Chapter 6
Conclusion and recommendations

6.1 Introduction

The emergence of abstract algebra in history has been observed to reveal a significant pattern in the development of mathematics. The levels of thought involved in the process seem to be significant and reveal a general trend of the development of thought in algebra. Hence careful consideration needs to be paid to the revelations arising from historical investigations so that these may help to contribute towards the encouragement of learning in students of algebra.

Although three distinct levels of development have been identified in the overall development of algebra, each one of these could be broken down into many sublevels. Instead of continually breaking down levels into more and more sublevels, in this study a spiral of learning is being considered. This allows for passage through sublevels right from very early stages and continues upward in different directions to levels high up within the overall abstract level.

The idea of levels of learning has been substantiated by many researchers and investigations undertaken in the past. Some of their findings as well as insights gained from the historical emergence of algebra are being united here to form a comprehensive theory of learning algebra at both secondary and tertiary levels of study.

6.2 The perceptual levels and sublevels of the spiral

6.2.1 The perceptual level in the history of algebra

There is a long history associated with the solution of polynomial equations. In ancient times the Greeks and Romans used impractical numerals and counting methods such as an abacus or counters. The three historical phases of algebra considered by the German GHF Nesselman (1842) were called rhetorical, syncopated and symbolic. The rhetorical level represents a perceptual stage very low down in the spiral of abstract algebra. It could scarcely be recognised as algebra at all except for the fact that by the symbolic stage similar problems began to appear with the introduction of some symbolism.

Francis Viete (1540-1603) was the first to make use of letters not only for unknowns but also for known quantities. His work gives evidence of the advancement of abstract algebra up the spiral. He presented rules for solving equations. These include the transferral of terms from one side of an equation to another and dividing all terms on both sides of an equation by the same thing. Performing these operations suggests the origin of abstract algebra. However, although Viete had clearly covered several rounds of the spiral, the fact that he was dealing with specific types of problems did suggest that he was still at the overall perceptual level of the spiral.

For a long time after Viete algebra was limited to solving equations of various degrees. However, even though it was necessary to pass through the perceptual, conceptual and abstract levels low down the spiral in order to reach methods of solving each type of equation, the fact that each type of problem had its own method of solution suggests that this is still the overall perceptual level.

6.2.2 The perceptual level of learning algebra

In the early stages of learning algebra, children too make use of counters or an abacus in their counting. Piaget believes that they need to coordinate their senses and perceptions to form practical concepts. These concepts may lie within the overall perceptual level of learning but represent an advancement to the abstract level low down the relevant spiral of learning. As a child advances in their study of numbers, he or she becomes capable of using symbolism to represent objects or events.

The perceptual level in one sense applies to very young children but could be applied to the introductory stage of learning of any concept in algebra. Freudenthal emphasises the importance of the first or perceptual level of learning. Although he refers to this as the pre-mathematical stage, he regards it as having vital significance. It is important, nevertheless, that activities performed at the perceptual level are geared towards promoting advancement to a further conceptual level of the spiral.

Van Hiele too believed that algebra should be taught by taking the various learning levels into account. His first level involves non-verbal thoughts and observations. It forms the lowest part of any section of the learning spiral. In his correspondence with Land he regards the perceptual level as one at which problems can be understood by means of being pointed to and which could lead to further development. At Land's perceptual level of exponential functions, different types of graphs were studied in isolation. However, this stage required some advancement up the spiral of learning algebra because the students already had to have background knowledge in order to make use of variables, understand experiments and plot graphs.

The perceptual level of Nixon also involves plenty of visual representations. Studying these at each level formed her introduction to any topic or sub-topic at various levels of learning. Even at an advanced level where visualisation may involve diagrammatic representations, the perceptual level seems to be significant. Vinner and others also regard concept image as being very important in mathematics. It helps students create mental pictures of concepts. As the new topics are studied in abstract algebra, previously learned concepts may form the relevant background to establish concept images leading to new concepts.

The perceptual level of learning seems to play a significant part of the learning of any algebraic topic. It involves the establishment of concept images which are necessary for both the understanding of topics and further advancement up the spiral of learning algebra.

