon bond option under the CN method over an unbounded domain.

<table>
<thead>
<tr>
<th>N</th>
<th>CN Method over the unbounded domain</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Flat</td>
<td>Upward</td>
<td>Downward</td>
<td>Humped</td>
</tr>
<tr>
<td>120</td>
<td>0.098817</td>
<td>0.09182</td>
<td>0.106204</td>
<td>0.078117</td>
</tr>
<tr>
<td>240</td>
<td>0.098863</td>
<td>0.091898</td>
<td>0.106275</td>
<td>0.078222</td>
</tr>
<tr>
<td>360</td>
<td>0.098889</td>
<td>0.091935</td>
<td>0.10631</td>
<td>0.078285</td>
</tr>
<tr>
<td>480</td>
<td>0.098896</td>
<td>0.091942</td>
<td>0.106314</td>
<td>0.078296</td>
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<tr>
<td>600</td>
<td>0.098898</td>
<td>0.091944</td>
<td>0.106316</td>
<td>0.078298</td>
</tr>
<tr>
<td>True(N=1200)</td>
<td>0.098901</td>
<td>0.091948</td>
<td>0.106323</td>
<td>0.078299</td>
</tr>
</tbody>
</table>

Table 4.2. Relative errors in percentage for the CN method over an unbounded domain

<table>
<thead>
<tr>
<th>N</th>
<th>Relative errors of CN Method over the unbounded domain</th>
<th></th>
<th></th>
<th></th>
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<td>Flat</td>
<td>Upward</td>
<td>Downward</td>
<td>Humped</td>
</tr>
<tr>
<td>120</td>
<td>-0.084933418</td>
<td>-0.139209118</td>
<td>-0.1119231</td>
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<td>360</td>
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<td>-0.014138426</td>
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</tr>
<tr>
<td>480</td>
<td>-0.005055561</td>
<td>-0.006525427</td>
<td>-0.00846477</td>
<td>-0.00383147</td>
</tr>
<tr>
<td>600</td>
<td>-0.003033336</td>
<td>-0.004350285</td>
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<td>-0.00127716</td>
</tr>
<tr>
<td>1200</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 4.3. Call option on American zero-coupon bond option under the CN method.

<table>
<thead>
<tr>
<th>N</th>
<th>Flat</th>
<th>Upward</th>
<th>Downward</th>
<th>Humped</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
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<td>True(N=1200)</td>
<td>0.098901</td>
<td>0.091948</td>
<td>0.106323</td>
<td>0.078299</td>
</tr>
</tbody>
</table>

Table 4.4. Relative errors of CN method

<table>
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<th>N</th>
<th>Flat</th>
<th>Upward</th>
<th>Downward</th>
<th>Humped</th>
</tr>
</thead>
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<td>360</td>
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<tr>
<td>480</td>
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</tr>
<tr>
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</tr>
<tr>
<td>1200</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</table>
Table 4.5. Call option on American zero-coupon bond option under the explicit FDM.

<table>
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<tr>
<th>( N )</th>
<th>( \text{Explicit FDM} )</th>
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<th>Upward</th>
<th>Downward</th>
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<td>0.098901</td>
<td>0.091948</td>
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<td>0.078299</td>
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Table 4.6. Relative errors in percentage for the explicit FDM.

<table>
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<tr>
<th>( N )</th>
<th>( \text{Relative errors of Explicit FDM} )</th>
<th>Flat</th>
<th>Upward</th>
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<td>-0.075042415</td>
<td>-0.01410795</td>
<td>-0.10983537</td>
</tr>
<tr>
<td>480</td>
<td></td>
<td>0.003033336</td>
<td>0.01740114</td>
<td>0.018810605</td>
<td>-0.02043449</td>
</tr>
<tr>
<td>600</td>
<td></td>
<td>-0.017188906</td>
<td>0.027189281</td>
<td>0.016929545</td>
<td>0.012771555</td>
</tr>
<tr>
<td>1200</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 4.1: Results of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Flat yield Curve

Figure 4.2: Results of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Upward yield Curve
Figure 4.3: Results of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for Downward yield Curve

