Chapter 5

A spiral theory of learning algebra

5.1 Introduction

Historically algebra began with a very restricted definition based on the concept of "arithmus". Viete synthesised the geometrical analysis of Pappus and the arithmetical methods of Diophantus. Thereafter, as a result of Viete's introduction of general symbols, a transition took place. Algebra came to be regarded as a new independent discipline.

Once algebra had become established as a separate subject, the solution of equations became its central theme. Three levels have been associated with its development into a study of structures. The first level, known as the intraoperational or perceptual level, involves the search for solutions of specific equations, each regarded as separate objects. The methods used were empirical and trial-and-error, characteristic of the perceptual level. However, these gradually became more theoretical though specific to certain classes of equations. Since mathematicians at this stage were so occupied with finding algorithms to solve various classes of equations, de Villiers (personal communication, 2005) believes that this could be called the "algorithmic stage". This initial period lasted for a long time. Only from the middle of the eighteenth century did the interoperational or conceptual level arise. This was when mathematicians began to seek more general methods of solving equations and to formulate the general problem of whether or not solutions exist. Equations began to be transformed from an unsolvable form to a reduced solvable form. Lagrange and Gauss were two very important mathematicians during this period.

Galois was the mathematician responsible for introducing the transoperational or abstract level of algebra. The theory of groups that he developed was the first thematised structure in mathematics. This led to the end of the theory of solving equations and the beginning of a new stage where structures predominate. As abstract algebra continued to grow, each further new development seemed to be reached after passing through one or several rounds of the spiral. Each round involves passing through the intraoperational or perceptual level, the interoperational or conceptual level and the transoperational or abstract level.

Even though the overall development of algebra may be considered to be subdivided into three levels, each of these levels could be broken up into many sublevels. Considering the spiral approach to learning, the overall perceptual, conceptual and abstract levels could all be broken down into many sublevels or rounds of the spiral. Even in antiquity it was necessary to pass through these three levels in order to represent a concrete number with a general symbol. Nevertheless this development took place within the overall perceptual level of algebra.

It seems that the historical development of algebra could reflect the way in which algebraic concepts develop in the minds of students. Two modes of historical development have been noted here. One of these is the "a posteriori" or descriptive development of mathematics which arises historically when real world or mathematical experiences and analysis of logically unrelated statements lead to mathematic theories. The other development can be described as "a priori" or constructive and relates to mathematics being created from, for example, the variation of existing axioms and establishing new mathematics. At a later stage it may be found that constructively developed mathematics does indeed have application in the real world. The "a posteriori" development is connected to both ancient and modern mathematics while "a priori" development began to emerge from the nineteenth century and is still prevalent in modern mathematics today.

5.2 The parallelism between the historical development of mathematics and some stage theories of learning

The three main stages of the historical development of the solution of algebraic equations have been studied in detail in Chapter 2 and briefly summarised in 5.1. As the spiral of learning is climbed, the three levels termed the intraoperational or perceptual level, the interoperational or conceptual level and the transoperational or abstract level can be seen to be repeated over and over again within each of the three overall levels. Piaget and Garcia (1989:167) observe how "Each of these trends reveals the same mechanisms that were identified for general development". For the pattern always seem to be the "discovery of transformations followed by the interpretation of such transformations as manifestations of a total structure" (Piaget & Garcia 1989:167).

In previous chapters the three levels have been identified as forming part of several different theories of learning. They have been seen to play an important part

in the forming of concepts and theories in history. Both in history and in the minds of individuals, knowledge seems to be built up as the rounds of a spiral. In each of these cases what appears to be relevant is "… a regular sequence of sublevels for each new construction" (Piaget & Garcia 1989:167). Another aspect that seems to be shared by both the historical and psychological situations is "…the way in which previous acquisitions are re-interpreted from the perspective of the newly attained stage" (Piaget & Garcia 1989:167). The consideration of the relationship between the historical and epistemological development of concepts seems to warrant serious consideration. Piaget and Garcia (1989:169) note how "The notion of a sequence of three stages, of the intra, inter, and trans variety, poses epistemological problems which necessitate a detailed analysis". Historically the first stage of algebra lasted very long and there was also a relatively long gap between the second and third stages. One reason for this could be ascribed to the fact that "structures require both a greater degree of reflective (or thematizing) abstraction and a more nearly complete generalization" (Piaget & Garcia 1989:169). This does show the importance of the first and second levels of learning preceding the third one. Sufficient time needs to be given to introducing new topics at the perceptual level so that concepts can be formed at the conceptual level. Furthermore, various opportunities of reflection and abstraction should be provided in order to facilitate the climb to the abstract level. This notion would apply to every round of the spiral, not merely to the overall three levels of a topic. Nixon (2002) made use of visualisation at the perceptual level, exploring patterns at the conceptual level and generalisation at the abstract level whenever a new section of a topic was introduced.

Piaget and Garcia (1989:171) observe how at every level of the spiral what can be found is "reflective thematization" which may be described as "… an exhaustive conceptualization of progressively constructed mathematical objects, and this even before such representational intuitions were developed into axioms". This suggests the image of a spiral growing up in many directions, with perceptual, conceptual and abstract levels being repeated at each round. Not only could this represent the growth of knowledge in history but also in a student as he/she expands his/her mathematical knowledge and experience.

Historically, growth took place in algebra when various mathematicians worked on the problems which predominated during that period of time. In the student too, growth comes about as a result of his/her activity and involvement in the learning process. A student grows as he/she works on problems that have been

presented to him or her. Piaget and Garcia (1989:172) observe that pupils' actions and manipulations lead to organisation of what they term "pre-structures" or "pre-algebraic systems", the simplest of which are groupings such as classification and seriation. The importance of these elementary systems is not only the fact that they form a round very low down on the spiral of abstract algebra but also that they are acquired through passing through the intra, inter and trans sublevels or the perceptual, conceptual and abstract levels situated in the nether regions of the learning spiral.

Although much time elapsed between the evolvement of the overall intra and interoperational levels as well as the inter and transoperational levels in history, many sublevels did exist during these periods. Thus the comparatively short time period in a student's growth from one stage to the next can be likened to sublevels within a particular level or rounds of the spiral within a certain stage of development. The characteristics associated with each of the three levels described in history certainly do seem to be similar to those associated with the learning levels of several learning theories that have been studied here.

Sawyer (1959:2) tried to teach students how modern algebra grows out of traditional elementary algebra. Freudenthal (1973:34) pointed out how the history of group theory shows how axiomatic systems arise. Freudenthal (1973) saw levels of learning in the emergence of group theory. Bell (1945) drew parallels between the development of history and the van Hiele levels of thought. Hull (1969:2) pointed out that the history of past mathematical discovery suggests a pattern of individual learning. Jones (NCTM 1989:15) also describes the historical development in mathematics in such a way that it seems comparable to a spiral, passing through many rounds or series of stages as it continues to expand in various directions:

> *The frequent occurrence of simultaneous discoveries in mathematics illustrates the growing and maturing nature of mathematical knowledge and the fact that frequently new discoveries are generated by earlier ones. Not only are new discoveries generated by earlier ones, but often earlier ones are so necessary as a preparation for the next stages that when the preparatory stages have been completed a number of persons will see the next step.*

Despite the evident link between the stages in the historical development of algebra and the associated learning levels, teaching a concept in the same order as it developed in history is not always necessarily the best path to follow. For example, de Villiers' example mentioned in the first chapter showed how he was able to

successfully teach a topic by changing the historical order. Group theory too has many applications and the students could be given the opportunity of trying to derive the relevant axioms if they were given the chance to examine various suitable examples. Bell (1945:212) observes that "Groups also may be derived from common algebra by the same technique of generalisation. But they were not so obtained originally".

In response to a question posed regarding how history can be of use to a researcher of mathematics education, Fauvel and van Maanen (1997:256) makes the following observations: To analyse the view "… that the development of an individual's mathematical understanding follows the historical development of mathematical ideas – may be appropriate". Smith (1958b:iii, iv) commented on the benefit of a student being able to appreciate the ongoing historical growth of the solution of equations and the development of numbers:

> In algebra he will see, partly by means of facsimiles, how the symbolism has grown, how the equation looked three thousand years ago, the way its method of expression has changed from age to age,

and later he adds that

> He will learn how the number concept has enlarged as new needs have manifested themselves, and how the world struggled with fractions and with the mysteries of such artificial forms as the negative and the imaginary number, and will thus have a clearer vision of mathematics as a growing science.

Cooke (1997:6) observes how number and shape were the prototypes of mathematics, both stemming from the human tendency to compare things and rank them in order. As the theory of equations developed, so new numbers had to be introduced to make the equations solvable.

The concept understanding scheme of Vinner, Tall and others emphasises the necessity of concept image leading to concept definition. It seems to be the input of example, or experiences at the perceptual level that leads to concept image at the conceptual level and concept definition at the abstract level. Thereafter continued growth up the spiral can take place in a similar manner, where definitions can form part of the perceptual level and lead to theorems and other applications at the abstract level. Relating their historical triads of intra, inter and transoperational levels and sublevels to the growth of learning in people, Piaget and Garcia (1989:173) remark:

If our triads thus contain nested – triadic sublevels, then there is no reason why this should not also be the case during that important pre-algebraic period when the subjects as yet incapable of systematic thematization, nevertheless constructs, on the level of action and practical know-how, what the observer can interpret only as a progressive formation of structures.