6.2.3 The parallelism between the perceptual levels of the historical and learning spirals of abstract algebra

Right from the beginning of algebra in ancient times such as when counters were used by ancient Greeks and Romans, a parallelism may be seen in the way in which concepts developed historically and in the minds of learners. Initially there was plenty of evidence of the "a posteriori" development of mathematics. This initially involves real world experience leading to mathematics but later involves mathematical experience and the analysis of logically unrelated statements. Later also the "a priori" or constructive development involving mathematics constructed from other mathematics by such activities as adding, deleting or replacing axioms emerged and still continues to grow today. However, the historical path of development need not be followed slavishly. Sometimes different illustrations or applications of the concept in question may help to build up the concept in a way it could have happened historically.

As was the case in ancient times, students first need to build up the image of a concept in order to associate a name or symbol with it. Sfard (1991:13) also recognises the cyclic process of the development of numbers. At the first stage mathematicians had to get used to certain operations on already known concrete objects or numbers. This reflects the way in which students develop algebraic concepts.

As abstract levels within the overall perceptual level of algebra were reached, new rules of operating with various objects were established. This can also be seen in the development of mathematical concepts in the minds of students. Just as in Viete's day when rules involving transposing terms from one side of an equation to another and dividing through equations by the same quantity were developed, so students can be led to discover how to solve simple first degree equations.

Following a similar pattern to the one involving first degree equations, students can consider the solution of those of a higher degree. As in ancient times these are dealt with separately and individually, each time the student is led to a way of solving an equation of a particular degree. The fact that each type is dealt with in isolation is a characteristic of the perceptual level. Every time a new method for a particular type of equation is encountered, a new abstract level within the overall perceptual level is reached. Dealing with each type of equation in isolation forms a necessary part of the perceptual level and serves to form a basis for further development of the overall conceptual level.

6.3 The conceptual levels and sublevels of the spiral

6.3.1 The conceptual levels in the history of algebra

During the seventeenth and the first half of the eighteenth centuries very little progress was made towards the development of the theory of groups. Then mathematicians began to pay attention to the new instrument of infinitesimal calculus created by Leibniz and Newton. It was when mathematicians like Euler, Lagrange and Gauss made use of this calculus that the overall perceptual level of algebra emerged at the middle of the eighteenth century.

The overall conceptual level of algebra was a time when certain mathematicians paid more attention to the study of numbers which would solve various sorts of equations. Lagrange was considered by Piaget and Garcia (1989:155) as being the main figure in the transition from the overall perceptual to the conceptual level of algebra. This is because he adopted a more general scope when he considered the nature of the solution of third and fourth degree equations and pondered over their success. He managed to show that all methods involved reducing the original equation and seeking the relationship between the reduced equation and the original one. When he started considering how many different values a polynomial is able to take when it is permuted in all possible ways, he was beginning to advance up the overall conceptual level of the spiral and was laying foundations for the subsequent abstract level.

Other mathematicians continued to work on the problem of a general method of solving equations and then Carl Frederich Gauss, who was born in 1777 in Germany, gave the first completely satisfactory proof of the fundamental theorem of algebra in 1797. He proved that "for any polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ with complex coefficients, $n \geq 1$ and $a_n \neq 0$, there is a complex number r such that $p(r) = 0$" (Temple 1981:385). One of the most original points of Gauss' work came to be known as the composition of forms. This was the very first operation that had ever been introduced into a non-numerical domain with properties that could not immediately be deduced from operations on numbers.

Transformations dominated algebra for a long time before the emergence of abstract algebra. Abel was another mathematician who greatly contributed to the advancement of algebra. He at first thought he could solve the quintic equation or polynomial of degree 5 but then found he could not. He reached the peak of the overall perceptual level when he began to work on the problem of whether or not a given equation could be solved by radicals.

Historians have expressed their amazement that Gauss and others were able to come so close to the concept of a group but were never able to actually define it. This only happened at the overall abstract level.

6.3.2 The conceptual level of learning algebra

According to Piaget and Garcia (1989:x) this level may be considered as the one at which there is a shift away from analysing objects to considering the relations or transformations that can be found between them. At this level there is a reconstruction or reorganisation of what arose at the previous level. Although it may become possible to explain how something happens, it may not be possible to explain why it happens. Students can reach conceptual levels at different rounds of the spiral occurring at the overall perceptual, conceptual and abstract levels of development.

Freudenthal (1973:123) regards mathematical activity, as one of organising various fields of experience and this is what happens at the perceptual level. He stresses how important it is for students to be involved in the establishment of definitions and theory. For otherwise he believes that they would have been forced to skip over a significant step in the learning process. Re-invention is a very important part of his theory and plays an important role at the perceptual level of learning.