Figure 4.4: Results of the Explicit FDM, CN method and CN method over an unbounded domain (UD) for humped yield Curve
**Figure 4.5:** Relative Errors estimation of the Explicit FDM, CN method and CN method with an unbounded domain for Flat yield Curve

![Graph](image1)

**Figure 4.6:** Relative Errors estimation of the Explicit FDM, CN method and CN method with an unbounded domain for Upward yield Curve

![Graph](image2)
**Figure 4.7:** Relative Errors estimation of the Explicit FDM, CN method and CN method with an unbounded domain for Downward yield Curve

![Graph](image1)

**Figure 4.8:** Relative Errors estimation of the Explicit FDM, CN method and CN method with an unbounded domain for Humped yield Curve

![Graph](image2)
Chapter 5:

General Conclusion

In this scientific work, by introducing forward price and applying the risk neutral valuation formula and referring mainly to [1], [11] and Jamshidian Work [43], we have derived a simple closed-form expression for pricing European option written on the zero-coupon and coupon-bearing bonds under the Hull-White extended Vasicek model. We draw two important findings:

- The price formula of a European bond option under the Hull-White extended Vasicek model is equivalent to Jamshidian formula [43]; consequently, the result resembles the Black-Scholes formula [10] and has the same interpretation.
- The price of the call option is greater than the price of put option for any yield curve to maturity.

As there is no analytical solution for American option, we have used numerical methods. After transformation from the Hull-White term structure equation to the diffusion equation, we have applied the finite difference method especially explicit, implicit and Crank-Nicolson methods. As FDMs require truncation of interval from infinite to finite one, we have built one method which remedies to that, the Crank-Nicolson method over an unbounded domain into which we get an initial boundary problem on a finite domain enclosed by the artificial boundary which is equivalent to the original problem. The Crank-Nicolson scheme has been used in order to find the numerical solution. We found that the CN method with an unbounded domain outperforms FDMs in term of both efficiently and accuracy when we price American Bond option.
Appendices

A.1. Matlab codes for European bond option under
The Hull – White extended Vasicek model

This program computes analytically the European Call and Put bond option for
the Hull-White extended Vasicek model referring to the formulas (3.23b) and
(3.23e) and generates the curve for both call and put prices for the flat, upward,
downward and humped yield curves.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Parameters
% t = Initial time
% T = Maturity date for the European option.
% S = Maturity date for the bond.
% K = Strike price
% sig = volatility term of the HW model
% pT = Bond price with maturity date T
% pS = Bond price with maturity date S
% fp = Forward price
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Clear all;
t = input('Enter the initial date :');
S = input('Enter the maturity date S for the bond :');
T = input('Enter the maturity date T :');
K = input('Enter the strike price :');
a = input('Enter the speed of reversion for the HW model :');
sig = input('Enter the volatility term for the HW model :');