The diagram below shows how number systems developed in history:
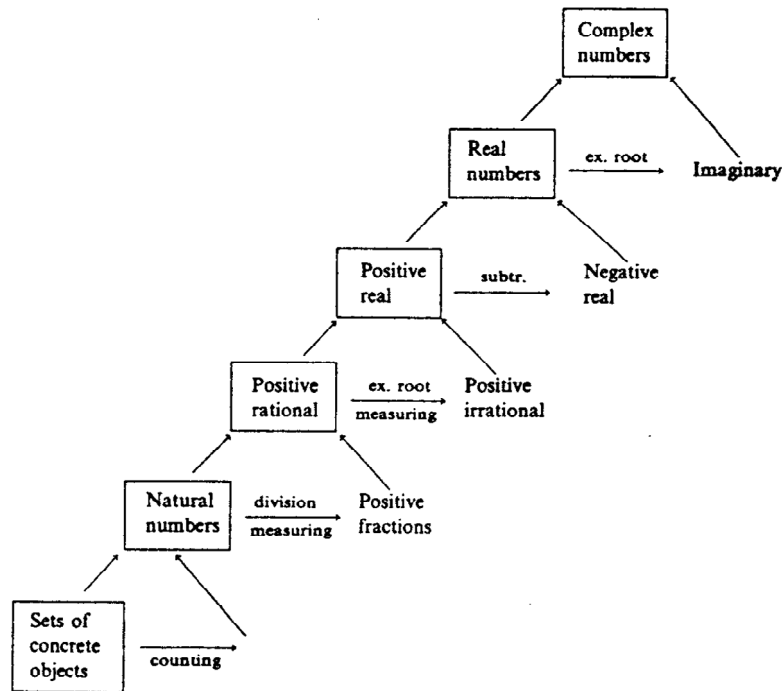
Development of the concept of number.   (Sfard 1991:13)

5.3   The characteristics associated with the perceptual, conceptual and abstract levels

5.3.1 Introduction

It seems evident that not only each relevant round of the spiral preceding a concept is necessary but also each stage of a particular round of a spiral is essential too. The type of characteristics identified with each of the perceptual, conceptual and abstract levels of the spiral will be considered in this sub-section of the chapter.

According to Gagne's (1970) theory of instruction:

The assumption is that a complex task can be analysed into a hierarchy of sub-tasks, and that failure to perform the complex tasks can be traced to lack of confidence in one or more of the subtasks. These in turn can be similarly analysed until one

reaches sub-skills which are within the learner's previous competence (Bell, Castello & Küchmann 1983:177).

By analyzing the errors of students in a previous test, Trembath and White (1978) established the following hierarchy designed to take the student upwards through the hierarchy of concepts in the section of calculus they were studying:
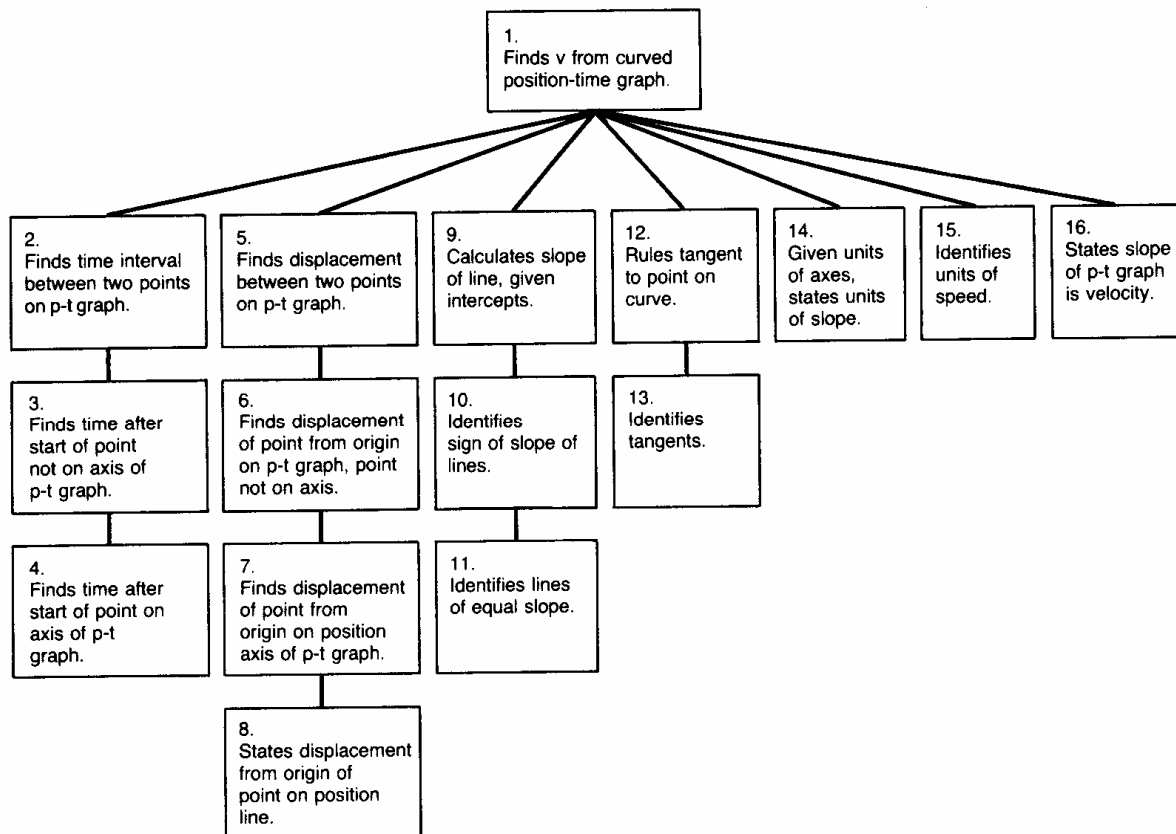
(Bell, Costello & Küchemann 1983:177)

Similarly, lecturers and teachers of algebra could carefully consider previous, current and subsequent rounds of the spiral regarding topics they are teaching their students.

5.3.2 The perceptual level

Whether concerning the historical development of algebra or the development of thought in students, the perceptual level refers to the early stage of a round of the spiral. It is the initial stage of each new concept during which plenty of input involving relevant objects or experiences is essential.

Piaget is very adamant that children should be provided with concrete materials and encourages the idea of their engaging in free play in the classroom. His intraoperational level and sublevels also involve experimenting with isolated

examples. Freudenthal (1973:127) also stresses the importance of the bottom level of learning and points out that its omission is one of the mistakes that has occurred in traditional education. He also describes it as a vital stepping-stone to the second level but sees it as pointless if it is not directed towards reaching the conceptual stage (Freudenthal 1973:68). De Villiers (2005) in his correspondence with the writer has pointed out that one of the criticisms that could be levelled at the Realistic Maths movement in the Netherlands is that they do not "reach or even attempt to reach the abstract level".
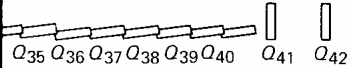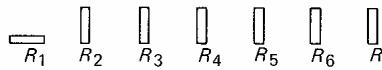
At the perceptual level, as a result of perceiving varying inputs, certain invariant properties persist in the mind and can lead to an abstraction. This can in turn head in the direction of a "second-order abstraction" where the properties become less perceptual and more functional. Thus Skemp (1971:20) observes that:

"A concept therefore requires for its formation a number of experiences which have something in common". He also claims that low order concepts do not require language. As concepts of a higher order are reached, they cannot be acquired by means of a definition. Instead they can be conveyed "… by arranging for students to encounter a suitable collection of examples" (Skemp 1971:30). Particularly in mathematics, "… these examples are almost invariably other concepts" so that " it must first be ensured that these are already formed in the mind of the learner" (Skemp 1971:30). The importance of passing through every round of the spiral is emphasized by Skemp (1971:33) in his comment "if a particular level is imperfectly understood, everything from then on is peril".