Freudenthal's ideas of the development of mathematical concepts are not as rigid as those of van Hiele. He perceives mathematics as continuously growing in various directions, which strongly suggests the spiral theory. The van Hieles also saw the conceptual level as one at which students were able to find interrelationships of properties to be found both within and amongst figures or other concepts being taught. However, they felt that definitions belonged to the conceptual level and theorems to the abstract level. This is not necessary in the spiral theory because both definitions and theorems form part of the abstract level but at different parts of the spiral. Land's theory of levels is similar to van Hiele's. She observes that it is difficult for students to rise to the conceptual level when so much necessary groundwork is lacking at the perceptual level. This also suggests that they are lacking other relevant previous rounds of the spiral.

In Nixon's study, the exploring of patterns was encouraged in the teaching of every topic or sub-topic in order to encourage the analysing of interrelationships between concepts at the perceptual level. Students claimed that they were becoming accustomed to searching for relationships and became eager to look for them whenever a new topic was presented.

According to the concept understanding scheme of Vinner and others, concept images are essential for the understanding of definitions and the construction of proofs. At the conceptual level it is also necessary to take time to establish properties as well as consider the relevant relationships and inter-relationships. The conceptual level of learning forms a very important link between the perceptual and abstract levels of learning. If students are expected to progress straight from the perceptual to the abstract levels of learning, then a significant level has been omitted and further development up the spiral will be hindered.

6.3.3 The parallelism between the conceptual levels of the historical and learning spirals of abstract algebra

The fact that both the perceptual and conceptual levels of abstract algebra lasted for long periods in history shows the importance of spending time passing through these levels in the acquisition of algebraic concepts. Nevertheless, the student obviously has considerably less time to dwell on and develop these concepts. However, history gives an indication that passing through these levels is an important contributory factor towards mathematical development.

As the conceptual level emerges, there is a change in emphasis from isolated examples to considering a more general scope of possibilities. It is a time to analyse and compare the situations arising from different cases. Students need to be made aware of the problem and be involved in the solving of it so that they can appreciate the results arrived at when the abstract level is reached.

Any new topic with which they are presented may depend on many preceding rounds of the spiral as certainly has always been the case in history. Learning algebra could become more relevant to them if they become aware of relevant preceding rounds of the spiral and the background involved in passing through the appropriate conceptual level of the topic at hand.

6.4 The abstract levels and sublevels of the spiral

6.4.1 The abstract levels in the history of algebra

The abstract level of abstract algebra was reached in the middle of the nineteenth century after Galois had defined the concept of a group. Freudenthal (1975:35) noted that by this time a vast stack of groups had in fact accumulated but all of these dealt with separate cases and group axioms had not as yet been established. Their discovery was a great achievement because they served as an instrument that could be used in many situations.

Galois managed to reach the overall abstract level of algebra before others who were working on the same problem. His definition of a group resulted from studying to find a way of distinguishing which polynomial equations were solvable by a formula and which were not. His main idea was to look at the symmetries of the polynomial. His theory has application in many areas of mathematics since so many problems in mathematics reduce to the solution of polynomial equations.

Piaget and Garcia (1989:156) see the second half of the nineteenth century as a period of great “leaps” in mathematics, particularly algebra. The spiral of group theory continued to spread out in other directions on the abstract level. Dedekind studied and gave structure to the concept of a field in 1910. Not only rational, real or complex numbers need to be used in the definition of a field but many entities could be found that also constitute fields. David Hilbert (1862–1943) defined a ring and many other mathematicians including Emmy Noether (1882–1935) continued to climb the spiral of abstract algebra within the overall abstract level.

6.4.2 The abstract level of learning algebra

Piaget and Garcia (1989) observe how the abstract level can be considered to be a function of what precedes. However, the identification of three different stages in algebra is more complex because the process of algebraization in mathematics does in itself represent an abstract stage. Freudenthal (1973:123) sees the abstract level as a time of putting results into a “linguistic pattern”. As students move up to the abstract level, they organise results acquired at the perceptual level. Freudenthal favours a broad approach which takes all previous rounds of the spiral into consideration rather than merely placing a pyramid on the top of whatever has come before.

Van Hiele regards the third level as a time when theorems may be established within an axiomatic system. Although he believes that definitions belong to a lower level, here they are regarded as part of the abstract level but lower down the part of a spiral leading up to the theorems dependent on them. Although reference models may not be needed for geometry, mathematical objects may form the objects of study at the perceptual level. This could suggest the “a priori” or “a posteriori” development of mathematics in history. Land’s uppermost level corresponds to the abstract level and to van Hiele’s highest level. Students become able to use symbols with insight and understanding in order to construct proofs and they are able to understand the importance of deductions, axioms, postulates and proofs.

