Chapter 4

The abstract level or the third learning level in the development of abstract algebra

4.1 Introduction

The abstract level or third learning level in the historical formation of the concept of a group is what Piaget terms the transoperational level. The word “trans” is a Latin word meaning “over”, “across” or “on the other side of”. This suggests that reaching the transoperational level is a sign of crossing over to the other side or reaching the upper level of thought of either the spiral as a whole or a round of the spiral. Reaching the abstract level is a desirable goal because

> *To think in the abstract means to overlook deliberately the disturbing influence of this multiplicity of particulars in favour of the few essentials or the main features of importance with regard to an object or situation* (Duminy & Sőhnge 1990:202).

Freudenthal (1973:35) observed that by the time the group concept was formulated, there was a vast stock of groups accumulated but all of these dealt with separate cases. In group theory all special results were consolidated in order to define a group. When groups emerged at the overall abstract or transoperational level, group axioms were established just as in geometry. However, the beauty of the group axioms is that, without referring to only one model, as is the case in geometry, they make it possible “… to work with one instrument in many situations, which makes life easier” (Freudenthal 1973:35). This gives a clear indication that the introduction of group theory marked the beginning of the overall abstract or transoperational level in algebra. Many centuries of work and many rounds of the spiral within the overall perceptual and conceptual levels took place in order to reach this turning point. Di Sessa observed how

> *In axiomatic mathematics it has long been recognised that beneath all the complex of definitions and theorems must exist a special layer which serves as foundation for all the rest* (Gentner & Stevens 1983:15).

The abstract level involves definitions and, as the spiral continues to be traversed, proofs of theorems appear. More definitions and theorems continue to be encountered in the upward journey of the spiral. This often presents a problem to university students. Moore (1994:249) claims that *at many colleges and universities students are expected to write proofs in real analysis, abstract algebra, and other advanced courses with no explicit instructions in how to write proof*.

Dubinsky et al (1994) notes that in abstract algebra students suddenly have to work with understanding concepts rather than learning algorithms. It is thus of critical importance that in the teaching of abstract algebra the teacher guides the students through all the relevant rounds of the spiral in order to reach the appropriate abstract level. Once students have passed through several rounds of the spiral, they could become more ready for a constructive approach.

4.2 The emergence of the abstract level in the history of abstract algebra

4.2.1 Introduction

Piaget and Garcia (1989:155) regard Gauss, Lagrange, Ruffini, Cauchy and several other mathematicians as being "… the last representatives of the interoperational period in the development of algebra and specifically in the history of the theory of algebraic equations". In psychological development he regards the transoperational stage as being the one when it becomes possible for learners to perform operations on operations. In the case of abstract algebra this corresponds to when it becomes possible to derive all possible permutations of n elements. This is the stage at which the student is able to introduce an ordering that relates to the permutations which have been executed.

The final level of the history of abstract algebra has been referred to by Bell in his description of the development of the concept of an abstract group as follows:

the emergence of general principles making further calculations, unless needed for some definite application, superfluous' (fourth level deduction) (Land 1990:29);

and *the formulation of postulates crystallizing in abstract form the structure of the system investigated (fifth level rigor)* (Land 1990:27).

This stage relates to the overall transoperational or abstract level of abstract algebra. It is the level at which the structure of the group was finally established.

The question of whether or not a quintic equation could be solved by radicals had finally been settled by Abel conclusively in 1824 when he proved that it was insoluble. The next problem which arose was finding some way of determining whether or not it was possible to solve a given equation by radicals. Abel had been working on this but died prematurely in 1829.

Evariste Galois, a young Frenchman who had also been working on this problem, was killed in a duel in 1832 when he was only twenty-one years old. Although for some time he had been seeking recognition for his mathematical theories and had submitted three memoirs to the academy of sciences in Paris, they were all rejected. His work seemed to have been entirely lost to the world of mathematics but then fortunately on 4th July 1843 Joseph Louiville began his address to the academy with these dramatic words:

> *I hope to interest the Academy in announcing that among the papers of Evariste Galois I have found a solution, as precise as it is profound, of this beautiful problem: whether or not it is soluble by radicals* (Stewart 1998:xvi)

4.2.2 The transoperational or abstract level of algebra in history

Evariste Galois, the man who took the theory of solving equations in algebra into the transoperational stage, was born at Bourg-la-Reine near Paris on 25 October 1811. His father was Nicholas-Gabriel Galois, a Republican and head of the village liberal party, who became a mayor after the return of Louis VIII to the throne in 1814. His mother's name was Adelaide-Marie (ńee Demante) and she was the daughter of a jurisconsult. She had a solid education in religion and the classics and she was able to read Latin fluently. He was offered a place at the college of Reims when he was ten years old but his mother preferred to keep him at home.

In October 1823 he went to the lyćee Louis-le-Grand where he did well for the first two years and obtained first prize in Latin. But then he became bored, was made to repeat the next year's classes and so became even more bored. At this point he became very interested in mathematics and read a copy of Legendre's "Elements de Gèomètrie", a classic text which broke away from the tradition of Eulidean Geometry taught in schools. It is said that he read it as if it were a novel and fully comprehended it in merely one reading. The school algebra texts were no match for that so he then turned to the original memoirs of Lagrange and Abel. In fact he was merely fifteen years old when he was reading material that was actually meant for professional mathematicians. However, he had lost interest in his schoolwork and his teachers did not understand him, thinking he was merely pretending to be ambitious and original.

Galois was a very untidy worker, who tended to work in his head and then merely write the results on paper. He took the competitive exam necessary for entrance to the "École Polytechnique" described as "the breeding ground of French mathematicians" (Stewart 1998:xviii) but failed. Two decades later Terquem attributed his failure to his being of superior intelligence to his examiner. In 1828 Galois entered the École Normale and attended advanced classes under Richard, a mathematician who believed Galois should have been allowed to enter the Polytechnique without examination. In 1829 Galois printed his first paper on continued fractions which was competent but did not show signs of genius. But Galois was making great discoveries in the theory of polynomial equations, some of which he submitted to the Academy of Sciences. The referee was Cauchy who had been working on a central theme in Galois' theory and had already published work on the behaviour of functions under permutations of variables. However, Cauchy rejected Galois' memoir as well as another one presented eight days later and the memoirs were never seen again.

In 1828 Galois suffered two more setbacks when his father committed suicide on 2nd July after a bitter dispute with the village priest and when a few days later he failed his final chance to enter the Polytechnique. In February 1830 Galois entered a competition for the greatly esteemed "Grand Prize in Mathematics" (Stewart 1998:ixx) and appeared to be more than worthy of winning the prize. The secretary, Fourier, took Galois' work home to read but died before doing so and the manuscript could strangely not be found amongst all his papers. Galois came to believe that the way in which his papers were continually being lost was not merely by chance. Instead "…He saw them as the effect of a society in which genius was condemned to an eternal denial of justice in favour of mediocrity; and he blamed the politically oppressive Bourbon regime" (Stewart 1998:ixx).

In 1830, when Charles X was faced with abdication, he attempted a coup d'ėtat and tried to curb freedom of the press. During the time of unrest, Guignault, the Director of the École Normale, locked in the students. Galois consequently wrote a scathing attack on him in the "Gazette des Écoles" and was subsequently expelled. On 13th January 1831 Galois began to privately offer a course on advanced algebra but had little success. Four days later his memoir "On the conditions of solubility of equations by radicals" (Stewart 1998:xx) was sent to the Academy.

Poisson and Lacroix were appointed as referees but when Galois queried the fact that he had received no response from them two months later, he was given no answer. Galois then joined the artillery of the National Guard, a republican organisation. On 9th May 1831 at a riotous banquet of protest, Galois proposed a toast to Louis-Philippe with an open knife in his hand and was arrested because he was accused of making a threat on the king's life but was set free on 15th June. Finally on 4th July Poisson declared his memoir to be "incomprehensible" (Stewart 1998:xxi) and made the following remark in his conclusion:

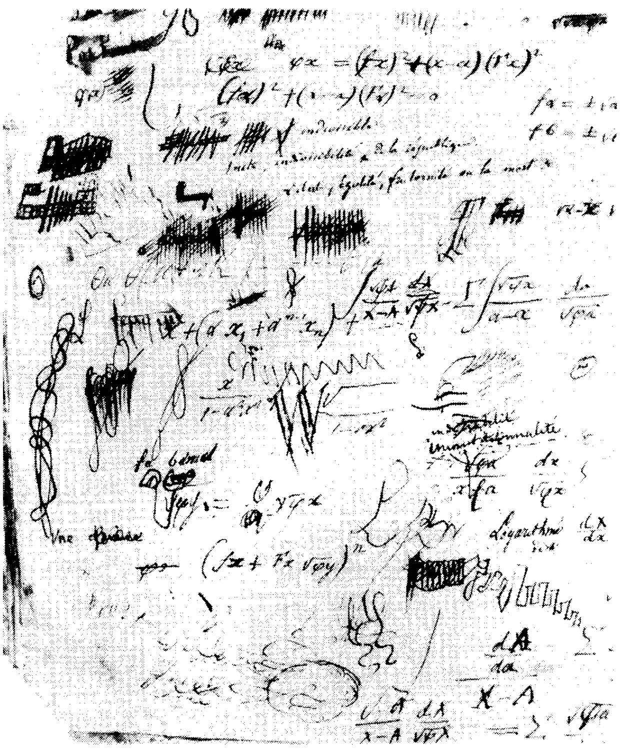
> *We have made every effort to understand Galois's proof. His reasoning is not sufficiently clear, sufficiently developed, for us to judge its correctness, and we can give no idea of it in this report* (Stewart 1998:xxi).