At van Hiele's perceptual level, the percepts consist of geometric shapes and figures low down the spiral. Students are able to recognise them but are not as yet able to identify their properties. Further up the spiral, pupils may have come to recognise the properties of various figures but would still be in the overall perceptual level until they stop considering figures in isolation and begin to compare their properties. Van Hiele (1959:15) did suggest three stages involved in promoting development from one level to another. He described the first of those as being a time when the symbols of the relevant field of study need to be developed. As far as algebra is concerned, van Hiele did suggest to Land (1990:131, 132) that she include a basic level where objects and situations can be pointed out or observed. When Nixon (2002) utilised Land's levels in the teaching of sequences and series in 2000, numerous examples of visualisation were provided to initiate the formation of concepts at each of van Hiele's levels. Visualisation can be very effectively used to

develop high level concepts or, in terms of the spiral analogy, concepts occurring high up the spiral. For example, the following example serves as a very good introduction to proof by mathematical induction:

Why They Fall and Why They Don't

|	
$P_n : \dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$ P_1 is true $P_k \Rightarrow P_{k+1}$	 $P_1 P_2 P_3 P_4 P_5 P_6 \cdots$ First domino is pushed over. Each falling domino pushes over the next one.
$Q_n : n^2 - n + 41$ is prime. $Q_1, Q_2, \ldots, Q_{40}$ are true. $Q_k \not\Rightarrow Q_{k+1}$	 $Q_{35}\, Q_{36}\, Q_{37}\, Q_{38}\, Q_{39}\, Q_{40} \quad Q_{41} \quad Q_{42}$ First 40 dominoes are pushed over. 41st domino remains standing.
$R_n : (a+b)^n = a^n + b^n$ R_1 is true. $R_k \not\Rightarrow R_{k+1}$	 $R_1 \quad R_2 \quad R_3 \quad R_4 \quad R_5 \quad R_6 \quad R_7$ First domino is pushed over but dominoes are spaced too far apart to push each other over.
$S_n : 1 + 2 + 3 + \ldots + n = \dfrac{n^2 + n - 6}{2}$ S_1 is false. $S_k \Rightarrow S_{k+1}$	 $S_1 S_2 S_3 S_4 S_5$ Spacing is just right but no one can push over the first domino.

(Fleming & Varberg 1989:429)

As mentioned previously, van Hiele did once claim in his correspondence with Land that algebra has no visual level. But he did add that it would be possible to add a first column where pointing could serve to identify introductory aspects. However, algebra comes from an abstraction from number systems and there are pictures which are commonly associated with number systems. The set of natural numbers $\mathbb{N}$ or {0; 1; 2; 3; …} can be presented on a number line as follows:
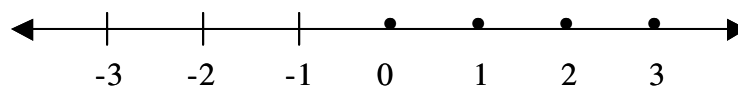

whilst the set of integers $\mathbb{Z}$ consisting of the numbers {… -2; -1; 0; 1; 2; …} can be depicted in the following manner.
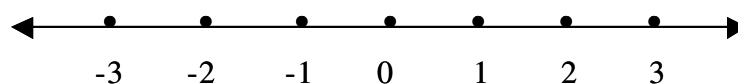
Both $\mathbb{Q}$ and $\mathbb{R}$ require number lines to represent them as follows:
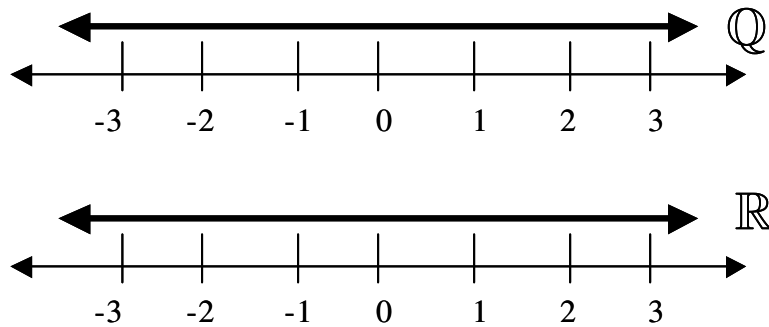
This is because the density property of rational numbers indicates that between any two rational numbers another one can always be found. However, there must be some gaps on the number line for $\mathbb{Q}$ to accommodate the irrational numbers on the number line representing $\mathbb{R}$. Furthermore, the set of complex numbers $\mathbb{C}$ can be represented in the Cartesian plane where the complex number $a + bi$ is indicated below:
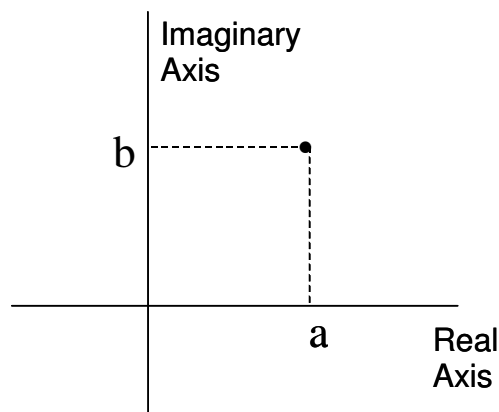
These very basic visual representations of number systems strongly suggest a perceptual level in algebra and seem to refute van Hiele's suggestion that there is no visual level in this subject. Vinner, Tall and others have a concept understanding scheme which involves concept image, concept definition and concept usage. At the first level input is required in order to establish a concept image at the second level. Thus the perceptual level incorporates relevant input used to establish a concept. Low down the spiral physical input could be necessary but higher up concept usage could serve as input. Concept usage incorporates definitions, axioms, theorems or practical applications all arising in the form of output from lower levels.

In the overall perceptual stage of algebraic equations many different types of a variety of equations were studied in isolation without consideration being given to

overall trends. Likewise in any specific topic in algebra, a new round of the spiral should be entered by studying appropriate instances which illustrate the relevant characteristics. This is an activity which needs to be organised by the teacher. Lower down the spiral, the examples used tend to involve physical, real world or mathematical objects and suggest the initial stage of a type of "a posteriori" or "descriptive" development. However, at a perceptual level further up the spiral, not only are mathematical experiences analysed but such activities as altering, replacing or deleting of axioms might take place. This is a situation of creating more mathematics out of mathematics and hence could be described as being of an "a priori" or constructive nature. The percepts may be definitions or even theorems. However, it is very important that, at whatever round of the spiral it is, students are given the opportunity of seeing and operating with the relevant percepts.

5.3.3 The conceptual level

The conceptual level is the second important level associated both with the historical development of algebra and the learning by students at any round of the spiral. This is the stage at which comparisons are made, common properties are sought and relationships are established.

Piaget and Garcia (1989:24,25) point out that at the interoperational or conceptual level, it is possible to know how something works without knowing the theory behind it. Correspondences and invariants are important at this stage. Freudenthal (1973:122,123) sees the conceptual level as a time of organising and reflecting about what has been experienced at the perceptual level. Thus the operational matter of the lower level becomes the object of analysis at the higher level.

Van Hiele has at times been criticised for the rigid inflexible nature of his levels. In both his original and later theories, formal definitions were considered to be part of the middle stage of development. However, the formal statement of definitions does involve a high level of abstraction and so is considered to form part of the abstract rather than the conceptual level here. It is nevertheless also true that definitions might also go through stages. For example, there has been a gradual historical evolution of the function concept to the modern set-theoretic one. Thus at the conceptual level it is possible for more intuitive and less formal type of definitions to be formed leading up to the formal ones at the abstract level.

Van Hiele does also regard relationships as being significant at the intermediate level. He believes that students are able to understand the relationship between the properties of different figures. In the spiral theory low down the spiral the formation of concepts would also have formed part of the perceptual level. As far as van Hiele's suggested three stages regarding moving from one level to the next are concerned, the second level involves a time of examining the properties and connections so that the student becomes aware of relevant relationships.

When Land adapted van Hiele's theory to teach functions, she did consider relationships and properties of exponential and logarithmic functions to form part of the intermediate stage. Solving equations could form part of this level but higher up the spiral once the appropriate concepts had been attained. In Nixon's study conducted regarding the formation of concepts regarding sequences and series, recognising relationships between different types of sequences and series was considered to be an intermediate level characteristic. The following activities also formed part of the conceptual level once the appropriate round of the spiral was reached: understanding statements relating properties of sequences and series, solving equations involving manipulation of symbols; formulating statements showing interrelationships between symbols. Exploring patterns is the type of activity that seems to encourage conceptual level thinking. Many such activities were organised by Nixon in her study to encourage students to rise to the relevant conceptual level. The illustration below provides such an example:
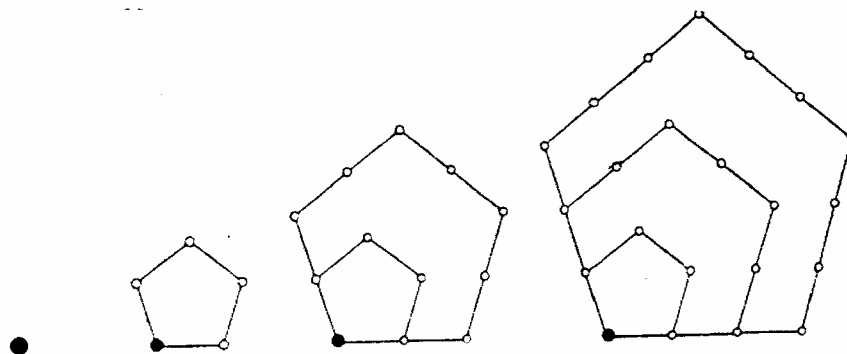
(Reid 1992:77 in Nixon 2002:LXXVII)

Vinner, Tall and others point out the importance of the concept image in mathematics. This results from input that is relevant to the situation and provided by the teacher. If concept images are not formed but merely the third level activity of definitions provided, then it is highly unlikely that the student will have a thorough understanding of the concepts and will struggle to utilise them in further contexts.