%%%%%%%%%%%%%%%% Calculations %%%%%%%%%%%%%%%%%
Rf=0.03;                        % Flat Yield curve
Ru=0.03+0.003*((T-t) ^ 0.5);    % Upward yield curve
Rd=0.03-0.003*((T-t) ^ 0.5);    % Downward yield curve
Rh=0.06*exp(-0.01*(T-t))-0.03*exp(-0.3*(T-t)); % Humped yield curve
\[ p_{Tf} = \exp\left(-1 \times (T-t) \times R_f\right); \]
\[ p_{Sf} = \exp\left(-1 \times (S-t) \times R_f\right); \]
\[ fpf = \frac{p_{Sf}}{p_{Tf}}; \]
\[ p_{Tu} = \exp\left(-1 \times (T-t) \times R_u\right); \]
\[ p_{Su} = \exp\left(-1 \times (S-t) \times R_u\right); \]
\[ fpu = \frac{p_{Sf}}{p_{Tu}}; \]
\[ p_{Td} = \exp\left(-1 \times (T-t) \times R_d\right); \]
\[ p_{Sd} = \exp\left(-1 \times (S-t) \times R_d\right); \]
\[ fpd = \frac{p_{Sd}}{p_{Td}}; \]
\[ p_{Th} = \exp\left(-1 \times (T-t) \times R_h\right); \]
\[ p_{Sh} = \exp\left(-1 \times (S-t) \times R_h\right); \]
\[ fph = \frac{p_{Sh}}{p_{Th}}; \]
\[ v = \sqrt{\left(1-\exp\left(-1 \times a \times (S-t)\right)\right)^2 \times \left(1-\exp\left(-2 \times a \times (T-t)\right)\right) \times \left(\sigma^2 / (2 \times a^3)\right)}; \]
\[ d_{1f} = \frac{(\log(fpf/K) + 0.5 \times v^2)}{v}; \]
\[ d_{2f} = \frac{(\log(fpf/K) - 0.5 \times v^2)}{v}; \]
\[ d_{1u} = \frac{(\log(fpu/K) + 0.5 \times v^2)}{v}; \]
\[ d_{2u} = \frac{(\log(fpu/K) - 0.5 \times v^2)}{v}; \]
\[ d_{1d} = \frac{(\log(fpd/K) + 0.5 \times v^2)}{v}; \]
\[ d_{2d} = \frac{(\log(fpd/K) - 0.5 \times v^2)}{v}; \]
\[ d_{1h} = \frac{(\log(fph/K) + 0.5 \times v^2)}{v}; \]
\[ d_{2h} = \frac{(\log(fph/K) - 0.5 \times v^2)}{v}; \]
\% European call bond option price for flat yield curve
\[ \text{phi\_callf} = p_{Sf} \times \text{normcdf}(d_{1f}) - p_{Tf} \times K \times \text{normcdf}(d_{2f}); \]
\% European put bond option price for flat yield curve
\[ \text{phi\_putf} = p_{Tf} \times K \times \text{normcdf}(-1 \times d_{2f}) - p_{Sf} \times \text{normcdf}(-1 \times d_{1f}); \]
\% European call bond option price for upward yield curve
\[ \text{phi\_callu} = p_{Su} \times \text{normcdf}(d_{1u}) - p_{Tu} \times K \times \text{normcdf}(d_{2u}); \]
\% European put bond option price for upward yield curve
phi_putu = pTu*K*normcdf(-1*d2u)-pSu*normcdf(-1*d1u);

% European call bond option price for downward yield curve
phi_calld = pSd*normcdf(d1d)-pTd*K*normcdf(d2d);
% European put bond option price for downward yield curve
phi_putd = pTd*K*normcdf(-1*d2d)-pSd*normcdf(-1*d1d);
% European call bond option price for humped yield curve
phi_callh = pSh*normcdf(d1h)-pTh*K*normcdf(d2h);
% European put bond option price for humped yield curve
phi_puth = pTh*K*normcdf(-1*d2h)-pSh*normcdf(-1*d1h);
% call/put matrix prices for all yield curves
prices = [phi_callf phi_callu phi_calld phi_callh; phi_putf phi_putu phi_putd phi_puth]
plot( R,Call,:b* , R,Put,-'ko','-linewidth',2.0);
xlabel('Yield Curves ');
ylabel('Prices ');
legend ('Call option', 'Put option');

A.2. Matlab codes for standard FDMs and the CN method over an unbounded domain

In Section 4.3, we have built boundary value problems for pricing American bond option under the Hull-White extended Vasicek model. We have used finite difference scheme for numerical analysis and we have seen that the solution can be done iteratively. Therefore, we choose the Successive Over Relaxation (SOR) method for the sake of solution due to its high speed of convergence (see [15]). Let us discuss it before building the Matlab 7.1 program.