On 14th July, Galois was convicted for six month's imprisonment for illegally wearing a uniform when he headed a Republican demonstration. He worked for a while on his mathematics, was transferred to a hospital during the cholera epidemic of 1832 and was soon on parole. He then fell in love with Stephanie-Felicie Poterin du Motel, the daughter of a nearby physician. It appears that Galois took his subsequent rejection badly and was later challenged to a duel, apparently because of the relationship he had had with this girl. Some believe that she was used as an excuse to eliminate a political opponent while others claim that his opponent was possibly a revolutionary comrade of Galois. The latter idea that the duel was precisely what it appeared to be seems to be substantiated by the following proclamation made by Galois before the duel took place:

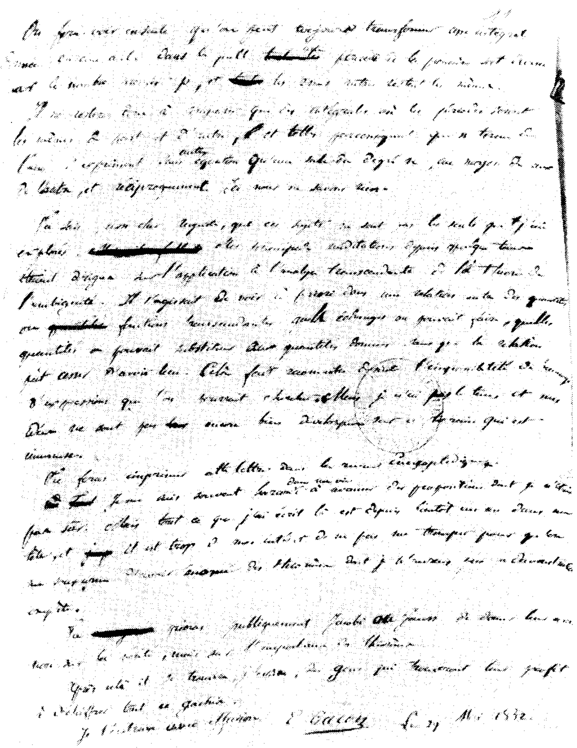
> *I beg patriots and my friends not to reproach me for dying otherwise than for my country. I die the victim of an infamous coquette. It is in a miserable brawl that my life is extinguished* (Stewart 1998:xxii).

On 29th May, the eve of his duel, Galois also wrote to his friend Auguste Chevalier. In his letter he gave an outline of his discoveries and Chevalier later had these published in the "Revue Encyclopédique". In the document there were scrawled remarks in the margins and comments that he had no time. Amongst other ideas including elliptic functions and the integration of algebraic functions, he outlined the connection between groups and polynomial equations. He stated that an equation is soluble by radicals provided that its group is soluble. Unfortunately this brilliant young mathematician was shot in the stomach on 30th May, died of peritonitis on 31st May and was buried in the common ditch at the cemetery of Montparnasse on 2nd June 1832.

The below pictures show doodles he had left on the table before he left for the duel (on the left) and the final page he wrote (on the right).



Stewart (1998:xxii)



Stewart (1998:xxiv)

As has been mentioned, Galois had been studying Abel's memoirs when he was a young teenager. Abel had managed to prove that the general polynomial could not be solved algebraically for $n > 4$. However, Abel's further aims were to determine all the equations of any given degree which were solvable algebraically and also to find whether or not a given equation is solvable algebraically. Fortunately Abel's proof, which involves permutation groups to some extent, was published early in the first volume of August Leopold Crelle's "Journal". Abel's proof captured the imagination of Galois and he was able to give complete answers to the questions posed by Abel. The concepts associated with Galois's results are called Galois theory. Galois was the one responsible for the term "group". He wrote: "A system of permutations such as, etc. is called a group. We shall represent this set by G" (Piaget & Garcia 1989:156). His findings can be seen to date right back to the early beginnings of algebra, termed the intraoperational or perceptual level of the subject as well as recent development at the interoperational or conceptual level. His work could be associated with various different rounds and stages of the spiral because: "Galois theory is a fascinating mixture of classical and modern mathematics, and it takes a certain amount of effort to get used to its manner of thought" (Stewart 1998:xxv).

John K Baumgart (NCTM 1989:255) observed that although Galois' accomplishments were highly significant and original "… they did not immediately make their full impact on his contemporaries because these men were slow to understand, appreciate, and publish Galois's work". This gives an indication that Galois had managed to rise from the interoperational to the transoperational or abstract level ahead of the others. Galois's aim was to study the solution of polynomial equations $f(t) = t^n + a_{n-1}t^{n-1} + ... + a_0 = 0$ and to find a way of distinguishing which of these were solvable by a formula and which were not. What is meant by a formula here is a radical expression, meaning anything that can be built up from the coefficients a_i by the operations of addition, subtraction, multiplication, division and finding n^{th} roots where $n = 2,3,4,....$ Using modern terminology, Galois' main idea was to look at the symmetries of the polynomial $f(t)$. These form a group which is known as a Galois group. The various properties of the Galois group reflect the solution of the polynomial equation. Since there are many problems in mathematics that reduce to the solution of polynomial equations, Galois's theory has application in many other areas of mathematics.

Galois invented the concept of a group when he was working on the solution of equations. Since by now abstract algebra has continued to advance higher and higher up the transoperational or abstract level spiral, by today's standards his approach may seem quite concrete as it belongs to the perceptual level of the overall abstract level. However, in his time it was extremely abstract as it was leading up to an entirely new overall level in the course of history.

To illustrate the way in which Galois built up his theory, the following illustration is provided. If the polynomial equation $t^4 - 6t^2 - 7 = 0$ is considered, it factorises into the form $(t^2 + 1)(t^2 - 7) = 0$. Continuing to solve this equation, it becomes

$$t^2 = -1 \; or \; t^2 = 7 \text{ and hence}$$
$$t = \pm\sqrt{-1} \; or \; t = \pm\sqrt{7}$$
$$t = \pm i$$

Thus there are two pairs of roots and it is possible to denote these by the Greek letters α, β, γ and δ as follows: $\alpha = i$; $\beta = -i$; $\gamma = \sqrt{7}$ and $\delta = -\sqrt{7}$. It is feasible to write down infinitely many valid equations with rational coefficients which are satisfied by these roots. Some examples could include $\alpha^2 + 1 = 0;\ \alpha + \beta = 0;\ \delta^2 - 7 = 0;$

$\gamma + \delta = 0;\ \alpha\gamma - \beta\delta = 0$. There are also infinitely many equations which are not solved by the above roots. If the pairs of roots are interchanged, in other words α and β or δ and γ, the resulting equations remain true. For example, corresponding to the above equations $\beta^2 + 1 = 0;\ \beta + \alpha = 0;\ \gamma^2 - 7 = 0;\ \delta + \gamma = 0;\ \beta\delta - \alpha\gamma = 0$ are all true for the given roots. However, interchanging the roots which do not form pairs, such as β and δ could, for example, give $\gamma^2 + 1 = 0$ which is false.

The operations used above are all permutations of the roots α, β, γ and δ. Using usual permutation notation for the interchange of α and β, let $R = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \gamma & \delta \end{pmatrix}$ and for the interchange of γ and δ, let $S = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \delta & \gamma \end{pmatrix}$. Two other permutations that preserve the valid equations are $T = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \delta & \gamma \end{pmatrix}$ and the identity permutation $I = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \end{pmatrix}$

The symbol ! is used to denote "factorial". For example, $n!$ means $n(n-1)(n-2)....(2)(1)$ and *3! = 3* x *2* x *1 = 6.* If all possible permutations of the roots α, β, γ and δ were to be considered there would be *4! = 4* x *3* x *2* x *1= 24* possible permutations of the four symbols α, β, γ and δ. However, only *R, S, T* and *I* above preserve valid equations and they are said to form the Klein 4 - group.

The group table could be represented as follows:

	I	R	S	T
I	I	R	S	T
R	R	I	T	S
S	S	T	I	R
T	T	S	R	I

To find any entry on the table, for example RS, we observe that

$$R = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \gamma & \delta \end{pmatrix} \text{ and } S = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \delta & \gamma \end{pmatrix}$$

Then since R takes α to β and S takes β to β, RS takes α to β;

R takes β to α and S takes α to α, RS takes β to α;

R takes γ to γ and S takes γ to δ, RS takes γ to δ;

R takes δ to δ and S takes δ to γ, RS takes δ to γ.

In this way, the result $RS = \begin{pmatrix} \alpha\ \beta\ \ \gamma\ \delta \\ \beta\ \alpha\ \ \delta\ \gamma \end{pmatrix} = T$ is obtained as indicated in the previous group table. The other table entries are obtained by continuing in the same way.

What struck Galois was that the structure of this group to some extent controls the way in which one goes about solving an equation. Herstein (1999:51) defines a subgroup H of a group G in the following manner. "A nonempty subject, H, of a group G is called a subgroup of G if, relative to the product in G, H itself forms a group". For example, $\mathrm{H} = \{\mathrm{I, R}\}$ is a subgroup of the group $\mathrm{G} = \{\mathrm{I, R, S, T}\}$ above. The group table for H appears as follows:

	I	R
I	I	R
R	R	I

Certain expressions such as $\alpha^2 + \beta^2 - 7\gamma\delta^2$ are fixed by the permutations in this group. For example, if we apply R to the above expression we obtain $\beta^2 + \alpha^2 - 7\gamma\delta^2$ which is the same as it was before. Considering the way R is formed, an expression is fixed by R if and only if it is symmetric in α and β. It can be shown that any polynomial in α, β, γ and δ can be written instead in the form of a polynomial in $(\alpha+\beta)$, $\alpha\beta$, γ and δ and the fact that $\alpha+\beta=0$ and $\alpha\beta=1$ can be used to eliminate the α and β altogether from the resulting expression.

Let g(t) be any quartic polynomial with explicit zeros α, β, γ and δ. $\mathbb{Q}$ represents the set of all rational numbers. In other words, $\mathbb{Q} = \left\{ \frac{m}{n} / m,n \in \mathbb{Z};\ n \neq 0 \right\}$ where $\mathbb{Z}$ is the set of integers or $\mathbb{Z} = \{...\ -2;\ -1;\ 0;\ 1;\ 2;\ ...\}$. Thus $\mathbb{Q}$ includes all integers and all fractions. When elements of $\mathbb{Q}$ are expressed in decimal form, they always either terminate or recur. $\mathbb{Q}(\gamma,\delta)$ consists of all rational expressions in γ and δ while $\mathbb{Q}\ (\alpha,\ \beta,\ \gamma,\ \delta)$ consists of all rational expressions in $\alpha,\ \beta,\ \gamma$ and δ. Here $\mathbb{Q}$ is contained in $\mathbb{Q}(\gamma,\delta)$ while $\mathbb{Q}(\gamma,\delta)$ is contained in $\mathbb{Q}\ (\alpha,\ \beta,\ \gamma,\ \delta)$ or $\mathbb{Q} \subseteq \mathbb{Q}(\gamma;\delta) \subseteq \mathbb{Q}(\alpha;\beta,\gamma;\delta)$. Considering $H = \{I, R\} \subseteq G$, it can be assumed that the expressions fixed by H are precisely those in $\mathbb{Q}(\gamma,\delta)$. Using the fact that the expressions $(\alpha+\beta)$ and $\alpha\beta$ are both fixed by H and hence lie in $\mathbb{Q}(\gamma,\delta)$, then since $(t-\alpha)(t-\beta)=t^2-(\alpha+\beta)t+\alpha\beta$, α and β both satisfy a quadratic equation with coefficients in $\mathbb{Q}(\gamma;\delta)$. Thus a formula for solving a quadratic equation to express α, β in terms of rational expressions in γ and δ can be used. Hence α and β can be obtained as radical expressions of γ and δ. The same trick can be repeated to find γ and δ, and by substituting these back into the formulae for α and γ all four zeros can be found to be radical expressions in rational numbers.