5.3.4 The abstract level

The abstract level represents the completion of a round of a cycle of learning. For this is the stage when the percepts and concepts have been formed and generalised. Once results have been established at this level, it is possible to utilise them and proceed further to a new round of the spiral.

In Piaget's original theory of development, the abstract level is associated with the age at which students begin secondary school algebra. However, it is necessary for them to have passed through the relevant perceptual and conceptual levels in order to appreciate the symbolism involved in algebra. He believed that in his original levels at the final stage it became possible for children to coordinate negations and reciprocities within a unified system. Thus he detected the elements of group theory in the way in which children think. In his later non-age related final level, Piaget saw the evolution of structure as a characteristic of the transoperational or abstract level. He and Garcia (1989) believed that it is reached when students become able to carry out operations on operations. It is necessary to pass through the lower levels in order to reach this point in the learning process.

Freudenthal (1973) saw the abstract level as a time of putting results of learning into a linguistic pattern. As students move from the conceptual level to the abstract level, the organisation of concepts attained at the previous level becomes the object of analysis. Freudenthal believes that re-invention plays such an important part at this level that, unless a student is able to reflect on activities belonging to lower levels, he is unable to successfully reach the higher level. The fact that he believes in a broad approach, rather than merely placing a "pyramid" (Freudenthal 1973:133) on the top strongly suggests the importance of the rounds of the spiral that lead up to the relevant topic at hand. In order to successfully reach and operate on the abstract level, many generalities should be gathered together rather than obtaining a structure by merely considering one example. Unless the student is able to investigate and establish definitions and results for him or herself, Freudenthal believes that he/she is skipping the relevant lower levels that lead to the abstract level. Consequently he/she would not be able to operate efficiently at the appropriate abstract level.

Van Hiele's uppermost level involves theorems and structures. Although, precise definitions belong to a lower level in his theory, here both formal definitions and theorems belong to the abstract level of a spiral but theorems generally lie higher up the spiral than any definitions upon which they depend. At this level students are

able to establish theorems within an axiomatic system. However, even different theorems could be regarded as forming parts of different abstract levels of the spiral. In order to reach them, either the "a posteriori" or "a priori" approach may be followed with the "a priori" approach becoming more dominant as higher parts of the spiral are reached.

Land too saw relevant properties of objects being studied becoming ordered at the abstract level. In Nixon's abstract level students used information about sequences and series to deduce more information. This of course is an abstract level fairly high up the level of learning the topic. Lower down the spiral an abstract level would have been obtained when students were able to establish the general term or a summation formula for various types of sequences or series. In this study visualisation, exploring patterns and encouraging generalisation were utilised for the learning of each new topic or sub-topic but here these are being considered as relevant perceptual, conceptual and abstract levels at various rounds of the upward path of the spiral. At the abstract level of a round of a spiral, concept definition and usage may be regarded as playing a relevant part. Concept definition results from the input leading to concept image and the linguistic form associated with the concept image. Without an interplay between concept image and concept definition, definitions cannot be understood properly. Rote learning of theorems hampers further concept usage and this in turn cannot lead to the relevant output of intellectual behaviour necessary for further concept usage involving the development of more theorems and applications.

5.3.5 Conclusion

Similar levels of learning may be detected in several different theories of how mathematical learning takes place. Besides the different examples already mentioned, de Villiers (2005) in his correspondence with the writer gives an indication of the levels he recognised in his study of Boolean Algebra. These seem to resemble to the perceptual, conceptual and abstract levels and are listed below:

Level 1: *Practical application. Solving switching circuit problems by trial and error, experimentation, truth tables, etc.*

Level 2: *Analysis. Deeper analysis of switching circuit theory discovering commutative, associative, etc. properties.*

Level 3: *Systematisation. Full axiomatisation of disconnected properties discovered at Level 2.*

All of these theories mentioned here indicate how important it is that all levels of learning are encountered. Omitting any level can cause gaps in the learning process and prevent further progress. In order to reach the abstract level of any round of the spiral, it is important to guide students through the relevant perceptual and conceptual levels.

5.4 The establishment of an integrated theory of learning algebra

5.4.1 The link between historical and epistemological levels of learning

The close link between the historical and epistemological levels of learning studied here gives an indication that in this way some light may be shed on the manner in which algebra is learnt by students. Fauvel and van Maarven (1997:258) observe how studying the historical development of concepts may be useful because

In this case the researcher applies history as a possible 'looking glass' on the mechanisms that put mathematical thought into motion. Such combinations of historical and psychological perspectives deserve serious attention.

5.4.2 The perceptual, conceptual and abstract levels

The history of the development of algebra had revealed three main levels of thought named by Piaget and Garcia (1989) the intraoperational level, interoperational and the transoperational levels. These levels, which are being called the perceptual, conceptual and abstract levels here, may be subdivided into many sublevels. In this way, the intra, inter and trans stages keep on re-appearing at various stages of the growth of knowledge concerning algebra. Piaget and Garcia (1989:29) recognise "the generality of this triplet, intra, inter and trans, and its occurrence at all sublevels as well as within global sequences". Since a mathematical topic cannot be completely dissociated from its historical development, "…the history of a concept gives some indication as to its epistemic significance" (Piaget & Garcia 1989:7). Even though a topic of mathematics may have gone through delays or even setbacks in its development, it is important to look at the stages involved in its final development and to attempt to explain the reasons for the sequence of stages.

Algebra did not develop historically as a linear sequence of stages as this would suggest that the early stages played no role in its continued development. Furthermore, each stage is dependent on what came before. The same is true in the acquiring of mathematical knowledge. Mathematical knowledge can be regarded as

being built up in the manner of an ever-growing spiral. Observing the history of algebra in general, the three overall levels which relate to the three main stages of mathematical development may also be associated with the three main levels of learning algebra here. Instead of the intra, inter and transoperational levels the terms perceptual, conceptual and abstract levels are being used to describe the characteristics which each stage possesses. Thus at the perceptual level percepts are formed, at the conceptual level concepts are formed and it is at the level of abstraction that abstraction takes place.

5.4.3 The spiral level theory

Alfinio Flores (1993:152) describes the main idea of a spiral curriculum as "to view the same concepts several times with greater depth and understanding each time the topic is revisited". But in this spiral theory the aim is to keep climbing up the spiral of a particular topic and encountering new concepts and results on the upward journey rather than continually revisiting the subject. Here the idea would also be that the student "begins at the bottom of the helix and is gradually helped by the teachers to progress around and up" (Thompson 1996:1). But, unfortunately what happens all too often with a spiral curriculum is that "a much more appropriate model in this case would be that of a circle rather than a spiral. Topics are revisited but each time we return to a subject, the level of sophistication of the debate remains the same" (Thompson 1996:1). The consideration of perceptual, conceptual and abstract levels involved at each round of the spiral should be able to help propel students upwards in the appropriate manner.

The spiral image is important as it reflects the way in which knowledge grows with each round of the spiral containing a perceptual, conceptual and abstract level. Since perceptual levels emerge from abstract levels, it is evident that knowledge is reorganised and expanded at the subsequent level. Eventually, after many rounds of the spiral have been traversed, the overall perceptual level becomes the overall conceptual level. Then, after many more rounds have been covered, the overall abstract level emerges and expansion continually takes place in many directions. One advantage of using the spiral image is that it is easier to refer to a round of a spiral somewhere high or low down a particular part of the spiral than a sublevel such as the one termed intra- inter- trans sublevel. Furthermore, spirals represent continual growth in all directions rather than continual subdivision of parts of a whole or an as yet unfinished part.

Ausubel (1978:58) regards the process of linking new information to pre-existing segments of cognitive structure as "subsumption". He refers to the "hierarchical organization of cognitive structure" and "a subordinate relationship of the new material to existing cognitive structure" (Ausubel 1978:59). This structure could represent the previous levels of the spiral. Ausubel distinguishes between two different types of subsumption. These include derivative subsumption and correlative subsumption. Derivative subsumption relates to when learning material is understood as an example of some established concept in the cognitive structure. In this case the meaning may be grasped quickly and with relative ease. However, as the spiral is climbed, it is by means of "a process of correlative subsumption" that new learning material can be seen to be "an extension, elaboration, modification, or qualification of previously learned propositions" (Ausubel 1978:59). Ausubel recognises what he terms a "Superordinate relationship to cognitive structure" when a new concept being learnt involves several established ideas. This is certainly the case in the learning of abstract algebra. Many concepts need to be established in the journey up the spiral and understanding of these is necessary for further progress upwards.