A.2.1. SOR method to Problems (4.22) and (4.38)

In this subsection, we adapt the standard SOR method to our pricing boundary value problem (4.22) and (4.38), and we build an algorithm ready to be
programmed. Let us assume we have been given a system of linear equations in $R^M$

$$Au - b = 0 \quad (A.1)$$

where $A = (a_{mn}), m, n = 1, ..., M$ is a square matrix and $b$ a column vector. The system above can also be written as

$$0 = -Au + b \quad (A.2)$$

Letting $S$ a nonsingular suitable matrix, we can write Equation (A.2) as

$$Su = Su - Au + b = (S - A)u + b,$$

it is followed that

$$u = S^{-1}(S - A)u + S^{-1}b$$

$$= (S^{-1}S - S^{-1}A)u + S^{-1}b$$

$$= (I - S^{-1}A)u + S^{-1}b,$$

which leads us to the iteration

$$u^{(k)} = (I - S^{-1}A)u^{(k-1)} + S^{-1}b, \quad (A.3)$$

where $k$ represents the number of iterations. Let the matrix $A$ additively be partitioned into

$$A = D - L - U,$$

with $D$ diagonal matrix, $L$ strict lower triangular matrix and $U$ strict upper triangular matrix. Then for the case where:

- $S = D$, we get the so-called *Jacobi method* and we have

$$S - A = S - D + L + U = L + U.$$  

The Equation (A.3) becomes

$$Du^{(k)} = (L + U)u^{(k-1)} + b.$$  

- $S = D - L$, we get the so-called *Gauss–Seidel method* and we have

$$S - A = U,$$

and from Equation (A.3), we deduce
Let us make a slight modification by introducing a parameter \( \omega_R \) such that

\[
S = \frac{1}{\omega_R} D - L,
\]
then we have

\[
S - A = S - D + L + U = \frac{1}{\omega_R} D - D + U = \left( \frac{1}{\omega_R} - 1 \right) D + U.
\]

Finally, from (A.3) we deduce

\[
\left( \frac{1}{\omega_R} D - L \right) u^{(k)} = \left[ \left( \frac{1}{\omega_R} - 1 \right) D + U \right] u^{(k-1)} + b,
\]
or equivalently

\[
u^{(k)} = \left( \frac{1}{\omega_R} D - L \right)^{-1} \left[ \left( \frac{1}{\omega_R} - 1 \right) D + U \right] u^{(k-1)} \left( \frac{1}{\omega_R} D - L \right)^{-1} b.
\] (A.4)

The above method is called \emph{Successive Over Relaxation} (SOR) method. \( \omega_R \) is called \emph{relaxation parameter} and the method converges for \( 1 \leq \omega_R \leq 2 \). If \( \omega_R = 1 \), we find the Gauss-Seidel method. For the sake of convergence of these above methods elaborated, we refer the reader to [15].

Problems (4.22) and (4.38) are not in the easy form of Equation (A.1). In order to generate the iterative solution to our problems, we need to make little modifications of the standard SOR method. To be concrete, let

\[
v = u - g
\]
and

\[
w = Au - b.
\]
Then, we have

\[
Av - w = A(u - g) - Au + b
= Au - Ag - Au + b
= -Ag + b.
\]

So, the problem becomes
\[ \begin{align*}
\begin{cases}
Av - w &= \tilde{b} \\
v &\geq 0,
\end{cases}
\end{align*} \]

with \( \tilde{b} = -Ag + b \).

From matrix algebra theory, we may reduce the equation (A.4) in terms of elements of the matrix A given by \( a_{mn} \) \( m, n = 1, \ldots, M \), in order to obtain

\[ v^{(k)}_m = v^{(k-1)}_m + \omega_R \frac{\tilde{b}_m - \sum_{n=1}^{m-1} a_{mn} v^{(k)}_m - a_{mm} v^{(k-1)}_m - \sum_{n=m+1}^{M} a_{mn} v^{(k-1)}_m}{a_{ii}} \]  

(A.6)

As in our problem \( A \) is the tridiagonal matrix with \( a_{m,m+1} = a_{m-1,m} = -\partial \eta \) and \( a_{mm} = 1 + 2\partial \eta \), then sums in (A.6) are given by

\[ \sum_{n=1}^{m-1} a_{mn} v^{(k)}_m = \begin{cases} 
0, & \text{for } m = 1 \\
(1 - \partial \eta) v^{(k)}_{m-1}, & \text{for } m = 2, 3, \ldots, M
\end{cases} \]

and

\[ \sum_{n=m+1}^{M} a_{mn} v^{(k-1)}_m = \begin{cases} 
0, & \text{for } m = M \\
(1 + \partial \eta) v^{(k-1)}_{m+1}, & \text{for } m = 1, 2, \ldots, M - 1
\end{cases} \]