In Nixon’s study each topic was subdivided into smaller subtopics and every time students were guided to pass through the perceptual, conceptual and abstract levels. They were encouraged to generalise and became more and more ready to do so. In fact, whenever they were presented with a new topic, they were in a hurry to draw comparisons, seek generalisations and establish results at the abstract level. They observed that they appreciated results they themselves had discovered far more than those presented to them in advance.

According to the concept understanding scheme, definitions form part of the abstract level of development. Vinner in Tall (1991) has pointed out that there is often a misconception that “concepts are mainly acquired by means of their definitions”. However, the concept image acquired at the conceptual stage is important for the understanding of a concept. Concept usage, which follows once an abstract level is reached and an adequate definition has been formulated, can form part of the perceptual level of a new round of the spiral.

6.4.3 The parallelism between the abstract levels of the historical and learning spirals of abstract algebra

The abstract level clearly plays a very vital role both in the historical development of algebra and the learning of the subject by students. De Villiers (1986:4) points out “the importance of axiomatisation in modern mathematics and its implication for teaching”. It is important that students are not merely presented with the final form of the axioms but become involved in their establishment.

In history the definition of a group arose from a vast stock of examples that came before it. The student of algebra too needs a variety of examples so that he or she can extract the vital information and learn not to limit results to only one situation. As in the case of Galois, they could study symmetries and learn to appreciate the concept of a group as being applicable to many situations in life. If they are led to the definition of a group or other algebraic topics through a variety of relevant perceptual and conceptual activities then it would become possible for them to appreciate what an exciting leap there is when the relevant abstract level is finally reached.

There is a danger that students of algebra are often presented with results without having passed through the previous rounds of the spiral. As was the case in history, they need to follow a broad approach in order to establish results on the abstract level. If they follow the outlines of the historical development or the way in which the topic could have developed historically then they can learn to appreciate what they are doing as a function of what precedes. Eventually, after much experience of passing through levels, they could hopefully reach the point of following a mathematics textbook on their own.

6.5 The implications for a comprehensive learning theory of algebra at all levels

Freudenthal (1978: 198) observes that “Teaching is the intentional promotion of learning processes”. The spiral theory seems to very aptly describe or represent both the growth of algebra since ancient times and the development of knowledge in students of algebra beginning from the very early stages. Just as each round of the spiral being covered in history led to reorganisation of knowledge and new growth in various directions, so too can students of algebra advance up the spiral of learning if they are given the opportunity to do so. Several important aspects of this theory were highlighted in Chapter 5 and these will be summarised here.

6.5.1 The link between historical and epistemological levels of learning

The link between the historical and epistemological development of algebra is something which could be borne in mind in the presentation of a topic. The methods, mistakes and problems encountered in the past could also serve to shed some light on how students acquire algebraic concepts. In history it can be seen that growth was stimulated by problems and likewise in students growth can be encouraged by appropriate problems to motivate the development of more theory. Either historical problems or, making use of the advantage of hindsight, more suitable problems could be chosen for the purpose.

6.5.2 Taking the perceptual, conceptual and abstract levels into consideration

Since the perceptual, conceptual and abstract levels have all been identified as levels of learning both historically and in the minds of learners, these should be taken into consideration in dealing with a topic. Students can be encouraged to pass through all of these levels as a result of the careful planning of the teacher or lecturer. Promoting passage through all of these levels is important because “pupil’s knowledge will remain a chaotic multiplicity if we do not succeed in lifting it out of the purely observational levels of thought” (Duminy & Sőhnge 1990:202).

It is important that the students become motivated by being made aware of the need for and relevance of groups, fields and abstract algebra in general. Merely beginning with definitions and axioms on the abstract level would not help the majority of students appreciate the beauty and structure of a group. However, providing various examples including games, geometric shapes and numbers as well as making them aware of some relevant applications of the theory would help make it more meaningful to them. Merely being told that what they are doing will be relevant later does not usually satisfy them. De Villiers, in his correspondence with Nixon (2005), refers to such activities as teaching factorisation before solving quadratic equations and completing the square before encountering a non-factorisable quadratic as “decontextualized” teaching. Alternatively, either historical problems or carefully chosen ones that might have led to the theory would help them see the value of proceeding further with the development and study of the appropriate theory.