5.4.4 Analysing the spiral of learning in the presentation of a topic

In the learning of algebra it would thus seem necessary for the teacher to carefully consider the topic to be taught and then try and picture, analyse or determine the spiral of learning that led up to that topic. If the topic is in algebra then low down in the spiral would come the symbolism of algebra. Once the teacher has sorted out what seem to be the relevant rounds of the spiral that lead to the topic, each one of these could be dealt with to a greater or lesser extent, depending on the background knowledge of the students in question. Although this might be time-consuming, it would help to ensure that the spiral has been properly climbed and that no significant gaps exist in the progress towards the necessary section. For older, more experienced students, the lowest part of the spiral may not need to be covered at all and other fairly low down rounds may be omitted if the students reveal that it is not necessary. The following example indicates how the author of the book on "Galois Theory" carefully thought about the spiral of topics in his book and gave a clear indication to his readers what chapters should be read in order to follow a particular line of thought or reach any particular chapter.
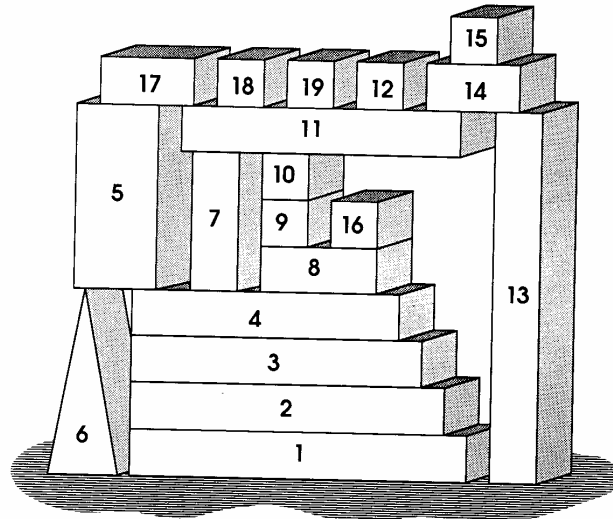
Structure of the book: Each chapter depends on those that support it.

(Stewart 1998:xii)

Once the lower rounds of the spiral related to a particular topic have been carefully considered, these need to be presented to students initially before the actual topic is reached. As has been stated previously, the teacher may know that it is not necessary to cover some of them. However, others may need to be dealt with in advance, possibly in a brief manner but preferably passing through the perceptual, conceptual and abstract levels in the process.

The following spiral diagram is derived from the basic diagram of Vinner in Tall (1991:171).
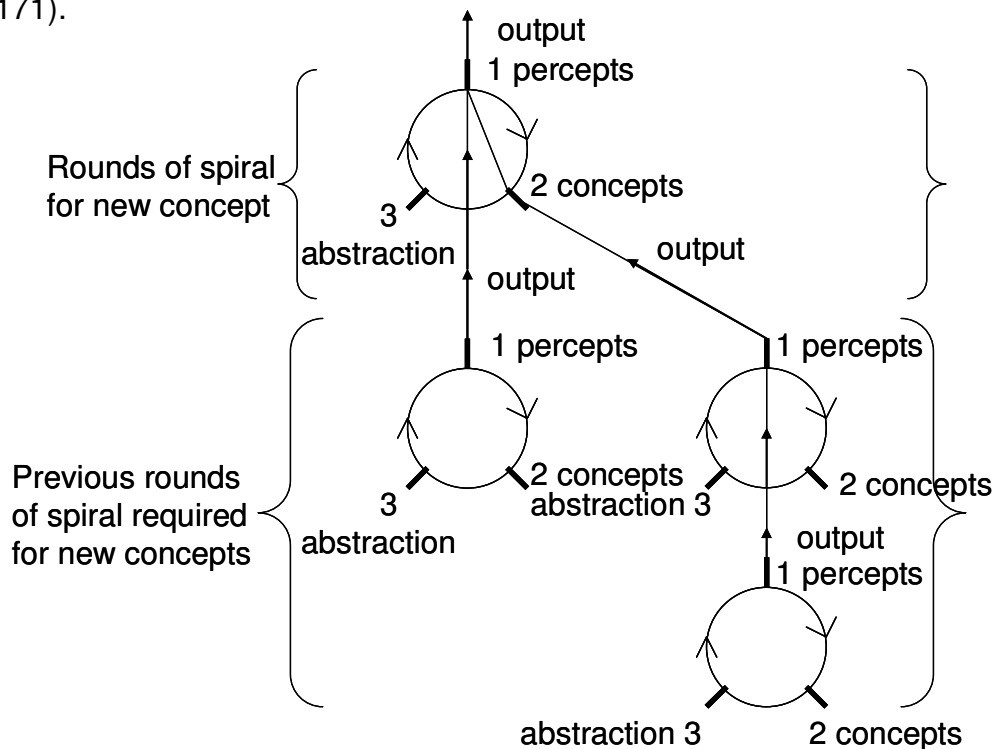
5.4.5 The "a posteriori" or "a priori" development of mathematics approaches

The two historical processes of the development of mathematics are relevant in the development of algebraic theory. These include: the "a posteriori" or descriptive type of axiomatisation described by de Villiers (1986:6) and illustrated below. His diagram has been adapted to incorporate the perceptual, conceptual and abstract levels as follows:

Perceptual level		Conceptual level		Abstract level
Logically unrelated statements	analysis →	Logical relationships analysed	Synthesis →	Theorems and axioms established

In addition, the "a priori" or "constructive" process of axiomatisation diagram of de Villiers (1986:5) has also been adapted to indicate the perceptual, conceptual and abstract levels:

Perceptual level		Conceptual level		Abstract level
Existing axioms	→	Variations of axioms	Further deduction →	New theorems

As has been reflected in the theory of concept development, students should be encouraged to become actively involved in the learning process. Freudenthal strongly advocates that they should be given the opportunity of re-inventing the theory themselves. This may or may not be done in the actual historical order that the original concept was developed in history. De Villiers (1986:iii) did not teach Boolean Algebra in the way it historically developed, but attempted to reconstruct it in "…the way in which professional mathematicians go about deductively ordering their research results".

5.4.6 The importance of not omitting stages of rounds or rounds of a spiral

As algebra is being taught, it is very important that all three stages are passed through in a particular round of the spiral. For example, if there is no input or activity

at the perceptual level, then students find it extremely difficult to form a concept image. This in turn means that definitions or theory established at the abstract level cannot be understood, prohibiting further output and advancement up the spiral. Furthermore, Vinner in Tall (1991:72) shows that sometimes theorems are proved before concept definitions have been grasped as follows:
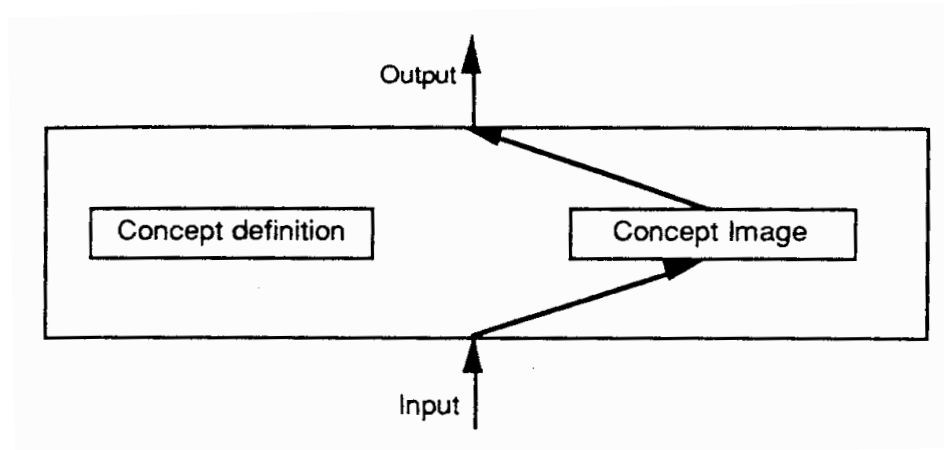
This causes serious problems for students attempting to study proofs such as those found in group theory, which depend on definitions and axioms. In order to prove theorems, according to the spiral theory, not only should definitions be understood but they should form the basis for the next perceptual level. Once they have been considered and analysed, a new conceptual and subsequently abstract level could be reached at which a theorem may be proved or a new application may be found.

5.4.7 The provision of opportunities of visualisation, exploring patterns and generalisation

Freudenthal claims that "No Mathematical idea has ever been published in the way it was discovered" (Freudenthal 1983:ix). Techniques are developed, definitions and propositions may be turned around until "the hot invention is turned into icy beauty" (Freudenthal 1983:ix). He believes that this didactical inversion could in fact be termed antididactical. However, recognizing the parallelism between the development of mathematical concepts and the way in which students develop concepts, he observes that "… the young learner is entitled to recapitulate in a fashion the learning process of mankind" (Freudenthal 1983:ix).

Patrick Thompson in (Steffe et al 1996:267) highlights the importance of imagery which is regarded here as pertaining to the perceptual level or the lowest part of every round of the spiral: "Mathematical reasoning at all levels is firmly

grounded in imagery" (Thompson in Steffe et al 1996:267). Piaget drew a distinction between three general types of images, depending on the actions of reasoning associated with it. The earliest images are those which could be associated with the perceptual level where people internalize objects by acting upon them. The second or type of image has been described as one which includes "ascription of meaning or significance" or one which contributes to "… the building of understanding and comprehension" (Steffe et al 1996:268). The third type of image is described as one which "… supports reasoning by way of quantitative relationships" (Steffe et al 1996: 268). (Kosslyn 1980:269) like Piaget, did not see images as mental pictures but rather as "highly processed perceptual encodings" (Steffe et al 1996:209).