Therefore, the equation (A.6) can be further reduced to

\begin{itemize}
  \item For the case where \( m = 2, \ldots, M - 1 \)
  \[ v^{(k)}_m = v^{(k-1)}_m + \omega_R \left[ \frac{\tilde{b}_m + \partial \eta v^{(k)}_{m-1} - (1 + 2\partial \eta) v^{(k-1)}_{m+1} + \partial \eta v^{(k-1)}_m}{1 + 2\partial \eta} \right] \]
  \end{itemize}  

(A.7)

\begin{itemize}
  \item For the case where \( m = 1 \)
  \[ v^{(k)}_m = v^{(k-1)}_m + \omega_R \left[ \frac{\tilde{b}_m + \partial \eta v^{(k-1)}_{m+1}}{1 + 2\partial \eta} - v^{(k-1)}_m \right] \]
  \end{itemize}

\begin{itemize}
  \item For the case where \( m = M \)
  \[ v^{(k)}_m = v^{(k-1)}_m + \omega_R \left[ \frac{\tilde{b}_m + \partial \eta v^{(k)}_{m-1}}{1 + 2\partial \eta} - v^{(k-1)}_m \right] \]
  \end{itemize}

Since \( v \geq 0 \) then we may write

\[ v^{(k)}_m = \max \left\{ 0, v^{(k-1)}_m + \omega_R \left[ \frac{\tilde{b}_m + \partial \eta (v^{(k)}_{m-1} + v^{(k-1)}_{m+1})}{1 + 2\partial \eta} - v^{(k-1)}_m \right] \right\} \]
By setting
\[ \rho_m = \frac{\tilde{b}_m + \partial \eta (v_{m-1}^{(k)} + v_{m+1}^{(k-1)})}{1 + 2 \partial \eta}, \]
and by adapting for \( v \geq 0 \) to \( u \geq g \) or equivalently \( u - g \geq 0 \), we arrive to the following algorithm for the adapted SOR model.

**Algorithm A.2.1: Adapted SOR model.**

**for** \( k = 1, 2, \ldots \)

**for** \( m = 1, 2, \ldots, M \)

**switch** \( m \)

**case** \( m = 1 \)

\[ \rho_m = \frac{\tilde{b}_m + \partial \eta v_{m+1}^{(k-1)}}{1 + 2 \partial \eta}; \]

**case** \( m = M \)

\[ \rho_m = \frac{\tilde{b}_m + \partial \eta v_{m-1}^{(k)}}{1 + 2 \partial \eta}; \]

**otherwise**

\[ \rho_m = \frac{\tilde{b}_m + \partial \eta (v_{m-1}^{(k)} + v_{m+1}^{(k-1)})}{1 + 2 \partial \eta}; \]

**end**

\[ u_m^{(k)} = \max \{ g_m^k, u_m^{(k-1)} + \omega_R [\rho_m - u_m^{(k-1)}] \}; \]

\[ u_m^{(k)} - u_m^{(k-1)} > \varepsilon, \quad \varepsilon > 0 \]

**end.**

Note that the test \( v_m^{(k)} - v_m^{(k-1)} > \varepsilon \) allow us to get off the loop and the algorithm above is also valid for Crank-Nicolson over an unbounded domain by replacing \( \partial \) by \( 1/2 \) which lead us to

\[ \rho_m = \frac{2 \tilde{b}_m + \eta (v_{m-1}^{(k)} + v_{m+1}^{(k-1)})}{2(1 + \eta)}. \]

We note furthermore, the above algorithm can be particularized to the European option by replacing the line

\[ u_m^{(k)} = \max \{ g_m^k, u_m^{(k-1)} + \omega_R [\rho_m - u_m^{(k-1)}] \} \]

by
Let us now propose an algorithm which performs pricing of American bond option. But before to do so, the following algorithm for testing early exercise is crucial.