Although these expressions may not have been found explicitly, the above illustration reflects the idea that certain information concerning the Galois group does necessarily imply their existence. Furthermore, the task can be completed when more information is given. In this way the subgroup structure of the Galois group G is closely connected to the possibility of finding the solutions of equations $g(t)=0$. Galois made the discovery that "this relationship is very deep and detailed" and was able to prove that an equation of the fifth degree does not have a solution because "the quintic has the wrong sort of Galois group" (Stewart 1998:xxix). The modern approach to Galois theory is very similar. However, the set $\mathbb{Q}\ (\alpha,\ \beta,\ \gamma,\ \delta)$ described above is considered as being contained in the set of complex numbers. The group $\{\mathrm{I}, \mathrm{R}, \mathrm{S}, \mathrm{T}\}$ described above is called the automorphism group of $\mathbb{Q}\ (\alpha,\ \beta,\ \gamma,\ \delta)$. Some, other modern additions concern fields, which were introduced after Galois' time and will be mentioned later.

Galois was thus responsible for introducing "… one of the first important modern advances in the theory of groups, and hence to him we owe much of our modern theory of algebraic equations of higher degree" (Smith 1953:499). Although Galois's most important memoir was written the year before his death, it was not published until 1846. Cauchy continued to publish many articles on group theory.

The modern definition of a group first appeared in 1854 in a paper produced by Arthur Cayley (1821-1895).

Definition. A **group** $\langle G,*\rangle$ is a set G, together with a binary operation $*$ on G, such that the following axioms are satisfied:

ς_1. The binary operation $*$ is associative

ς_2. There is an element e in G such that $e*x=x*e=x$ for all $x\in G$. This element e is an **identity element** for * on G.

ς 3. For each a in G, there is an element a' in G with the property that $a' * a = a * a' = e$ The element a^1 is an **inverse of a with respect to** *. (Fraleigh 1977:18).

He developed the theory of finite groups and listed all possible multiplication tables that can be formed for groups of eight elements. Cayley and others worked on developing algebras obeying different structural laws satisfied by those concerning common algebra. In this way, he "…opened the flood gates of modern abstract algebra" (Eves 1990:510). Here further growth up the spiral may be seen within the transoperational or abstract level of the history of equations. It was found that an enormous variety of systems could be studied by weakening, deleting or replacing some of the postulates of common algebra. This is a typical example of the constructive or "a priori" development of mathematics where new mathematics is created from existing mathematics. In 1854 Cayley published an article with the title "On the Theory of Groups as Depending on the Symbolic Equation $\theta^n = 1$" (NCTM 1989:255). This is important as it contains what is believed to be the first definition of a finite abstract group. Another important result contained in the article is what is now known as Cayley's theorem. This states that "…every finite group is isomorphic to a regular permutation group" (NCTM 1989:255).

The spiral of group theory which had emerged at the abstract level of the theory of equations, continued to grow and develop in many directions. In 1870 Camille Jordan (1838-1922) published a presentation on the basic elements of Galois Theory entitled "Traité des substitutions". This covered the results of Lagrange, Ruffini, Abel, Galois, Cauchy and Serret as well as some of his own work done on the subject. In the same year Leopold Kronecker (1823-1891) was responsible for giving a set of axioms defining finite abelian groups. Working in an "a priori" manner with a completely arbitrary abstract set of elements, he was able to derive the usual group properties, including the existence of an identity element and inverses for a set.

In the mean time, the spiral of group theory had continued to spread out in other directions on the abstract level. Each time through, it would pass through the perceptual, conceptual and abstract levels which make up one round of the spiral. In a treatise of 1831 Gauss had published the geometric representation of complex numbers although he had already described this in a letter to F.W.Bessel as far back as 1811. Then in 1837 William Rowan Hamilton, born in Ireland in 1805, gave a purely arithmetical definition of a complex number $a+bi$. He defined a complex number as an ordered pair (a; b) of real numbers, which was subject to the standard rules of combining these pairs. Hamilton continued to apply his ideas to rotations and vectors in the plane and in 1843 he reached a new abstract sublevel of the spiral when he generalised from ordered pairs to ordered n-tples. He concentrated on quadruples or quaternions and in this way extended the algebra of vectors in a plane to vectors in space. The concept of a complex number $a+bi$ was thus extended to $a+bi+cj+dk$ (a, b, c and d real numbers) where $i^2 = j^2 = k^2 = ijk = -1$. The most remarkable property of quaternions was the fact that the commutative law did not hold. This law means that the order of operations is important. It took Hamilton fifteen years to realise that it was possible to create a useful and consistent mathematical system that did not obey the commutative law.

Grassman, also worked independently on a more general theory of n-tples at the same time as Hamilton but did not get as much publicity. Quaternions did not, however, turn out to be as practical as Hamilton had believed and, as the spiral expanded further they were replaced by later inventions which were found easier to apply. Nevertheless, Hamilton had made a great contribution to the "a priori" development of mathematics because once mathematicians realised that the commutative law was not an essential axiom, they began to experiment with new systems and change other axioms as well.

The study of algebraic equations reached a new peak in the second half of the nineteenth century. The concept of a field had been used by both Abel and Galois in an intuitive manner in their work concerning, polynomial equations. It was Richard Dedekind (1831-1916) who gave the earliest concrete formalisation of the theory of fields. Although Galois had reached the transoperational or abstract level of algebra when he introduced the concept of a group, he was still in the interoperational or conceptual stage regarding a field. Both Dedekind and Kronecker acknowledge that the Galois theory of equations had been "… the inspiration for their own general and semi-arithmetical approach to algebra" (Bell 1945:212). Two of Galois's basic concepts regarding domains of rationality were the lowest levels of the rounds of the spiral which led them on to new heights.

Piaget and Garcia (1989:156) regard the second half of the nineteenth century as being "a period during which one of the great historical "leaps" can be observed". This was the final stage in the study of algebraic equations. One of the fundamental processes associated with this stage is the re-interpretation of symbols. Gauss had worked on quadratic forms and his composition of forms had been the first operation in which numbers were not directly involved. Nevertheless each of these forms was essentially a relation in which numbers were represented by coefficients and variables. Subsequently it was shown that the properties of the polynomial functions and consequently algebraic equations are not dependant on the coefficients and variables being numbers. Progress was made further up the spiral which was already in the transoperational or abstract level in the history of equations. Here "a priori" development took place when the fundamental questions being asked concerning the study of polynomial equations of different degrees changed from being what kind of number determines the zero values or properties of polynomials to what are the relevant properties of numbers that need to be taken into account in this situation. It had become evident that properties were in fact very general and were not restricted merely to numbers. This is because there are many sets of elements that do possess the same sort of properties as numbers have. Consequently, instead of the properties common to such classes of elements only defining a specific domain of mathematical objects, they define a structure which is common to many domains. Dedekind (1831-1916) was the one who studied this structure and he gave it the name of "field".

Dedekind found that when studying polynomials and algebraic equations it is possible to "abstract from numbers as such and consider as "coefficients" only those classes of elements that fulfil the following conditions:

I. *Between the elements of a set two laws of internal composition hold*
   a,b → a+b; a,b → ab. These are called addition and multiplication, respectively.
II. *These two laws of composition form a group (excluding, in the case of multiplication, the neutral element of addition, commonly called zero).*
III. *These two operations satisfy the following conditions, called distributivity*
   a(b+c)=ab + ac
   (b+c)a=ba+ca (Piaget & Garcia 1989:157)

Up until then the whole rational, real or complex number system had been used. However, not only numbers need to be considered here but many entities could be found that in fact satisfy these rules and hence constitute fields. For example the polynomial expression $a_0 + a_1x + ... + a_nx^n$ could be considered where the a_i are all rational numbers. Let $b_0 + b_1x + ... + b_nx^n$ be another such polynomial expression. Then, if $P=(a_0;a_1;...a_n)$ and $Q=(b_0;b_1;...b_n)$, we can define the sum and product of P and Q in the following manner:

$P+Q=(a_0+b_0;a_1+b_1;...\ a_n+b_n)$

$PQ=(a_0b_0;a_0b_1+a_1b_0;...;a_0b_n+a_1b_{n-1}...+a_nb_0)$

Using these definitions, it can be shown that the polynomials constitute a field. There are other examples which also form fields such as the congruence classes modulo a prime number in $\mathbb{Z}$, algebraic numbers, Veronesse's formal series and Hensel's p-adic numbers. However, these will not be discussed in detail here.

Dedekind's concept of a field had already been used by Abel and Galois but neither of them had defined a field as a set. Although they had considered the elements of a set and given precise definitions of them, nowhere in their work did the set itself explicitly appear. This was similar to the situation which had previously arisen when Gauss, Ruffini, Cauchy and Abel were still at the interoperational stage of the evolution of solving algebraic equations. It was Galois who managed to climb the spiral into the transoperational level. However,

> *…in respect to the development of the notion of field, Galois remained at the interoperational stage. Dedekind took the next step when he succeeded in identifying and thematizing the structure of algebraic fields, thus ushering in the transoperational stage* (Piaget & Garcia 1989:158).

Furthermore Stillwell (2002:422) remarked that

> *The concept of a field was implicit in the work of Abel and Galois in the theory of equations, but it became explicit when Dedekind introduced number fields of finite degree as the setting for algebraic number theory.*

Felix Klein (1849-1925) established what became known as the "Erlanger Programme" and

> *it appeared right at the time when group theory was invading almost every domain of mathematics, and some mathematicians were beginning to feel that all mathematics is nothing but some aspect of group theory* (NCTM 1989:255).