Thompson believes that the two aspects of imagery that have a significant influence on the development of mathematical reasoning are their immediate understanding of situations encountered in schooling and "… more global aspects of their development of mental operations" (Steffe et al 1996:273). As students advance up the spiral, they continually need to build up mental images as, for example, Thompson "… demonstrated how advanced mathematics students' impoverished images of rate obstructed their understanding of derivative, integral, and relationships between them" and furthermore "… students cannot constitute the situations that their visible mathematics is supposed to be about with sufficient richness to support their reasoning" resulting in their being "reduced to forming figural associations between a teacher's notational actions and superficial characteristics of a problem statement's linguistic presentation" (Steffe et al 1996: 281). This is an unfortunate situation resulting from too little attention being paid to the perceptual level. However, by coordinating images it is desirable to help these become the substance of concepts.

In order to ensure that each stage of a round of a spiral is covered, the teacher needs to provide appropriate activities to promote passing through the relevant stages. As has been mentioned in previous chapters, Nixon (2002) conducted a study regarding the influence of encouraging visualisation, exploring patterns and generalisation on progress through the van Hiele levels in the teaching of sequences and series. These results seem to suggest that visualisation is closely associated with the perceptual level, exploring patterns is linked to the conceptual level and generalisation corresponds with the abstract level. Thus, providing these activities in progress from each van Hiele level to the next, helped to ensure that each sublevel or lower round of the spiral was adequately covered before the next

level was attained. For example, in order to climb from the third van Hiele level or Level 2 to the fourth van Hiele level or Level 3, activities of visualisation and exploring patterns were provided in order to reach the generalisation required at the relevant level. The success of considering visualisation and exploring patterns in order to reach high levels of learning in Nixon's study gave an indication of how necessary the perceptual and conceptual levels of each round of the spiral continue to be in order to make the abstract level of each stage attainable. At higher levels it is mathematical objects such as definitions, axioms or theorems that play the role of percepts. For it is the intuitions linked to what has been established at the abstract level that become the starting point of a new turn of the spiral. The constructive approach becomes more feasible too. In fact, the experienced mathematician is able to form a concept image as soon as definitions or axioms are presented because of all the levels encountered before.

5.5   Illustrations of spirals involved in various algebraic topics

In teaching topics of algebra it is very necessary to analyse the topic carefully and determine what earlier rounds of the spiral are relevant for the topic at hand. Depending on the group of students, the teacher may deem it unnecessary to go down to all the apparent relevant preceding rounds. However, these should all be well established in order for further progress up the spiral to take place. Either an "a posteriori" or "a priori" approach to the development of a topic may be considered. The topics need not necessarily be taught in the exact historical sequence in which they occurred but in such a way that students are encouraged to pass through the relevant perceptual, conceptual and abstract rounds of the spiral.

5.5.1 Introductory Algebra

In an attempt to overcome "students' inability to spontaneously operate with or on the unknown" (Linchevski & Herscovics 1996:39), an alternative approach was followed aimed at overcoming the "cognitive gap" (Linchevski & Herscovics 1996:39). Problems arise for students beginning algbra because they have such a limited view of algebraic expressions that they see the process of solving equations as part of a ritual of the solution process rather than obtaining numerical solutions. An attempt was made to let the students find intuitive procedures and use these as a basis for further learning.

During the research it was found that students had difficulty with the various uses of letters in algebra including "letter as unknown, as generalized number, as function variable, etc" (Linchevski & Herscovics 1996:42). Their findings were that in order for students "to be able to accept algebraic expressions, their interpretation of the literal symbol must be fairly advanced. Collis (1975) notes that such expressions require students to be able to map more than the number onto a literal symbol (viewed as a pronumeral), although it is only later that they perceive it as having acquired all numerical properties, at which point it becomes a "generalized number" (Linchevski & Herscovics 1996:42). However, they believed that in the framework of equations the letter can be grasped at a much lower level. The process of substituting a number for a literal symbol is more meaningful because it can be seen as a search for an appropriate number.

The method they used for solving an equation such as $5n + 17 = 7n + 3$ was to decompose it into the form $5n + 14 + 3 = 5n + 2n + 3$, implying that $14 = 2n$ from which students could easily deduce that n equals 7. In their initial approach, instead of introducing grouping in the usual way it is done with algebraic expressions, it was done with cancellation of identical terms. Linchevski and Herscovics (1996:61) observe that "It is interesting to note that our cancellation procedure is very close to the historical development of algebra". Although only a restricted class of questions was dealt with in this research, it was felt that the "more incremental approach" Linchevski and Herscovics 1996:63) did serve as a successful approach as it initially involved the use of fairly primitive procedures of a perceptual nature and did not rise too soon to the generalised process at the abstract level.

5.5.2 Number systems

In one approach used in order to teach number systems to students, it is helpful to show them how the lack of closure under certain operations can lead to the development of another one. For example, the fact that in the set of numbers $\{0; 1; 2; .....\}$ it is not possible to subtract a larger number from a smaller one necessitates the introduction of the set of integers $\mathbb{Z} = \{... -2; -1; 0; 1; 2; ...\}$. However, in the set of integers it is not possible to divide any two elements to obtain an integral result so this leads to the inclusion of fractions to form the set of rational numbers $\mathbb{Q} = \left\{ \dfrac{m}{n} / m, n \in \mathbb{Z}; n \neq 0 \right\}$. Square roots of non-perfect squares lead to the inclusion of

the irrational numbers to give the set $\mathbb{R}$ of real numbers. Finally, the inability to take the square root of negative numbers leads to the introduction of Imaginary numbers I and hence the set $\mathbb{C}$ of complex numbers comprising of the union of $\mathbb{R}$ and I.

Instead of closure, the idea of introducing new numbers in order to solve various types of equations could be shown. Thus the solution of $x + 3 = 3$ requires the introduction of a zero and $x + 2 = 1$ leads to the need for a negative number. The solution of $2x + 5 = 0$ shows the need for rational numbers, $x^2 - 7 = 0$ necessitates the introduction of irrational numbers and $x^2 + 1 = 0$ leads to the need for complex numbers.

The complex numbers can thus be subdivided in the following manner:
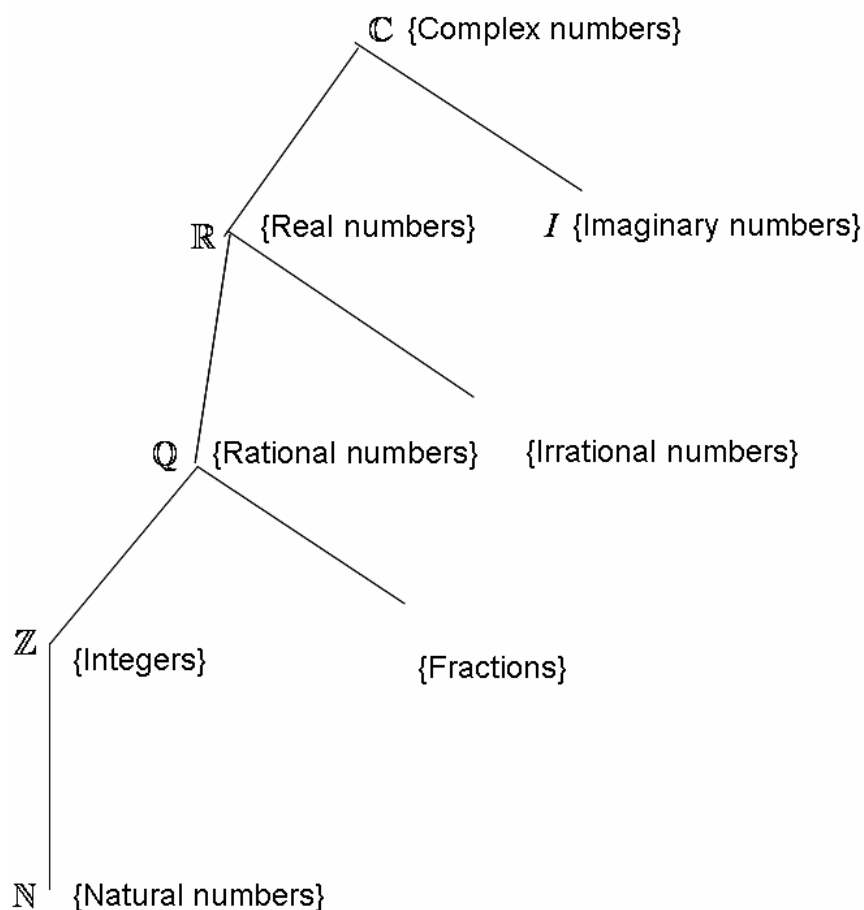
5.5.3 The solution of cubic equations

Although the algebraic solutions of cubic equations of the form $ax^3 + bx^2 + cx + d = 0$ can be found by numerical methods, more complex ones reveal the usefulness of the remainder and factor theorems. The remainder theorem states

that if a polynomial $f(x)$ is divided by $ax-b$ then the remainder is given by $f\left(\dfrac{b}{a}\right)$.

The factor theorem states that if $f\left(\dfrac{b}{a}\right)=0$ then $ax-b$ is a factor of $f(x)$. This enables $f(x)$ to be factorised and the equation to be solved.