**Algorithm A.2.2. Test for early exercise.**

*Input* $K, a, T, \sigma, \varepsilon$

*Calculations*

\[
\psi(T) = e^{-aT}
\]

\[
\tau_{\text{end}} = \frac{\sigma^2}{4a} \left( 1 - \psi^2(T) \right)
\]

\[
\gamma(T) = \frac{1}{\alpha} (\psi(T) - 1)
\]

\[
\beta(T) = \frac{\sigma^2}{4\alpha^3} \left[ 1 - 2 \ln(1 + a\gamma(T)) - (1 - a\gamma(T))^2 \right]
\]

*for* $m = 1, \ldots, M - 1$

\[
V(z_m, 0) = \exp \left( \frac{4az_m + \sigma^2(1 - \psi^2(T))}{4a\gamma(T)} \gamma(T) - \beta(T) \right) u(z_m, 0)
\]

**switch** option

*case* call

\[
i_f = \max\{i : |V(z_m, 0) + K - CB(m)| < \varepsilon\}
\]

$p_c(0) > p_c(i_f)$ *Stopping region*

*case* put

\[
i_f = \max\{i : |V(z_m, 0) + p_c(m) - K| < \varepsilon\}
\]

$p_c(0) < p_c(i_f)$ *Stopping region*

**end.**

The algorithm A.2.2 above evaluates the data at the final time $\tau_{\text{end}}$, which corresponds to $t = 0$. For the remaining times, we proceed the same way. The algorithm for pricing American bond option is then given by

**Algorithm A.2.3. Pricing American Bond option under the Hull-White Model.**

*Input* $K, a, T, \sigma, N, M, \text{type of option, method}
These above algorithms can serve as a practical tool for programming the American bond option in any programming language such as Maple, Matlab, C++, Pascal and so on. In our case, we do it in Matlab 7.1 as we have skills in that language.

A.2.2. Matlab Codes

This program computes the American bond option price and the optimal exercise boundary. We use the explicit finite difference and the standard Crank-Nicolson methods, and the Crank-Nicolson method over an unbounded domain to approximate the pricing boundary value problem for American bond option. The SOR method is used to solve them.

```matlab
Compute \Delta t, \Delta t, \Delta z, \Delta t^*, \psi, \gamma, \beta, x, \eta 
choose \epsilon, \omega_R, type of bond 
if method = FDM
    choose \partial
    create the vector b for the problem (4.22)
    perform Algorithm A.2.1
else if method = CN over an unbounded domain
    create the vector b for the Problem (4.38)
    perform Algorithm A.2.1 for the Problem (4.38)
end if
perform Algorithm A.2.2
end.

These above algorithms can serve as a practical tool for programming the American bond option in any programming language such as Maple, Matlab, C++, Pascal and so on. In our case, we do it in Matlab 7.1 as we have skills in that language.

A.2.2. Matlab Codes

This program computes the American bond option price and the optimal exercise boundary. We use the explicit finite difference and the standard Crank-Nicolson methods, and the Crank-Nicolson method over an unbounded domain to approximate the pricing boundary value problem for American bond option. The SOR method is used to solve them.

```
Clear all;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

MAIN PROGRAM %%%%%%%%%%%%%%%%%

T = input('Enter the maturity time T of the European option :');
S = input('Enter the maturity time S of the bond :');
K = input('Enter the strike price K (0<K<1):');
a = input('Enter the speed of reversion for the HW model :');
sig = input('Enter the volatility term for the HW model :');
N = input('Enter the number of time steps :');
% Define parameters
M=N;
dt=T/N;
dt_star=-dt;
dtau=-sig^2/2*a(1-exp(a*dt));  % according to Remark 4.2.2. and % considere that the time t starts from 0.
dz=2*sqrt(dtau);               % according to Remark 4.2.2.
gamma = -1/a(1-exp(a*dt));   % From (4.13)
beta = -dt - 2*gamma +0.5*gamma*(1+exp(a*dt));  % From (4.14)
etta = dtau/(dz^2);          % From (4.21)

