Chapter 3
The conceptual level or the intermediate learning level and sublevels in the development of abstract algebra

3.1 Introduction

The intermediate level of thought is the one termed interoperational by Piaget. The change from the first stage of thought or intraoperational level to the second level is marked "… by a total reinterpretation of the conceptual foundations" (Piaget & Garcia 1989:109) rather than an increase in the amount of knowledge. The intermediate thinking level is characterised by efforts to find relationships and search for transformations according to various forms of correspondence. This level of thought was reached in algebra during the second half of the eighteenth century when mathematicians began to form algebraic problems of greater generality including the fundamental theorem of algebra.

The intermediate stage of thought in van Hiele's thinking levels could be related to part of his original third level (Level 2) or his later second level. At this stage students are able to not only understand properties of classes and figures but also interrelate properties of classes and figures. Van Hiele describes his original intermediate level as the level at which definitions are understood whilst proofs of theorems are placed on the subsequent level. Here formal definitions will be considered to form part of the third or abstract level of a spiral. However, at the intermediate level students would still be at the intuitive stage. After topics have been encountered at the perceptual level and the concept has been firmly established at the conceptual level, formal definitions would become possible at the abstract level. Definitions will thus be rooted in the perceptual level and operating with them informally at the conceptual level will lead to formalisation at the abstract level. Both formal definitions and proofs will be considered as being abstract level activities although theorems would generally be higher up the learning spiral than the definitions upon which they are based.

Even though not all theories consist of three levels of thought, the intermediate level (or levels) of thought of various topics are considered here. Parallels are drawn between the development of the second level of thought in history, in the minds of learners and the way in which the subject is presented. Sublevels of the interoperational level are also considered in this chapter. However, in the spiral theory, sublevels may be regarded as branches of the spiral.

Rising up from the perceptual level to the conceptual level or sublevel is an important step in the acquisition of knowledge because “It is very important that the thinking person should be able to escape from the multiplicity of tangible, time-bound particulars of the physical presence of objects” (Duminy & Sőhnge 1986:202). With time and experience, individuals could learn to create their own mental pictures when starting at the abstract level but initially this does not seem to be the ideal method.

3.2 The interoperational or conceptual level in the history of abstract algebra

3.2.1 Introduction

There was very little progress towards the development of the theory of groups during the seventeenth and the first half of the eighteenth centuries. Although many attempts were made to find a formula for solving equations of degree higher than four, each problem had its own method of solution and so this could be called the intraoperational period. It was during this time that mathematicians began to pay attention to the new instrument of infinitesimal calculus which was created by Leibniz and Newton. When mathematicians like Euler, Lagrange and Gauss made use of this calculus, it led to the interoperational stage in abstract algebra beginning from the middle of the eighteenth century. It was Evariste Galois who later introduced modern algebra at the transoperational level and “...the most important predecessors of Galois were Lagrange, Gauss and Abel” (van der Waerden 1985:76).

Freudenthal (1973:123) discussed the history of the group concept and believed the following activities belonged to van Hiele’s descriptive level of groups: “… in the first half of the nineteenth century instinctive operations with groups in some cases up to a high level of consciousness of principles”. Bell too singled out the activities relevant to van Hiele’s various levels of thought associated with his original model. Here the attributes associated with the original second and third levels all seem to be relevant to the conceptual level of the spiral theory. Bell (1945:246) observes:

> *… then the recognition of certain features common to all (second level – descriptive); next the search for further instances, their detailed calculation and classification (third level – theoretical)* (Land 1990:27).

The historical development of groups reveals characteristics of both the descriptive and constructive development of history. Van Hiele seems to see a close link between the evolution of mathematical topics such as group theory and the general pattern of human thought development. For he writes: "… the levels are situated not in the subject matter but in the thinking of man" (van Hiele 1986:41).

3.2.2 The historical emergence of the interoperational or conceptual level in abstract algebra

At the historical beginning of the conceptual level of algebra, the great mathematicians of the time, such as Newton, Leibniz, the Bernoulis, Euler, Lagrange and Laplace began to devote their time to infinitesimal calculus and promoted its development. However, Freudenthal's research seems to indicate that although the universities of the day had accepted Cartesianism and even the Copernician system, in the eighteenth century there was no place where people could learn calculus. This feature seems to not have had a precedent in history and Freudenthal observes how the universities of the eighteenth century were in fact quite inert. Freudenthal (1973:11) remarks on this astonishing situation: "How could this happen? Was it is so difficult to sell calculus to the universities that had with much pain converted to Cartesianism and even accepted the Copernican system?". By then infinitesimal calculus had surpassed algebra but virtually no leading mathematicians taught at the universities. Instead they were regarded as academics and belonged to learned societies. Fortunately oral teaching was not the only means of communication as books were introduced. People could read calculus in books and some did try to do so after Newton because it was needed for astronomy. However, more people learned algebra than calculus. The fact that printing had been invented meant that many leading scientists were happy that they did not need to establish schools. However, this caused serious damage to the tradition of science which had to be rectified in the nineteenth century. It hampered the development of algebra too. Active science returned to the universities as the École Polytechnique was founded in France and the Humboldt reforms were taking place in Germany. However, the pace of change was not as quick at all places or in all branches of teaching.