Many lower levels of the spiral lead up to the solution of cubic equations. Students need to know and appreciate algebraic symbolism, manipulation and substitution. They need to know how to solve first degree equations of the form $ax+b=0$ and second degree equations of the form $ax^2+bx+c=0$. They also should have the prior knowledge of performing division of polynomial expressions. The teacher would need to make sure they are acquainted with all of this background knowledge before proceeding any further. If necessary, it would be appropriate to teach some of these topics, taking the perceptual, conceptual and abstract levels into consideration.

The students would need to be introduced to the idea of the Euclidean algorithm. Considering numerical examples at the perceptual level could help to establish the concept. For example, $3\overline{)13}$ with quotient 4 and 12 subtracted leaving 1, or $13\div3=4$ remainder 1 and students should note that $13=3\times4+1$.

After building up examples such as these, they could come to appreciate the fact that $f(x)=(ax-b)Q(x)+R$ where $f(x)$ is a polynomial expression which is divided by $(ax-b)$, giving a quotient of $Q(x)$ and a remainder of R. Their knowledge of substitution and manipulation of symbols would help them to appreciate that

$$
\begin{aligned}
f\left(\frac{b}{a}\right) &= \left[a\left(\frac{b}{a}\right)-b\right]Q\left(\frac{b}{a}\right)+R \\
&= (b-b)Q\left(\frac{b}{a}\right)+R \\
&= (0)Q\left(\frac{b}{a}\right)+R \\
&= 0+R \\
&= R
\end{aligned}
$$

They could then practise finding remainders in this manner by themselves and discover that if $f\left(\dfrac{b}{a}\right)=0$ then $(ax-b)$ is a factor of $f(x)$. Once they have found out how to find such a factor $(ax-b)$, they could then proceed to factorise $f(x)$ fully (as

a result of their knowledge acquired at lower rounds of the spiral) and finally reach the point of being able to solve a cubic equation. This topic illustrates just how important lower rounds of the spiral can be in the acquisition of new knowledge.

5.5.4 Introduction to group theory

In order to introduce students to group theory for the first time, various different approaches may be followed, although these need not represent the way in which this topic actually did develop historically. However, it should take the student through the relevant perceptual, conceptual and abstract levels.

Huetinck (1996:342) discusses a method of introducing group theory by means of a game called "It's a SNAP". After being taught the concepts of an operation and, a nonnumerical set, both of which form relevant parts of the lower spiral, the students are introduced to the transformations of the equilateral triangle with vertices A, B and C where the re-orientation is the operation and the triangle vertices form the nonnumerical set, called the configurations. The student are led to discover six different configurations, illustrated below which have previously been referred to in section 2.4.6 (iv). Here it is emphasised that the new orientation always begins with the initial configuration.
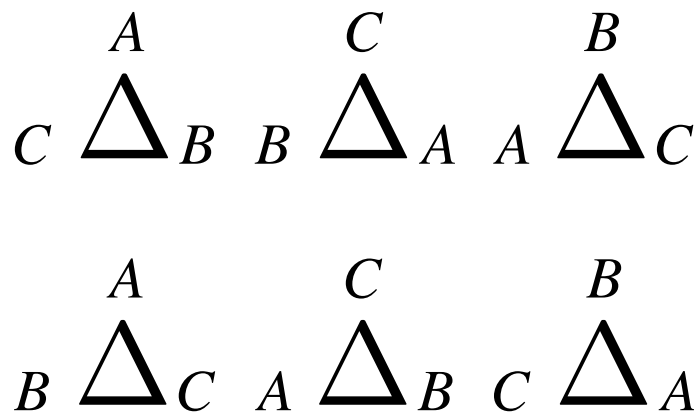
Possible orientation for an equilateral triangle   (Huetinck 1996:342)

After further discussion the game of "It's a SNAP" can be introduced. The rules are as follows:

1. *Every rubber band connects one peg in the top row with one peg in the middle row.*
2. *Each peg is included only once.*
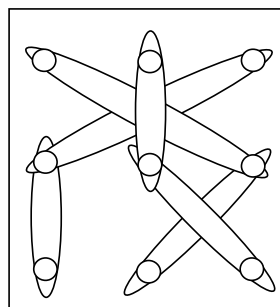3. *All three rubber bands must be used*   (Huetinck 1996:342).

Six possible patterns for the three rubber bands, called A, B, C, D, E and F are illustrated below:
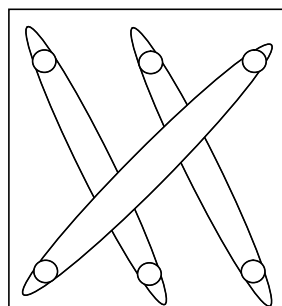

A

B

C

D

E

F
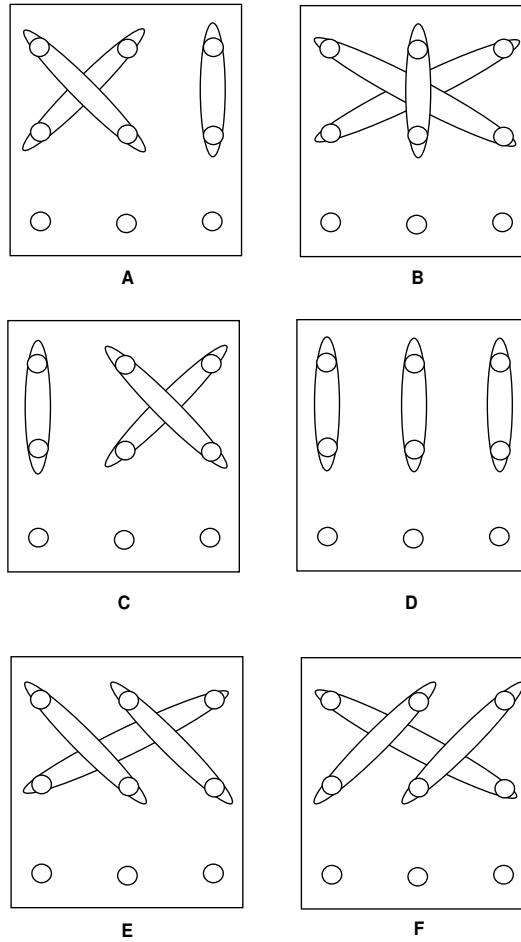
(Huetinck 1996:343)

If the student were to perform an operation such as B snap C then he/she would obtain the result:

After removing the middle pegs, the result can be seen to be as follows:

E

This means that B snap C equals E.

A table of results can be set up as follows:

Bottom Two Rows							
	SNAP	A	B	C	D	E	F
Top Two Rows	A	D	E	F	A	B	C
	B	F	D	E	B	C	A
	C	E	F	D	C	A	B
	D	A	B	C	D	E	F
	E	C	A	B	E	F	D
	F	B	C	A	F	D	E

Huetinck (1996:344)

This table reveals the group properties that the set is closed under the defined operation, the operation is associative, a unique identity element exists and each element of the set has a unique inverse.

The students could then also set up a table for $\mathbb{Z}_6$ under addition as follows and make comparisons at the conceptual level.

|	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

In this way students become actively involved in the learning process and are led through the perceptual and conceptual levels to form a definition of a group at the abstract level.

5.5.5 Lattice diagrams in group theory

Once the concept of a group has been defined and understood, the student may have reached an abstract level but will need to continue to pass through further

perceptual and conceptual levels in order to establish further results at abstract levels further up the spiral.

For example, consider the group of symmetries of a square
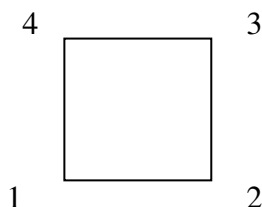
where ρ; stands for rotations, μ: for mirror images in perpendicular sides and δ for diagonal flips.
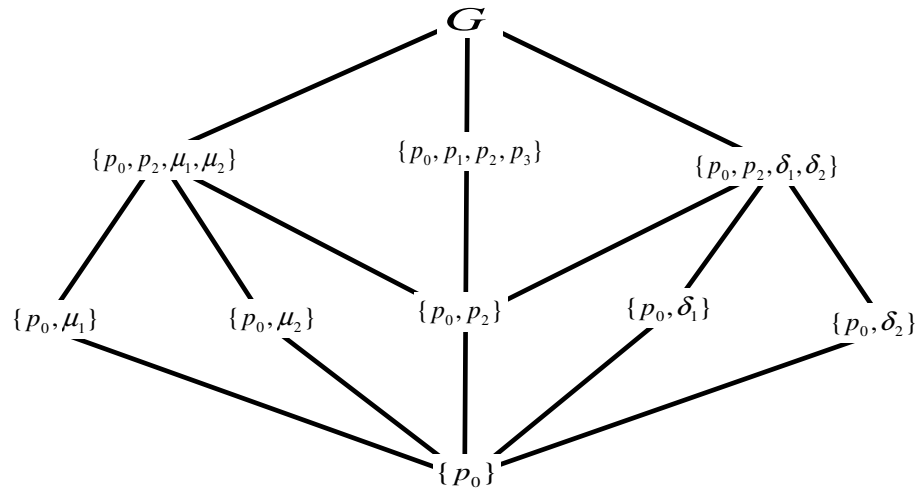
The eight permutations involved here are:

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \qquad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \qquad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix},$$

The corresponding group table shows plenty of wonderful examples of symmetry which the students could investigate on their own.