Rf=0.03;                  % From (2.2a)
Ru=0.03+0.003*(dt_star^0.5);           % From (2.2b)
Rd=0.03-0.003*(dt_star^0.5);           % From (2.2c)
Rh=0.06*exp(0.01*dt)-0.03*exp(0.3*dt); % From (2.2d)

z_min = M/2*dz;
z_max = -M/2*dz;
z   = (z_min:dz:z_max)';
tau = 0:dtau:tau_max;

%%% Compute American bond option through Explicit and CN FDM

[p,t,V] = Explicit_and_CN_FDM(K,T,r,delta,sigma,type,N);
%%% Compute American bond option through CN method over the unbounded domain

[p,t,V] = CN_Over_Unbounded_Domain(K,T,r,delta,sigma,type,N);

%%% Compute free boundary %%%
pf = FreeBoundary(S,t,V,K,type);

%%%%%%% FUNCTIONS %%%%%%%%

%%% Explicit and Cranck - Nicolson FDMs %%%

function [p,t,V] = Explicit_and_CN_FDM(K,T,a,sigma,type,N);

% Input :
% K     = Strike price
% T     = Time to maturity
% sigma = Volatility term of the Hull - White model
% a     = Speed reversion term of the Hull - White model
% type  = type of an option, call or put

% Output :
% p = range of bond price
% t = range of time from 0 to T
% V = corresponding option price

thetaVa = input('Enter theta Variant as define in (4.19) which must be 0 for explicit FDM or 1/2 for CN method :')
omega = 1;
eps = 1e-6;
% For performance reasons we compute one matrix with all the g values
Z = repmat(z,1,N+1);
Y = repmat(tau,N+1,1);
G = g(Z,Y);
u = zeros(M+1,N+1);

% boundary conditions
u(:,1)   = G(:,1);
u(1,:)   = G(1,:);
u(M+1,:) = G(end,:);

b = zeros(M-1,1); % righthandside is needed in core algorithm
vnew = zeros(M-1,1);

% Core algorithm
for j = 2:N+1
    % create righthandside b
    for k = 1:N-1
        switch k
        case 1
            b(k) = u(2,j-1)+eta*(1-thetaVa)*(u(1,j-1)-2*u(2,j-1)+w(3,j-1)) + eta*thetaVa*u(1,j);
        case m-1
            b(k) = u(M,j-1)+eta*(1-thetaVa)*(u(M-1,j-1)-2*u(M,j-1)+u(M+1,j-1)) + eta*thetaVa*u(M+1,j);
        otherwise
            b(k) = u(k+1,j-1)+eta*(1-thetaVa)*(u(k,j-1)-2*u(k+1,j-1)+u(k+2,j-1));
        end
    end
    v = max(u(2:M,j-1),G(2:M,j));     % initialize vector v
    iter = 1; % the variable iter is introduced to manage the
    % SOR iteration
    while iter == 1
        for k = 1:M-1
            switch k
            case 1
                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
            case m-1
                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
            otherwise
                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
            end
        end
        % SOR iteration
        while iter == 1
            for k = 1:M-1
                switch k
                case 1
                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                case m-1
                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                otherwise
                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                end
            end
            % SOR iteration
            while iter == 1
                for k = 1:M-1
                    switch k
                    case 1
                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                    case m-1
                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                    otherwise
                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                    end
                end
                % SOR iteration
                while iter == 1
                    for k = 1:M-1
                        switch k
                        case 1
                            rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                        case m-1
                            rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                        otherwise
                            rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                        end
                    end
                    % SOR iteration
                    while iter == 1
                        for k = 1:M-1
                            switch k
                            case 1
                                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                            case m-1
                                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                            otherwise
                                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                            end
                        end
                        % SOR iteration
                        while iter == 1
                            for k = 1:M-1
                                switch k
                                case 1
                                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                case m-1
                                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                otherwise
                                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                end
                            end
                            % SOR iteration
                            while iter == 1
                                for k = 1:M-1
                                    switch k
                                    case 1
                                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                    case m-1
                                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                    otherwise
                                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                    end
                                end
                                % SOR iteration
                                while iter == 1
                                    for k = 1:M-1
                                        switch k
                                        case 1
                                            rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                        case m-1
                                            rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                        otherwise
                                            rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                        end
                                    end
                                    % SOR iteration
                                    while iter == 1
                                        for k = 1:M-1
                                            switch k
                                            case 1
                                                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                            case m-1
                                                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                            otherwise
                                                rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                            end
                                        end
                                        % SOR iteration
                                        while iter == 1
                                            for k = 1:M-1
                                                switch k
                                                case 1
                                                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                                case m-1
                                                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                                otherwise
                                                    rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                                end
                                            end
                                            % SOR iteration
                                            while iter == 1
                                                for k = 1:M-1
                                                    switch k
                                                    case 1
                                                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                                    case m-1
                                                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                                    otherwise
                                                        rho = (b(k)+eta*thetaVa*v(k+1))/(1+2*eta*thetaVa);
                                                    end
                                                end
                                            end
                                        end
                                    end
                                end
                            end
                        end
                    end
                end
            end
        end
    end
end
case m-1
    rho = (b(k)+eta*thetaVa*vnew(k-1))/(1+2*eta*thetaVa);
    otherwise
    rho = (b(k)+eta*thetaVa*(vnew(k-1)+v(k+1)))/(1+2*eta*thetaVa);
end
vnew(k) = max(G(k+1,j),v(k)+omega*(rho-v(k)));
end
if norm(v-vnew) <= eps
    iter = 0;
else
    v = vnew;
end
end
u(2:M,j) = vnew;
end