6.5.3 The concept of an ever-growing spiral of levels

The spiral image helps to efficiently reflect the way in which knowledge continually grows where each round of the spiral contains a perceptual, conceptual and abstract level. The way in which each perceptual level arises from a previous abstract level gives an indication of how knowledge is reorganised and expanded as the spiral is climbed.

6.5.4 Analysing the spiral of levels of learning in the presentation of a topic

Teachers and lecturers of algebra need to take the relevant lower regions of the spiral into consideration in their presentation of a topic. In the case of algebra, symbolism would be one of the relevant low rounds of the spiral. The teacher would need to consider which of these topics need to be covered to a greater or lesser extent, depending on the background knowledge of the students. This exercise might be time consuming but would help to ensure that there are no serious gaps prohibiting further progress up the spiral.

6.5.5 The "a posteriori" or "a priori" development of mathematics approaches

Both the "a posteriori" and the "a priori" approaches to the historical development of mathematics are relevant and help to lead the student through the way in which concepts were originally developed. Following them or finding an approach that reflects that mode of thought would help students to follow the line of thinking and appreciate the development of mathematical theories. In the "a posteriori" or the descriptive historical development, real world (or mathematical) experiences and logically unrelated statements are analysed and lead to the establishment of results at the abstract level. However, in the "a priori" or constructive development existing axioms are varied and investigated in order to lead to new theorems. This shows how, as the spiral is climbed, existing axioms can form part of the perceptual level and, after passage through the perceptual level, new results may be formed at the abstract level.

6.5.6 The importance of not omitting stages of rounds or rounds of a spiral

When algebra is being taught, it is evidently of vital importance that the perceptual, conceptual and abstract levels are passed through at every round and that no essential rounds of the spiral are missing from the past. The omission of relevant previous rounds or levels at the current round would leave serious gaps in the progress of students and prevent them from continuing to successfully climb up the spiral. Duminy and Sőhnge (1990:202) observe: "What is really important, however, is the schematic formation of these facts into a system and then application of that system to achieve even higher ordination and arrangements". In the development of learning, the constant application of this principle leads to comprehensive and useful thinking processes.

6.5.7 The provision of opportunities of visualisation, exploring patterns and generalisation

In the study conducted by Nixon (2002) it became clear that providing activities involving visualisation and exploring patterns as well as encouraging generalisation helped to promote advancement through the perceptual, conceptual and abstract levels. Whenever a new topic or sub-topic is taught in algebra, a teacher could give careful consideration to incorporating activities of visualisation at the perceptual level, exploring patterns at the conceptual level and generalisation at the abstract level until the students become capable of establishing their own concept images.

6.6 Recommendations

Taking into consideration all the findings emerging from this study, the following recommendations could be made in this context:

- The link between the historical and epistemological development of algebra be carefully considered in order to shed some light on how students of algebra develop conceptual understanding. The perceptual, conceptual and abstract levels of algebra that may be detected in a topic's historical development be considered in its presentation to students. In addition the "a posteriori" or descriptive or "a priori" or constructive developmental historical line of approaches be followed to help students follow the historical or possible historical line of approaches that led to a topic.
- The spiral theory of perceptual, conceptual and abstract levels continually being repeated as the spiral is traversed should be taken into account at all levels. A teacher or lecturer of algebra should carefully consider the relevant rounds of a spiral preceding any topic to be presented.
- An effort should be made to carefully ensure that no relevant previous rounds of the spiral are lacking or that neither the perceptual, conceptual nor abstract level of the present round is being omitted. To promote this,

opportunities of visualisation, exploring patterns and generalisation should be provided at the perceptual, conceptual and abstract levels respectively.

Further research which could be relevant in this area could be:

- Researching both the perceptual, conceptual and abstract levels involved in the historical development of definitions, axiomatisation and proof in various topics of mathematics and the spiral approach in the learning of various other topics of mathematics apart from algebra.
- Carefully analysing and identifying key concepts in elementary algebra and abstract algebra and conducting studies at various levels of algebra to assess the effectiveness of the spiral level approach. In addition, carrying out subsequent qualitative and quantitative studies of school and university students' intuitions and misconceptions with regard to key concepts would be appropriate.
- Assessing the worth of the spiral level theory when started at the very beginning of studying algebra and observing the value of the spiral level approach when applied for the first time at university level. Furthermore, to determine whether in general, once students have become accustomed to following the spiral approach for a reasonable length of time, they become independent and are able to continue climbing the spiral on their own without needing continual input from the teacher or lecturer.