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_2	δ_1	μ_1	μ_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_1	δ_2	μ_2	μ_1
μ_1	μ_1	δ_1	μ_2	δ_2	ρ_0	ρ_2	ρ_1	ρ_3
μ_2	μ_2	δ_2	μ_1	δ_1	ρ_2	ρ_0	ρ_3	ρ_1
δ_1	δ_1	μ_2	δ_2	μ_1	ρ_3	ρ_1	ρ_0	ρ_2
δ_2	δ_2	μ_1	δ_1	μ_2	ρ_1	ρ_3	ρ_2	ρ_0

Group table for D_4. Fraleigh (1977:41).

Lattice Diagram for D$_4$ (Fraleigh 1977:42)

For example $\{\rho_0,\rho_1\}$, $\{\rho_0,\mu_1\}$, $\{\rho_0,\rho_1,\rho_2,\rho_3\}$ can be seen to form subgroups, under the group operation whilst $\{\mu_1,\mu_2,\delta_1,\delta_2\}$ obviously does not as it does not even contain the group identity element. After studying the group table, the students could themselves then establish a lattice diagram for the subgroups of D_4 shown above. This would also help to illustrate the result that the number of elements of a subgroup in a group divides the order of a group. For example, here the subgroups contain 1, 2, 4 or 8 elements, each of which divides 8.

5.5.6 The second homomorphism theorem for groups

Once students have reached this stage of algebra, they are clearly in the overall abstract or transoperational level of the topic. Nevertheless, in order to establish further results in their climb up the spiral, they need to continue passing through the perceptual, conceptual and abstract levels. Below is an example of a group theory theorem presented to third year level students:

Prove the Second Homomorphism Theorem for Groups, Theorem 2.7.3 on p. 86; see also p. 88 Problem 5, which guides you through the steps of the proof The theorem states the following:

Let H be a subgroup of a group G and N a normal subgroup of G. Then

$$HN = \{hn \,|\, h \in H, \, n \in N\}$$

is a subgroup of G, $H \cap N$ is a normal subgroup of H, and

$$H/(H \cap N) \simeq (HN)/N.$$

(Heidema 2004:29)

There are several rounds of the group theory spiral that precede this problem. Some of the definitions needed here are stated below:

Let G, G^1 be two groups, then the mapping $\psi : G \to G^1$ is a homomorphism if

$\psi(ab) = \psi(a)\psi(b)$ *for all* $a, b \in G$ (Herstein 1999:68);

The homomorphism $\psi : G \to G^1$ is called monomorphism if ψ is 1-1. A monomorphism that is onto is called an isomorphism (Herstein 1999:68);

Two groups G and G^1 are said to be isomorphic if there is an isomorphism of G onto G^1. We shall denote that G and G^1 are isomorphic by writing $G \approx G^1$ (Herstein 1999:68);

The subgroup N of G is said to be a normal subgroup of G if $a^{-1}Na \subset N$ for every $a \in G$ (Herstein 1999:70).

Heidema (2004:29) provides the following hint and diagram to encourage students to pass through the relevant perceptual and conceptual levels in order to appreciate the results established at the abstract level.

Contemplate the following picture; it exhibits all the secrets of this problem.

He continues to remind them that for the last part they need to

define a function $\psi : (HN)/N \to H/(H \cap N)$ *and prove that it is an isomorphism.*
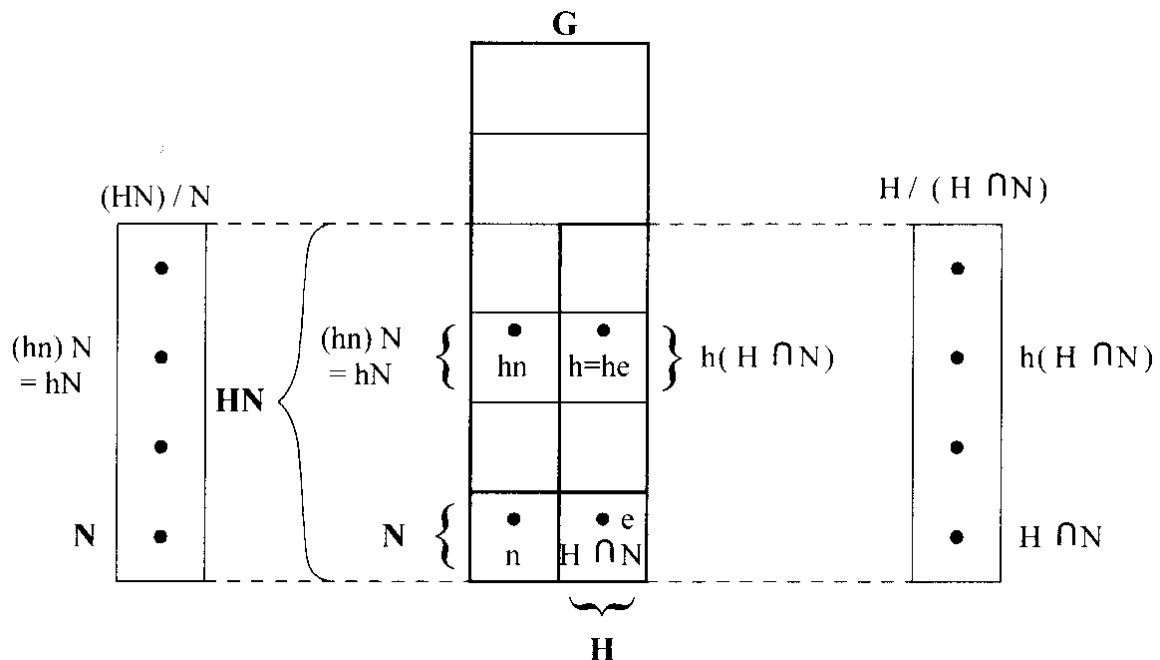
$HN = \{hn \mid h \in H, n \in N\} = \bigcup_{h \in H} hN$, so HN is the union of all those left cosets of N (in G) in which there are elements of H. (For "left coset", see p. 64 Problem 5.)

(Heidema 2004:29).

Problem 5 in Herstein (1999:64) is stated below:

> *Let G be a group and H a subgroup of G. Define, for $a, b \in G$, $a \sim b$ if $a^{-1}b \in H$. Prove*
>
> *that this defines an equivalence relation on G, and show that $[a] = aH = \{ah \mid h \in H\}$.*
>
> *The sets aH are called left cosets of H in G.*

Heidema (2004:29) concludes with the comment:

The element $(hn)\,N\,(= hN)$ of $(HN)/N$ corresponds to the element $h\,(H \cap N)$ of $H/(H \cap N)$. This should tell you how to define ψ.]

5.6 Conclusion

Historically, abstract algebra has developed through three main stages termed the intraoperational or perceptual, interoperational or conceptual and the transoperational or abstract levels. These same three levels are evident in the way in which students acquire concepts. As these levels can be seen to be repeated over and over again as more advanced topics are reached, it is convenient to represent the learning process by means of a spiral, growing outwards in many directions. Each time the abstract level is reached, further output leads to new perceptual levels and the cycle is repeated.

Sawyer (1959:2) observes how:

The traditional high school syllabus – algebra, geometry and trigonometry – contains little or nothing discovered since the year 1650 AD. Even if we bring in calculus and differential equations, the date 1750 AD covers most of that.

This shows how school pupils are generally taught algebraic topics which belong to the overall perceptual and conceptual levels of algebra.

Skemp (1971) complained about approaches to undergraduate teaching that "tend to give students the product of mathematical thought rather than the process of mathematical thinking" (Tall 1991:3). However, the spiral of learning is very relevant in the acquisition of algebra concepts. The low down rounds of the spiral need to be taken into account whenever a new topic is taught. Foundations need to be properly established and taken into careful consideration. In order to facilitate effective learning, it would seem to be appropriate to organise suitable activities that encourage appropriate passage through the relevant perceptual, conceptual and abstract levels.

Even students at the university level need guidance in their learning of algebra. Often they are encountering modern algebra for the first time. Sawyer (1959:2) observes that

Modern higher algebra was developed round about the years 1900 to 1930 AD. Anyone who tries to learn modern algebra on the basis of traditional algebra faces some of the difficulties that Rip van Winkle would have experienced, had his awakening been delayed until the twentieth century. Rip would only overcome that sense of strangeness by riding around in airplanes until he was quite blasé about the whole business.

This suggests that students need to be guided up the spiral when they are at both school and even university level. They need to be familiar with the lower regions of their topics and become involved in the establishment of new knowledge. This would be preferable to presenting "… the final form of the deduced theory rather than enabling the student to participate in the full creative cycle" (Tall 1991:3). Eventually, after travelling through many rounds of the spiral, the capable student becomes able to follow the constructive approach and form his/her own concept images from reading axioms, definitions and theorems provided in a textbook.