% Transformation to original dimensions

switch R
    case 'Rf'
        p=exp(t*Rf);
    case 'Ru'
        p=exp(t*Ru);
    case 'Rd'
        p=exp(t*Rd);
    case 'Rh'
        p=exp(t*Rh);
end

t = T-2*tau/sigma^2;
V = K*exp(-.5*(q_delta-1)*x)*exp(-(.25*(q_delta-1)^2+q)*tau)*w;
% re-arrange t and V in increasing time order
  t = fliplr(t);
  V = fliplr(V);
% Define function g as a nested function
function boundary = g(x,tau)

    abbl = exp(((q_delta-1)^2+4*q)*tau/4);
    abbl = exp((q_delta-1)*x/2);
    abbl = exp((q_delta+1)*x/2);

    switch type
        case 'put'
            boundary = abbl.*max(abbl2-abbl3,0);
        case 'call'
            boundary = abbl.*max(abbl3-abbl2,0);
    end

%%% Cranck Nicolson method over an unbounded domain %%%

function [p,t,V] = CN_Over_Unbounded_Domain(K,T,r,delta,sigma,type,N);
eta1=1+1/eta;
eta2=1-eta/1+eta;
for k=1:N
    lambda0=2*sqrt(dtau/pi)*(sqrt(k-1)-sqrt(k));
    for l=1:k
        lambda=2*sqrt(dtau/pi)*(sqrt(k-l-1)-sqrt(k-l));
        lambda1=(u(l+1,M)-u(l,M))/dtau*lamda;
        lambda2=lambda0+lambda1;
    end
    u(k+1,M)-eta1*u(k+1,M-1)=eta2*u(k,M)+eta1*u(k,M-1)+2*eta1*dz*lambda2;
end

%%%%%%%%%%%%%%%%%%%% Free Boundary Problem %%%%%%%%%%%%%
function pf = FreeBoundary(p,t,V,K,type)

    pf = zeros(1,length(t));
    eps = K*1e-5;

    switch type
case 'put'
    for j = 1:length(t)
        pf(j) = S(find(abs(V(:,j)-K+S)< eps, 1, 'last'));
    end
end
case 'call'
    for j = 1:length(t)
        pf(j) = p(find(abs(V(:,j)+K-p)< eps, 1, 'first'));
    end

% By Mr. Mukendi Mpanda/ October 2012 %
References