The concept of a transformation of a set S onto itself is a central theme in Klein's application of groups to geometry. His study of groups of transformations that preserve certain invariants can be seen in certain aspects of South African secondary school geometry. These include the study of isometric transformations that preserve congruence and dilations that preserve similarity. This sort of application of groups would help students appreciate the worth and beauty of group theory.

David Hilbert (1862-1943) introduced the term “ring”. The difference between a field and a ring is that an inverse for multiplication is not postulated. The integers $\mathbb{Z} = \{... -2;\ -1;\ 0;\ 1;\ 2;...\}$ are the simplest case of a ring. The set of integers is a group under the operation of addition and is closed under multiplication. Multiplication is not assumed to be commutative in a general ring. An ideal is “A subring N of a ring R satisfying $rN \subseteq N$ *and* $Nr \subseteq N$ for all $r \in R$” (Fraleigh 1977:232). Even though the term “ring” was introduced by Hilbert, the structures of rings and ideals had already been known by Kronecker and Dedekind. Systematic use of these concepts is made by algebraic geometry. For example, Max Noether made use of the theory of rings generated by polynomials.

During the second decade of the twentieth century Emmy Noether (1882-1935) managed to successfully show the applicability of abstract algebra to other disciplines and another direction in which the spiral of abstract algebra could grow. Her thesis completed a stage in the development of algebra. This is because she succeeded in unifying the two theories of integrable algebraic functions (polynomials) and ideals of whole algebraic numbers. Stilwell (2002:423) remarked that “The next level of abstraction was reached in the twentieth century” by Emmy Noether who during the 1920’s “developed common properties of different structures, such as groups and rings”. Just before 1920, O. Zariski made use of the notion of local rings (rings which possess a single maximal ideal). It was the first time that structure was studied from a purely algebraic viewpoint.

4.2.3 Sublevels of the transoperational or abstract level in history

In reaching the transoperational or abstract level of algebra, Galois built his work on centuries of work that had gone before him. There were mathematicians of his time who had advanced high up the spiral in the interoperational or conceptual level and he was able to rise from where they were to the overall transoperational or abstract level of solving equations. Cooke (1997:386) remarks how “Galois’s approach to the subject required several pieces of background”.

There were other discoveries made at various sublevels or rounds of the spiral upon which Galois built his theory. For example, these included: Abel’s notion of a domain of rationality generated by a given set of numbers, Lagrange’s consideration of the permutation of the roots of an equation, Ruffini’s work and Cauchy’s publications on groups of substitutions (though he did not call them groups).

However, Fauvel and Gray (1988:507) observe that: ”These beautiful results [of Lagrange and Abel] were however only the prelude to a much greater discovery. It was reserved for Galois to put the theory of equations on its definite footing”.

As mentioned before, Galois managed to show that there corresponds a group of substitutions to each equation. This group reflects the equation’s essential characteristics and especially those that have to do with its solution or finding of radicals by means of other auxiliary equations. Given any algebraic equation, it is enough to know one of its characteristic properties in order to find its group and then deduce its other properties. Observing the progress of groups, it is evident that “… the problem of solution by radicals which not long ago seemed to be the sole object of the theory of equations, only appears as the first link in a long chain of questions” (Fauvel & Gray 1988:507).

Even within the transoperational level the intra, inter and trans sublevels or many rounds of the spiral may be detected. Piaget (1989:168) refers to “a sequence of sublevels of the type intra, inter and trans” and notes that “… one finds the same process or work in the phases of construction toward a higher level as in the sequence of the main stages”. Although Galois was the one to introduce the transoperational level of solving equations, the first groups introduced by Galois concerned permutations only and not the group itself. Thus, considering this from the point of view of groups, the relations involved were of an intraoperational level. Felix Klein (1849-1925) then “… brilliantly applied them to geometry” (Eves 1990:492). The group of transformations such as the projective transformations play a constitutive role. In this case the structure is of the interoperational variety since the transformations themselves are components of the group. Finally the transoperational stage of groups themselves may be seen in the way that the concept of an abstract group was elaborated even further so that it referred to any class such as the one operating on vector spaces. Furthermore, this group transformation approach has been extended to include the two non-Euclidean geometries (elliptical and hyperbolic) and constitutes the mainstay of modern algebraic and topological transformations.

Thus after Galois had developed the first thematized structure in mathematics, although the period of equations and their solutions had come to an end, at the same time a new round of the spiral began during which structures predominated. Piaget and Garcia (1989:16) refers to this as “a long transoperational period”.

4.2.4 Conclusion

Algebra began with a very limited domain of trying to solve equations. This remained the case for many centuries. However, later on this turned out to be merely a small and restricted part of its later domain. Once Galois had established groups, Dedekind introduced fields and many other further developments took place. In the “a priori” development of algebra, many new systems were developed including

> … *groupoids, quasigroups, loops, semigroups, monoids, groups, rings, integral domains, lattices, division rings, Boolean rings, Boolean algebras, fields, vector spaces, Jordan algebras, and Lie algebras, the last two being examples of non-associative algebras* (Eves 1990:510).

Thus, after beginning as the study of solving equations, the spiral rose up and grew to such an extent that at the transoperational or abstract level it

> …*gradually acquired its true identity up to the moment when it surfaced as the study of structures, several centuries after inception* (Piaget & Garcia 1989:166).

4.3 Axiomatisation

During the twentieth century much effort was made to examine the logical foundations and structure of mathematics. This has led to what is called the creation of axiomatics or the study of postulate sets and their properties. As a result, many basic concepts of mathematics have undergone evolution and generalisation. The fundamental topics of abstract algebra, set theory and topology were greatly developed. Mathematical logic and philosophy have developed and computers have affected mathematics too. Nevertheless, even now during the transoperational or abstract level of mathematics, many new ideas can be traced back to the bottom of the spiral. Eves (1990:660) observes:

> …. *like so much of mathematics, most of these modern considerations trace their origins back to the work of the ancient Greeks, and, in particular, to the great “Elements” of Euclid* (Eves 1990:606).

Euclid’s Elements were such an early and enormous attempt at a presentation of the postulational method that it is not surprising that it was not free of errors. He has been criticised for assumptions he made and definitions he gave. For example, he defined a point as “that which has no part” and a line as “length without breadth” (Eves 1990:607). The difference between the Greek and modern conception of mathematics can be found in the primitive terms. The Greeks did not list primitive terms. They saw geometry as not merely an abstract study but an attempted logical analysis of "idealised physical space" (Eves 1990:607). They saw points and lines as idealisations of very small particles and very thin threads and tried to reflect this in their definitions.

Freudenthal (1973:28) believed that "… Both ancient and modern axiomatics originated in geometry, and for this reason geometry conserved certain archaic features longer than other axiomatised domains did". For example, in the axiomatics of Hilbert regarding points and lines in projective geometry, the words "point" and "line" are continually used. However, the definition of a group may be given without using the word "group" itself. Freudenthal observed that as long as ideas can be illustrated intuitively, not so much attention is given to the language used but as ideas become more abstract, more careful linguistic expression is required. The perfecting of mathematical language is a continuous process and the conscious attempt to make language a vehicle of exact expression is called formalising.

Modern axiomatics began in about 1870. In geometry right from antiquity the idea of a deductive system based on fundamental hypotheses and axioms was a task that was well understood. Non-Euclidean Geometry was discovered in about 1830 and could have started modern mathematics. In the eighteen-forties and fifties attempts had been made to axiomatise projective and complex projective geometry. Moritz Pasch (1843-1930) "… became the builder of the first irreproachable axiomatics of Euclidean geometry. He taught mathematicians how to formulate axioms" (Freudenthal 1973:33). However, Hilbert's "Foundations of Geometry" (1899) became even more popular.

From the time of the ancient Greeks until the beginning of the twentieth century "axiom" represents something which cannot be proved but which needs no proof because it is used as a foundation or something that can be presupposed in any proof. Soon after Hilbert had managed to found Euclidean geometry axiomatically, axiomatics began to turn in another direction. Hilbert had already begun to anticipate this in his book. For he had begun to consider what would happen if certain axioms were dropped. If nothing changed then the axiom was unnecessary but if it did then a weaker system would arise. Freudenthal remarked how "Properly said, such axiomatic systems were known of before, but up to that time normally referred to axiomatics if he meant them" Freudenthal (1973:74). Nowadays the postulates which are imposed on a set in order to make it a group are referred to as axioms or group axioms. This represents quite a large step ahead of the traditional meaning of an axiom.

Bell (1945:186) recognised three new approaches to number in 1801 and in the 1830's that hinted at the general concept of mathematical structure. He saw this in Gauss's concept of congruences, Galois' introduction of group theory and Kronecker's revolutionary programme in the 1880's for basing all mathematics on the natural numbers. He believed that

> *From the standpoint of mathematics as a whole, the methodology of deliberate generalisation and abstraction, culminating in the twentieth century in a rapidly growing mathematics of structure, is doubtless the most significant contribution of all the successive attempts to extend the number concept* (Bell 1945:186,189).

He noted how at each stage of the progression from the natural numbers to other types of numbers, all of the fields of mathematics that were affected were both enriched and extended.

Referring to modern axiomatics, Freudenthal observes that "How such axiomatic systems can arise is marvellously shown by the history of the group concept" (Freudenthal 1973:35). Group theory arises as a special way of organising the special results shared by all groups. For example, under the operation of multiplication, associative multiplication together with the existence of one and inverse elements are called group axioms. Group axioms possess a special property not possessed by the axioms of geometry. This is because they make it possible "… to work with one instrument in many situations, which makes life a lot easier" (Freudenthal 1973:35).

The field concept has been seen to have a history analogous to the history of a group. For centuries people were familiar with the rational numbers and their four operations of addition, subtraction, multiplication and division. Next came the system of real numbers with the same type of operations and laws although it was less familiar and less formalised. The complex numbers followed which were not very different from the former ones except they lacked the order property. Freudenthal believes that it was only when the general concept of a field was developed in 1910 and when p-adic number fields emerged as a new example could all fields be brought under one heading. He remarked that "What is called abstract or modern algebra actually starts here" (Freudenthal 1973:36).

Piaget and Garcia (1989:171) recognise algebra as the "science of the general structures common to all branches of mathematics, including mathematical logic".

However, in order to reach these structures, two important preliminary stages were required. The first one, termed the intraoperational or perceptual level, was when particular systems were analysed in terms of certain limited and static properties they possessed. The second one, termed the interoperational or conceptual level, involved transformations which were possible because of an abstract and general symbolism. Quite some time after this, when Galois introduced the concept of a group, there arose 'structures' in the contemporary sense of the term (Piaget & Garcia 1989:171). In order to keep rising from the initial level to axiomatisation at the highest level, Piaget and Garcia (1989:171) believe that continuous reflective thematization or continual ongoing conceptualisation of progressive mathematical objects needs to take place. This can happen by continually passing through the relevant perceptual, conceptual and abstract levels contained in many rounds of the spiral from the overall perceptual level to the overall abstract level.

De Villiers (1986:4) points out "the importance of axiomatisation in modern mathematics and its implication for teaching". However, it has often been the case that students have been merely presented with axioms without knowing why they exist or where they come from. Freudenthal and others have expressed the view that students should become actively involved in the establishment of axioms in the learning process. De Villiers differentiates between constructive axiomatisation and descriptive axiomatisation. In his correspondence with the writer in 2005, he points out how reasoning backwards occurs during descriptive axiomatisation and also reveals the way in which axioms arose in history. The example he provided is as follows:

> *Suppose to prove that the midpoints of a quadrilateral is a parallelogram, one can easily prove it in terms of the triangle midpoint theorem, but asking in turn how that can be proved, etc. eventually leads to some necessary starting points, i.e. the axioms.*

Constructive or "a priori" axiomatisation is said to take place when

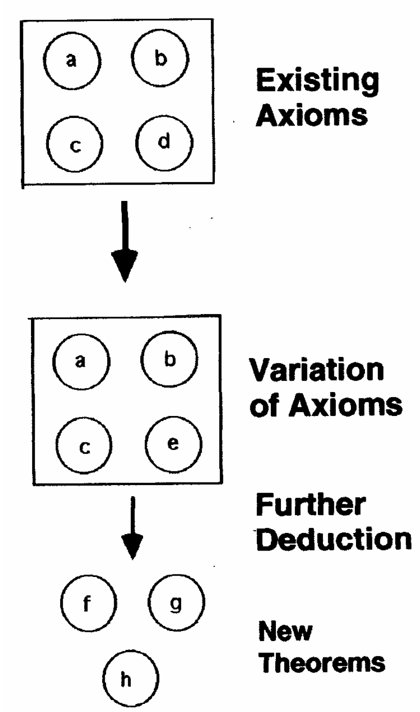
> *a given set of axioms is changed through the exclusion, generalisation, replacement or addition of axioms (or subsets of axioms) to that set, from which totally new content is then constructed in a logical deductive way* (de Villiers 1986:4).

Bell originally seems to have also believed that this is a very striking and relevant form of axiomatisation. He observed

The full import of the abstract formalisation appears only when it is taken as the point of departure for the deliberate creation of new mathematics. Certain postulates in the original set are suppressed or contradicted, and the consequences of the modified set are then worked out as were those of the original (Bell 1945:147).

Usually the choice of axioms is not completely arbitrary as this does not generally lead to any results of major importance.

The “a priori” or “constructive process of axiomatisation is represented below:

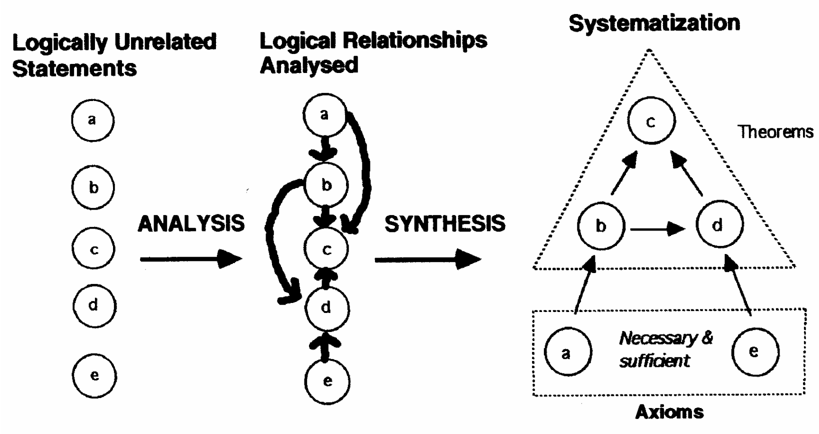


(de Villiers 1986:5)

The three levels of the spiral can be seen in the above representation. At the perceptual level existing axioms are observed. The conceptual level may be associated with the variation of axioms. Finally at the abstract level further deduction leading to new theorems takes place. The “a priori” aspect of development of mathematics can be seen in the way that mathematics leads to new mathematics. In the case of Descriptive Axiomatisation a set of axioms is selected from a set of statements that already exists. This type of axiomatisation may be described in the following manner:

For many sets of mathematical ideas (properties), there is often a subset from which all the other ideas (properties) may be deduced. (Indeed there may be many such subsets for each set). Such a subset is said to form an axiom system and the ideas or properties belonging to this subset are called axioms. Axiomatization is the process of selecting just such a subset from a given set of mathematical properties. (Scandura 1971:55)

The below diagram represents the “a posteriori” type of axiomatisation:



(de Villiers 1986:6)

Once again the three levels of the spiral can be detected in the diagram. At the perceptual level logically unrelated statements are considered. At the conceptual level logical relationships are analysed. Finally at the abstract level systemisation takes place and axioms as well as theorems are established. The “a posteriori” development of mathematics tends to be associated with the earlier development of mathematics. It often exposes the mathematical structure not as yet observed in a familiar area and so can be identified with the process of abstraction,

Systematisation involves “the **selection** or **variation** of axioms (descriptive or constructive axiomatisation) as well as the logical deductive **ordering** of axioms, definitions and theorems” (de Villiers 1986:8). In descriptive axiomatisation, however, the statements to be chosen as axioms and the logical ordering of these is often suggested by logically analysing relationships between statements. Constructive axiomatisation may be seen to encourage the creation of new knowledge whilst descriptive axiomatisation involves reorganisation of existing knowledge. Both types of axiomatisation have their merits and de Villiers (1986:7,8) points out several important functions of descriptive axiomatisation. These include: the identification of hidden assumptions, the integration and economical presentation of unrelated statements, theorems or concepts; the identification of inconsistencies in previous results; the provision of a broad overview of a topic; applications both within and outside of mathematics; the possible reorganisation of areas into alternative systems of a more elegant, powerful and economical nature than the existing ones.

Bell (1945:266) saw two main differences between mathematics in the twentieth century as opposed to the nineteenth century as being deliberately pursuing abstractness in which relations are significant and “…intense preoccupation with the foundations on which the whole intricate superstructure of modern mathematics exists". Freudenthal (1973) sees a stress on conceptual development rather than algorithms as being a distinguishing feature of modern mathematics. Steen (NCTM 1989:477) observed how

> *Throughout the twentieth century mathematics has grown in extent and diversified in form. Now, near the end of the century, what appeared of greatest importance one hundred years ago is just a small part of the entire mathematical landscape.*

This shows how the spiral has continued to grow and expand over the past century.

Axiomatisation has led to great developments in algebra and revealed common roots shared between algebra and analysis. However, the overall perceptual level of the spiral is still relevant to abstract algebra today. For "Classical mathematics has remained rooted in the Newtonian mathematics of analysis, a synthesis of algebra and geometry applied to the study of how things change" (NCTM 1989:478).

4.4 Proof

Proof in mathematics can be described in various ways. Tieszen in Detlefson (1992:60,68,69) gives the following explanations in this regard. He observes that proof is of

> *… a fulfilment of a mathematical intention; a realisation of a mathematical expectation; the solution of a mathematical problem; a programme which satisfies a particular specification*.

De Villiers (1990:1) observes how proof has almost always been seen as a means of verifying mathematical statements. However, as Tieszen pointed out above, proof fulfils various roles in mathematics. De Villiers (1990:1) recognises different functions of proof in mathematics which he substantiates by means of the following relevant quotations:

> *A proof is only a testing process that we apply to these suggestions of our intuitions* (Wilder 1944:318);
>
> *A proof* is *only meaningful when it answers the student's doubts, when it provides what is not obvious* (Kline 1977:151);
>
> *the necessity, the functionality, of proof can only surface in situations in which the students meet uncertainty about the truth of mathematical propositions* (Alibert & Thomas in Tall 1991:31);

> *a proof is an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing* (Hanna 1981:20).
>
> *Why do we bother to prove theorems? I make the claim here that the answer is: so that we may convince people (including ourselves)… we may regard a proof as an argument sufficient to convince a reasonable sceptic* (Volmink 1990:8,10).

Furthermore,

> *to progress in rigour, the first step is to doubt the rigour one believes in at this moment. Without this doubt there is no letting other people prescribe oneself new criteria of rigour* (Freudenthal 1973:151).

Consequently de Villers (1990:2) proposes a model consisting of five functions of proof. These include verification, explanation, systemisation, discovery and communication. He later added a sixth one, termed the function of an intellectual challenge, to the list. Each one of these aspects will be considered in more detail in subsequent paragraphs.

As far as verification is concerned, de Villiers (1990:2) observes that "Proof is not necessarily a prerequisite for conviction – conviction is more frequently a prerequisite for proof". Segal (2000:193) observes how if practising mathematicians with international reputations sometimes find formal proofs that they themselves have written unconvincing then obviously students too would find problems in this area. Bell (1978:24) substantiates this idea with the comment "Conviction is normally reached by quite other means than that following a logical proof" (Segal 2000:193). This would imply in this case that a logical proof alone does not provide a means of conviction. Another problem that can arise regarding conviction is that arguments that convince may not in fact be valid. Students may think that their proof verifies some result when in fact it is erroneous.

The second function of proof mentioned above is explanation. This suggests that a proof should provide some sort of insight regarding the truth of the result. De Villiers (1990:2) observes that although a quasi-empirical verification which could involve such activities as numerical substitutions or accurate geometric constructions may inspire some confidence as to the truth of the result, it does not necessarily explain why it is true.

Hadas and Hershkowitz (1998:3-26) believe that the need for explanation when findings are convincing but surprising can stimulate the student to question why the result is true. The systemisation aspect of proof involves the organisation of results into a deductive system involving axioms and theorems. In this case the results are known to be true and are unified into a coherent whole. Moore (1994:251) describes how an experiment was conducted in which Pierce was required to teach students short deductive proofs in which inferences were based largely on preceding relevant definitions and axioms. He found that students struggled with proofs because of their lack of appreciation and knowledge of the whole deductive system. The students did not know definitions, had little intuitive understanding of concepts, had inadequate concept images, were unable to utilise or generate examples of their own and, moreover, did not know how to make use of definitions or axioms to obtain the overall structure required for proofs.

Discovery or invention is the next aspect of proof that was mentioned above. There are several ways in which discovery can take place. These include intuition and the afore-mentioned quasi-empirical methods before they have been verified by means of proofs and formal deductive processes. When results are discovered by means of considering individual cases, attempting to prove them encourages the person in question to generalise the discovery and consider the relevant conditions, axioms or definitions involved. De Villiers (1999:6) describes proof as

> *…not merely a means of verifying an already-discovered result, but often also a means of exploring, analysing, discovering and inventing new results*.

Communication is another significant function of proof. Segal (2000:193) refers to “The Private and Public aspects of Proof”. It is not sufficient for a proof to merely satisfy the one constructing it because it has to be accepted by the mathematical community too. Sowder (1997) remarks “no researcher actually writes formal, Russell-Whitehead types of proofs, but relies on the confirmations of colleagues and reviewers”. He later continues: “the prevailing view of proof, particularly as it occurs in actual mathematical practice, does seem to have changed from verifying-absolutely to convincing-the-mathematical community” (Segal 2000:194). Proof encourages discussion, interchange of ideas and debate. Proofs have to be carefully thought through before they are conveyed to others. Sharing ideas regarding proofs thus helps prevent people from going astray and being misled.

De Villiers does observe how these different functions of proof are closely linked and has also observed that proof can be considered “…as a means of intellectual challenge” (de Villiers 1999:11). In addition he acknowledges that (Rentz 1981 and van Asch, 1993) also mention “…an aesthetic function or that of memorisation and algorithmization". Thus proof serves many functions in mathematics.

Proof is something which can create many problems for students "before they attain familiarity with the working of the mathematical culture" Tall (1991:19). In van Hiele's original levels of thinking, proof forms part of the fourth level while in his more recent three level adaptation it lies on the uppermost level. However, definitions form part of the third or middle level of his original five level theory. But since formal definitions involve abstraction themselves, here they are considered to be obtained on the abstract level of the spiral. Formal definitions and axioms then form part of the abstract level of a higher round of the spiral and can lead to the establishment of theorems and further definitions at the next and subsequent abstract levels. Thus rising immediately from definitions to theorems without giving through the perceptual, conceptual and then abstract levels can lead to problems of understanding.

De Villiers (2002:2) expresses concern about the way in which: definitions are often presented in such a way that: students might be led to believe that there is only one correct definition for a particular concept as definitions are discoveries rather than human inventions; are not presented in such a way that they arise naturally from previous knowledge. Consequently students often tend to lack concept images of definitions and are unable to use them to reconstruct steps in a proof. Moore (1994:260,261) mentions the organisation or skeleton of a proof, particularly how it begins, how it ends, and how the beginning is linked to the ending by rules of logic and definitions, axioms and / or theorems.

Although proof has primarily been associated with logical deductions from, for example, axioms of groups or fields, non formal proving type of activities can also be seen to take place at the perceptual and conceptual levels too. For example, de Villiers, in his correspondence with the writer, gives the following illustration involving analysis on the conceptual level. It involves explaining why a negative times a negative is a positive using the distributive law. Eventually this result could be generalised at the abstract level by the replacement of specific numbers with variables.

"$(-3)\ x\ ((5 + neg2) = (-3)\ x\ (3) = -9$

BUT

$(-3)\ x\ (5 + neg2) = (-3\ x\ 5) + (-3\ x\ neg2) = -9$

$-15 + (-3\ \text{x}\ neg2) = -9$

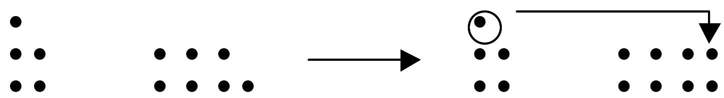
therefore

$(-3\ x\ neg2) = +6$"

As the spiral of learning is climbed, not only do established proofs lead to further ones but they can also develop through levels. Thus a proof that is acceptable at Level 1 may be analysed and become more formalised as higher levels are attained. De Villiers (2005) in his correspondence with the writer has suggested the following sort of illustrations in algebra, geometry and Boolean algebra:

Algebra

Level 1 Eg. Odd number + Odd Number = Even number.



Odd + Odd = Even

Level 2 $(2n-1)+(2m-1)=2(n+m-1)$

Level 3 Formal proof of the statement made at Level 2.

Geometry

Level 1/2 The angles of a rectangle are all equal because they map onto each other by folding around axes of symmetry.

Level 2/3 Using formal definition of rectangles and congruency this is now proved.

Boolean Algebra

Level 1 Practical testing (experimental)

Level 2 Truth tables (checking all cases)

Level 3 Formal derivation from axioms.

One common problem which occurs with proof is that in the process of trying to prove something, it may be essential to prove some necessary lemma that has been assumed true. In his book on Boolean Algebra de Villiers was able to reason backwards in his approach to the topic. This type of practice can be encouraged as a possible means of approaching mathematical proof in general.

Proof is a very necessary part of mathematics. Tall (1991:19) remarks: "Of course it is essential in advanced mathematics to take the step from (generic) explanation to formal proof". Gila Hanna in Tall (1991:60) mentions four relevant points which should be kept in mind when teaching proof: formalism and justification are necessary to enhance learning; providing mathematical experiences is not enough but rather reflection on experiences leads to growth; develop in the student the ability to tolerate ambiguity and appreciate pictorial explanations; help students to judge the necessary amount of rigour needed to be applied. Proof needs to be taught with careful thought and consideration because "… a teaching activity that includes formal or informal reasoning can be judged to be of value only to the degree that it promotes understanding" (Tall 1991:60).

4.5 The third or final level of various theories of thinking levels

The abstract level of thought, which has been illustrated in the history of mathematics, can also be recognised in human intellectual development. It involves abstraction and includes such activities as defining, axiomatising and proving. Even low down on the spiral of learning, an abstract level can be seen to be attained when some form of structure of subject matter is established. The characteristics associated with the abstract level in various theories of thought development are considered in this section of the chapter.

4.5.1 Piaget's third level of development

(i) Piaget's original third level of development

The original third level of development of Piaget coincides with adolescence. This is the stage at which a child becomes able to work with abstract ideas. Furthermore, a child should be able to engage in deductive hypothesis testing. It also becomes possible to use vocal rather than concrete propositions in problem solving. An example of this would be deducing that if $a > b$ and $b > c$ then $a > c$. At this stage it should also be possible for a child to understand metaphors.

(ii) Mathematics associated with Piaget's original third level of development

As far as mathematical topics are concerned, there are various topics associated with this level. However, these are largely of a very basic or general nature. They include numeric symbols being replaced by algebraic ones, algebra, logic, vector diagrams and formal science. In some of these cases the activity level could be considered as forming part of the abstract level round of the spiral very low down on the spiral.

It is when students begin secondary school that they are formally introduced to the subject of algebra. In order to be successful in the use of algebraic symbolism, students need to be able to see patterns and express generalisations. In other words they need to have passed through the perceptual and conceptual levels reaching up to the relevant abstract level of symbolism. The age at which pupils enter high school coincides with the onset of formal operational thought in Piaget's original theory. This is the stage when students should be able to abstract meaning from events and experiences. For example, students should be able to appreciate that all odd numbers are given by substituting all the natural values for n in the expression $2n-1$.

The fact that students have to change from using a number like 6 to represent 6 objects to the algebraic notation of x to represent any number of objects makes it necessary for students to represent a number in an abstract way. As they have passed through school, they have progressed from using a certain number of counters to represent a number, to a numerical symbol and subsequently to an algebraic symbol. However, many students find algebra extremely abstract and become very frustrated when they do not understand what they are supposed to do. They do not seem to have reached the relevant abstract level of the spiral. In terms of Piaget's original model it appears that they have not reached the stage of formal operational thought. For example, they might be able to understand mathematical equations using numbers only but do not see the point of using letters to make them more applicable in a broader sense. It would seem that in many cases children would not have passed through all the relevant rounds of the spiral leading up to the algebra they are studying.

The transoperational level may be related to Piaget's original highest level of thoughts. At this stage. "… There are not only transformations, but also synthesis between them, leading all the way to the building of "structures" albeit only at the level of actions and without thematization" (Piaget & Garcia 1989:178). Piaget believes that one of the most noteworthy of these structures unites inversions and reciprocities – i.e. the two forms of reversibility (Piaget & Garcia 1989:178). He notes that it includes the total set of parts without being limited to disjointed relations or classes that are to be structured. He believes that the structures obtained in this case could be regarded as an authentic group, which he calls *INRC* since for any operation such as $p \to q$, it can be left identical (*I*), can be converted to negation $(N) = p \bullet \bar{q}$, transformed to its reciprocal $R = q \to p \ (\bar{p} \to \bar{q})$ or also to what he terms its correlative (*C*) $= \bar{p} \bullet q$. Piaget and Garcia (1989:178) observe that hence *NR* = *C*, *NC* = *R*, *CR* = *N* and *NCR* = *I.* This formulation may belong to the observer but Piaget and Garcia (1989:178) remark how "… subjects from the ages of eleven and twelve can construct such synthesis where it is necessary to coordinate negations and reciprocities within a unified system".

Thus it is evident that Piaget even saw the elements of group theory in the way in which thoughts develop in students. He considered the problem of what brings this change from percepts to concepts and then finally to the abstract level. He noted that even at a young age (or low down the spiral of learning) students were able to make use of the simplest groupings such as seriations and classifications at the interoperational level or a conceptual level and move from the grouping characteristic of the intermediate level to the synthesis associated with the transoperational or abstract level.

The stages of development of Piaget's original theory are so deeply entrenched in the school syllabus that "… it is difficult to see them as being otherwise" (Zevenbergen 1993:9). According to Piaget's original theory it is cognitive maturity that is used as an explanation as to why some children might be unable to do certain mathematics. Piaget's original theory did give a correspondence between changes in cognition and biological changes. This would mean that teachers should adopt their approach to the stage associated with the majority of their pupils. According to Piaget's original theory, at certain stages "… people were biologically ready for learning certain things and that there were only certain things which could speed up the process" (Zevenbergen 1993:10).

Piaget's later third level

At a later stage of his life, the third level Piaget introduced was called the transoperational level which "… is easy to define as a function of what precedes" (Piaget & Garcia 1989:178). At this stage, there are not only transformations as at the interoperational level but also synthesis between them which leads further on to the development of structures. Piaget and Garcia (1989:142) describe how "The transoperational stages are characterised by the evolution of structures when internal relationships correspond to interoperational transformations". Piaget and Garcia (1989) claim that the intra, inter and transoperational stages represent three different ways of organising knowledge. However, they claim that the identification of the three different stages are more complex in algebra than they are in geometry because the process of algebraization of mathematics does in itself constitute a transoperational stage. This gives an idea of the range of levels of the spiral that need to be covered in the process of learning a topic in algebra.

Once the intra and interoperational acquisitions have been subordinated to sets of transformations generating the figures,

> *the primary and final victory of endogenous factors elaborating the structures which no longer consist in "figures" (such as a group) but integrate all reliable constructions within total systems* (Piaget & Garcia 1989:139).

The concept of a group, which is established at the overall abstract level of algebra, is not dependant on any particular objects or situations but may be found to be relevant in many contexts. Furthermore, discovery of the group structure was by no means a final result because as groups themselves become the object of study, this can lead for example to more theory on groups, the definition of a field and more theory related to fields. Piaget and Garcia (1989:139) remark that structures such as group structures, "… once elaborated and hence intrinsically necessary, can in turn be treated as data, as a kind of "pseudo-exogenous" reality, thus becoming a potential object of new intra type analyses". This alludes to the repetitive pattern of the intra – inter – transoperational triad as the learning spiral is traversed.

As far as psychological development is concerned, the transoperational level is reached once the student is able to carry out operations on operations. An example of this would be when a child becomes able to derive all possible permutations of n elements at the very time when he or she starts to be able to systemize or introduce an ordering with respect to permutations executed. Piaget and Garcia (1989:155), referring to this example of permutations, state that "… the set of permutations derives from a seriation of solutions". Galois himself introduced the idea of a group as a result of the action of grouping. He observed how "When we wish to group substitutions we shall have them all originate from some permutation" (Piaget & Garcia 1989:156). In this historical statement, the change in thinking from the interoperational level to the transoperational level can be seen as operations on elements progressing to operations on operations.

Later on at a higher level of the overall transoperational level of the spiral, many sets of elements were studied and were found to possess the same properties as numbers do. These properties were found not to be specific numbers but a specific domain of mathematical objects. Dedekind reached the intraoperational or abstract level of the spiral when he named such a structure a field. The development of groups and fields seems to give a clear reflection of the way in which concepts develop in the minds of learners. Both the descriptive and constructive historical approaches play a part in the development of abstract algebra.

4.5.2 Freudenthal's third thinking level

(i) Characteristics of this level

According to Freudenthal, the activity which belongs to the third or uppermost level is when results are "… put into a linguistic pattern" (Freudenthal 1973:123). This certainly does correspond to the level of linguistics or abstractions which Piaget refers to as the transoperational level.

At the transoperational or abstract level of mathematics, Freudenthal (1973:123) observes that the mathematics is not just a collection of mathematical derivations with no obvious common principles. Instead the whole field becomes organised and language formulations become the matter of reflection. As students move from the intermediate level to the upper level, the organisation of the lower level becomes the subject of the higher level. Freudenthal (1973:123) believes that the relation "… between one level and the next is overwhelmingly logical and accessible to logical analysis". Freudenthal strongly believes that reinvention plays a very important role at all levels of mathematics. In fact, he claims that if a student is not able to reflect on his or her own activity, then "… the higher level remains inaccessible" (Freudenthal 1973:130).

(ii) Mathematics at Freudenthal's third thinking level

Freudenthal (1973:133) believes that in the modern day students should learn how to mathematize the unmathematical so that it may be organised into a structure upon which it is possible to perform refinements. He believes in a broad approach rather than merely placing a pyramid on the top of what came before. This suggests the importance of all the rounds of the spiral leading up to the relevant abstract level.

Freudenthal claims that almost every reasonably formal theory of natural numbers presupposes some familiarity with the principle of complete induction. He believes that such a theory belongs to an advanced stage of school instruction. The three levels of thought involved in the learning of complete induction are clearly indicated by Freudenthal. As mentioned before, at the lowest level it is acted out while at the second or conceptual level it becomes a subject matter upon which learners can reflect. Once it reaches the uppermost level, it is placed into a linguistic pattern. Another example of levels in teaching mathematics given by Freudenthal (1977) involves operations with integers in which activities are provided on the first perceptual level, making them the subject of reflection occurs at the second or conceptual level and formulating rules follows on the third or abstract level. In a similar manner fractions may be operated with intuitively on a perceptual level, simplified and operated with at the conceptual level and further formalised at the abstract level. Freudenthal disapproves of the way in which pupils may, for example in geometry, be shown various examples of a parallelogram and then immediately required to define it. He believes that students should rather be given the opportunity of inventing the definition or else "… a level is passed by, and the student is deprived of the opportunity to invent that definition" (Piaget & Garcia 1989:134). Instead the student should be given the opportunity at the perceptual level of investigating properties and connections between properties to prepare him or her for finding one property from which all others can be derived. Then he or she will truly have passed through the conceptual level and will be ready to define at the abstract level.

Freudenthal gave another example of considering levels when teaching that $\sqrt{2}$ is an irrational number. The old *method* would be to immediately show that if $\sqrt{2}$ were rational, it would be expressed as the ratio of two integers. In other words $\sqrt{2}$ would be expressed in the form $\frac{p}{q}$ where *p* and *q* are integers, *q* is non-zero and $\frac{p}{q}$ is in its simplest form. By squaring both sides and considering divisibility, a contradiction can be reached. However, the formulation of the question here immediately omits the first level. Freudenthal believes that if instead the question were formulated as an instruction to work out $\sqrt{2}$ the problem would not be possible as the student could not have insight available to follow the instruction. He rather suggests considering a graph related to the solution of an equation like $x^3 + 3x^2 + x - 4 = 0$. The graph reveals that there is a solution although it is not integral or fractional. An approximate value may be obtained by substituting fractional values and finding which are too small or too large and approximating the solution. In this way the students begin to develop a feeling for an irrational number.

Freudenthal mentions the words comprehension and apprehension as describing two ways of acquiring generalities. He compares comprehension which involves "Generalities by gathering many together" (Freudenthal 1973:197) as opposed to apprehension involving "… seizing a structure, albeit by an example, by one example" (Freudenthal 1973:197). He believes that numerous instances and repetitions are necessary to develop concepts. This leads naturally to the establishment of properties, drawing of comparisons and then ",,, phases of abstraction or formalisation" (Freudenthal 1973:241) so that students become aware that "The mathematical language is neither an arbitrary invention nor a jargon detached from any content" (Freudenthal 1973:241). It is important that students become involved in re-inventing mathematics because "… the joy of discovering motivates activity in general" (Freudenthal 1973:187).

4.5.3 Van Hiele's upper level (levels) of learning

(i) Van Hiele's original upper level involving informal deduction, formal deduction and rigour.

Informal deduction (3rd level – Level 2)

Most of the characteristics attributed to this level have been considered to belong to the intermediate or conceptual level of learning. However, here precise definitions are considered to be understood and accepted by students only at the abstract level. This is because this understanding follows not only from recognising the concept and its properties in various contexts at level 1 but also comparing and interrelating properties at level 2. In this way the student is led to re-invent the definition for him or herself.

Formal deduction (4th level – Level 3)

Once students have arrived at this level, they become capable of establishing theorems within an axiomatic system and also rely on proofs to establish the truth of statements in mathematics. It becomes possible to distinguish between undefined terms, axioms, definitions and theorems. Scholars are also able to construct original proofs. At this level the type of reasoning leads to the establishment of second order relationships. It is necessary for pupils to reach the formal deduction level if they are to attain some success in high school geometry courses.

In the spiral theory, this level could be regarded as being a level of abstraction higher up on the scale than the abstract level of the definitions upon which the theorems depend. For once definitions have been established, they become the objects of study at a new perceptual level. Once inter-relationships between them have been investigated, this would lead to a new abstract level higher up the learning spiral.

Rigour (5th level – Level 4)

This is the uppermost level at which students reason formally about mathematical systems. Reference models are no longer needed for studying geometry. Formal reasoning involving manipulating statements such as definitions, axioms and theorems becomes possible, strongly suggesting that this is an abstract level high up on the learning spiral. The level of rigour may be described as the level of thought of the mathematician where reasoning leads to the establishment, development and drawing of comparisons between axiomatic systems of geometry. It is commonly believed that this is such an abstract level of geometry that it is unattainable to many. The impression is created that this is the case because it is an abstract level so high in the learning spiral that only the truly mathematically minded can in fact reach it.

(ii) Van Hiele's later third level or theoretical level

The theoretical level of the later van Hiele model seems to closely resemble Piaget's transoperational level or the level of abstractions. For this is the level at which most relations can be expressed in terms of language and mathematical symbols. Theories are seen to be the creations of one or a few people. At this level definitions and proofs are viewed as being infallible. The student is, moreover, able to use deductive reasoning in order to prove relationships.

It is interesting how in their later lives both Piaget and van Hiele narrowed down the number of levels in their learning theories to three. These levels seem to be remarkably similar and closely resemble Freudenthal's notion of levels too. Once the abstract level has been attained, there always seems to be the possibility of results obtained being further investigated and analysed in order to reach a new and higher abstract level of the spiral.

4.5.4 Land's third or highest level of learning

As in the case with van Hiele, Land's third level of learning here will be considered to incorporate some of her theoretical informal level as well as her formal theoretical level.

The informal theoretical level (3rd level – Level 2)

Although most aspects of this level have already been considered in the conceptual level of learning, there is one characteristic which seems to be too abstract for the present scheme and it has not been considered as forming part of the intermediate level here. This includes the accurate and concise definition of words. Students would need to have gone through the initial stage of recognising the concept in various contexts and the intermediate stage of investigating and interrelating the relevant properties in order to lead to a solid appreciation of definitions at the abstract level. As has been mentioned with regard to van Hiele's levels, once this level has been attained, definitions can form the objects of study leading to further results at a higher abstract level.

The formal theoretical level (4th level – Level 3)

At this level objects include the partial ordering of properties. In her study the properties being considered were those relating to exponential and logarithmic functions. The significance of deductions, postulates, theorems and proofs are understood. Students become capable of using symbols with insight and understanding in order to construct a proof or solve a problem. In addition, verbal summaries of arguments by students become possible. Two types of questions that students could be asked at this level include:

$prove\ \log(x-\sqrt{x^2-1}) = -\log(x+\sqrt{x^2-1})$

$solve\ for\ x:\ \log_x 1 < -1$ (Nixon 2002:57)

4.5.5 Nixon's third or highest level of learning

As was the case with van Hiele and Land, the third level here includes part of the theoretical informal level and all of the formal theoretical level.

3rd level – Level 2 - the informal theoretical level

The main sort of characteristic from this level which is being associated with the abstract level is the ability to define concepts accurately and precisely. This would need to follow the perceptual and conceptual level stages. Some of the other characteristics which are included in the second level are there because they could form part of the conceptual level though not at the bottom of the spiral but at some round further up it. Some of these sort of characteristics include solving an equation involving manipulation of symbols, following a derivation of a formula or a deductive argument.

4th level – Level 3 - the formal theoretical level

The characteristics mentioned at this level are related to sequences and series but could be adapted to apply to relevant topics in algebra. They include: using information about sequences and series in order to deduce more information; appreciating the importance of deducing; making use of definitions for the establishment of arguments; understanding the significance of postulates, theorems and proofs; having the ability to use symbols with insight and understanding with a view to solving a problem or constructing a proof; being able to formulate arguments based on diagrams or visual representations; having the ability to make a verbal summary of an argument.

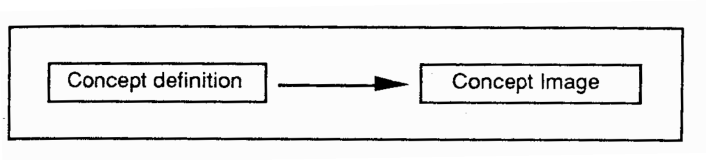
4.5.6 Concept definition and concept usage

Concept definition forms part of the abstract level of the development of mathematical concepts. Vinner in Tall (1991:65) observes that it is responsible for a serious problem in the learning of mathematics because “It represents, perhaps, more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition”. Mathematics courses at school are generally built up from a sequence of axioms, definitions and theorems. Vinner in Tall (1991:61) observes how the presentation of mathematics in textbooks and classrooms generally assumes that “Concepts are mainly acquired by means of their definitions”. However, this generally becomes possible only for capable students who have already passed through several previous rounds of the spiral. Furthermore, definitions are supposed to be as economical as possible and to be used by students to prove theorems and solve problems. Another characteristic of definitions is that they are arbitrary as they are man made and result from some concept being given a name.

Fodor (1980) in Tall (1991:67) argues that definitions do not perform the same function in non-technical situations as they do in technical ones. However, in technical situations it is important to acquire a concept image for a concept definition. Vinner in Tall (1991:68) defines a concept image as “… something non-verbal associated in our mind with the concept name”. This image could be a visual representation of the concept or a series of impressions and experiences which have been acquired with regard to the concept. Vinner refers to what he describes as an “evoked concept image” (Tall 1991:68). For example, the word “function” could evoke the image of the expression $y = f(x)$; a type of function machine sometimes used to teach functions, one-to-one or many-to-one mapping diagrams, a table of values, a graph of a function or graphs of particular functions like $y = x$, $y = \cos x$ or $y = \log x$.

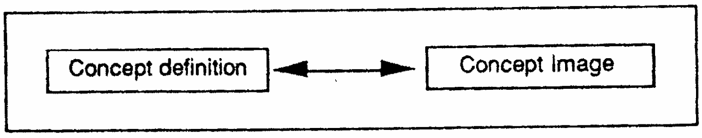
In order to acquire a concept it is necessary to obtain a concept image for it. Obviously merely knowing a concept definition by heart does not imply understanding of the concept. In order to give meaning to the words used for a concept, it is essential that time be taken to give meaning to it. Vinner in Tall (1991:69) points out how the thought habits in everyday life are not the same as those in technical contexts.

It is very important that there be an interplay between concept definition and concept image. At first concept image should lead to concept definition and not concept definition merely lead to concept image as depicted below:



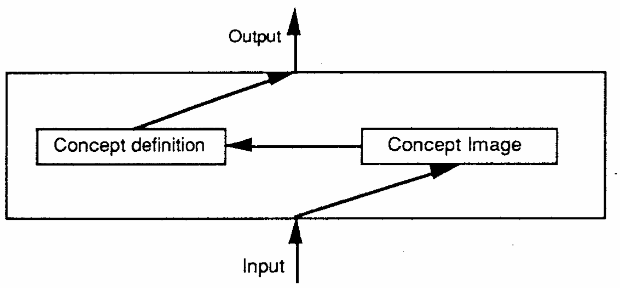
(Vinner in Tall 1991:71)

Higher up the spiral this could serve as an analogy with “a priori” axiomatisation. However, ideally what should result is an interplay between concept definition and concept image as follows:



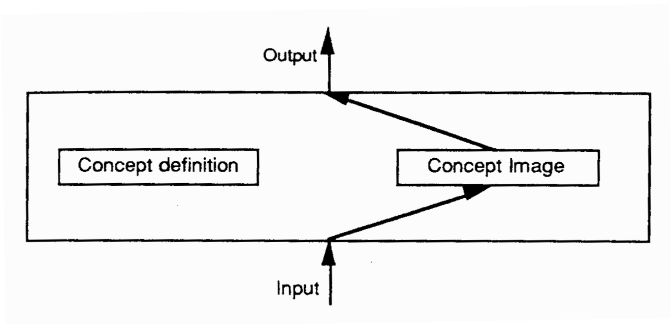
(Vinner in Tall 1991:70)

The diagram below gives one of Vinner’s examples of how intuitive thought can be channelled ino deductive thought. Here the concept image is first formed before the definition is sought and this corresponds to “a posteriori” axiomatising.



(Vinner in Tall 1991:72)

The abstract level of learning is very important. When it is omitted, the learning situation could be represented as follows:

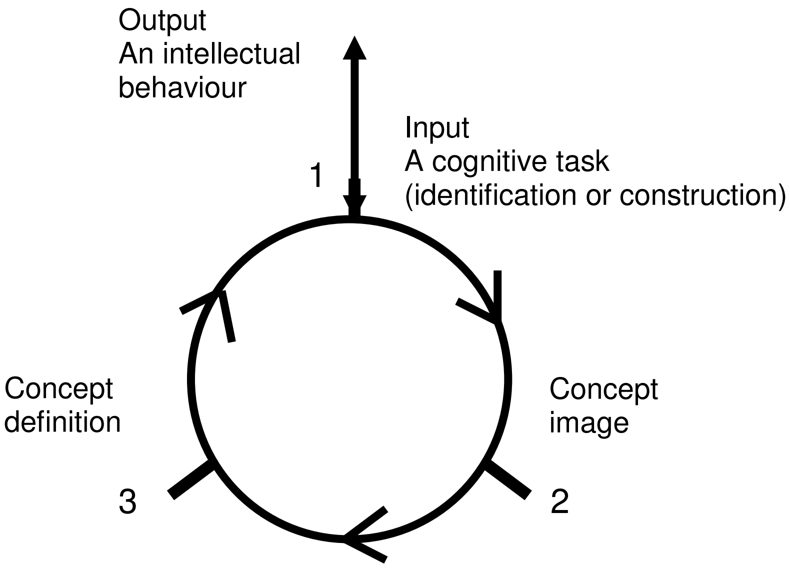


(Vinner in Tall 1991:73)

This causes serious problems because very often proofs depend on definitions. Thus if students do not know definitions they are unable to provide proofs. Students complain that their lack of understanding of definitions prevents them from understanding theorems. But it is very important that the abstract level be reached where it is possible to formulate a definition before further advancement is made.

Concept usage involves both theorems and application of what has been learnt. If concept usage involves a theorem, then it is necessary to operate with definitions and other relevant properties on the perceptual level in order to reach the conceptual and finally abstract levels needed to establish the theorem. Initial applications of results established at the abstract level would at first be of a perceptual nature. These could then lead to deeper insights at the next conceptual level and further results at the corresponding abstract level. In this way the spiral of learning would continue to evolve with concept usage emerging from the abstract level and leading to the next perceptual level.

The diagram presented by Vinner in Tall (1991:171) could be adapted to incorporate the idea of a spiral of learning as follows:



4.5.7 Conclusion

The attainment of the abstract level is significant because in the growth of learning

> *There must be insights, having their origin in the observational and being able to operate abstractly but with the possibility of descending into the observational layer for support whenever necessary* (Duminy & Sőhnge 1990:205).

The abstract level represents the establishment of structure in a spiral of learning. In the overall development of abstract algebra, it can be seen in the emergence of the concept of a group. However, this was merely the beginning of a whole new phase of mathematics. Within the overall abstract level of algebra many new rounds of the spiral can be seen shooting off in various directions.

Although any new topic needs to begin at the perceptual level and pass through the conceptual level, it is the attainment of the abstract level that is the ultimate aim. Not only does it raise the student to new heights of learning, but it opens the door to new possibilities. Results established at the abstract level may form the subject of investigation at a new perceptual level and lead off in an “a priori” or “a posteriori” manner to new discoveries at further conceptual and abstract levels of the spiral.

Encouraging students to participate and pass through the perceptual, conceptual and abstract levels of learning helps to establish a mode of investigation and a way of thought. Ultimately they become accustomed to the processes involved so that they could become independent in their study and learn to study and follow a mathematics textbook on their own. Furthermore, it would help to establish an awareness of the endless possibilities of growth of mathematics as one discovery continually can lead up the spiral to another.