After many mathematicians had helped to prepare the way, Newton and Leibniz were responsible for: "… the epoch-making creation of the calculus" towards the end of the seventeenth century. Isaac Newton was born in Woolsthorpe hamlet on Christmas day, 1642 and died in 1727. Recent research has shown that Newton developed his calculus (reaching the point of finding the tangent and radius of curvature at some arbitrary point of a curve) at Cambridge University in 1666. This was during the short temporary opening of the university after it had been closed as a result of the spread of bubonic plague. From 1673 to 1683 Newton's university lectures were devoted to algebra and the theory of equations.

The main substance of the lectures given by Newton from 1673 to 1683 is contained in "Arithmetica Universitalis". Many important results concerning the theory of equations are to be found in this book. These include: the fact that imaginary roots of real polynomial equations occur in conjugate pairs; rules enabling the upper bound to the roots of a real polynomial equation to be found; formulae he had derived for expressing the sum of nth powers of the roots of a polynomial equation in terms of the polynomial's coefficients; an extension of Descartes' rule of signs in order to give limits to the number of imaginary roots possessed by a real polynomial equation. Newton (1642-1727) made so many contributions to mathematics and science that Alexander Pope (1688-1744) praised him for revealing Nature's laws by writing that "God said, 'Let Newton be', and all was light" (Eves 1990:402). However, in contrast to this can be found the following humble statement that Newton has been recorded to have made regarding his achievements: "… if he had seen further than other men, it was only because he had stood on the shoulders of giants" (Eves 1990:403). This seems to suggest that he was able to climb up sections of the spiral of learning and reach more advanced levels because of all the rounds of the spiral that had been traversed before his time.

Gottfried Wilhelm Leibniz, born in 1646, was a great genius and the rival of Newton in the invention of calculus. The last seven years of his life (1709 – 1716) were embittered by controversy others brought upon both him and Newton. This was concerned with whether he and Newton had discovered the calculus independently. Leibniz had invented his calculus at some stage between 1673 and 1676. The long letter $\int$ for the modern integral sign, derived from the first letter of the Latin word "summa" meaning "sum" and indicating the sum of Cavalieri's indivisibles, was first used on 29 October 1675. Leibniz was partly responsible for raising algebra further up the spiral to new abstract levels, resulting in a rise from the overall intraoperational or perceptual level to the interoperational or conceptual level of abstract algebra. This was because "Leibniz had a remarkable feeling for mathematical form and was very sensitive to the potentialities of a well-devised symbolism" (Eves 1990:405). Two important contributions Leibniz made to the development of equations were: the theory of determinants with reference to systems of simultaneous linear equations in 1693 (a similar consideration had, however, been made by Seki Kōwa in Japan ten years before the time); the expansion of $(a+b+c...+n)^r$ by generalisation of the binomial theorem to a multinomial theorem.

The Bernoulli family of Switzerland was one of the most distinguished families in the history of mathematics. From the late seventeenth century onwards an unusually large number of thirteen capable mathematicians was produced from this family. They were some of the first mathematicians to realise the great power of calculus and apply it to a variety of problems. It was arranged for Leonhard Euler, born in Basel, Switzerland in 1707 and whose father had studied under Jakob Bernoulli, to study under Johann Bernoulli. Soon after Euler reached twenty years of age, he became the chief mathematician of St Petersburg Academy formed by Peter the Great. Euler published 530 books on many branches of mathematics even though he became blind in his latter years.

Euler is another mathematician who took abstract algebra up the spiral into the interoperational or conceptual stage. He introduced the notations: $f(x)$ for functional notation; $\sum$ for the summation sign and i for the imaginary unit $\sqrt{-1}$. The important foundations he laid as he climbed the learning spiral of algebra are reflected in the following ways: his algebra books "served more than any other writings as models in style, scope and notation for many of the college textbooks today"; "so many of the great mathematicians coming after him have admitted their indebtedness to him" (Eves 1990:435).

At the beginning of the nineteenth century plenty of attention was paid to applied mathematics. The Frenchmen Fourier, Poisson and Cauchy were all significant at this time when it was an honour to belong to the mathematical society called "Ecòle Polytechnique". However, Freudenthal observes how "… the overestimation of applied and underestimation of pure mathematics did much harm to mathematics as a whole in the 19th century France" although "… the shift towards applications was enormously beneficial" and also contributed to "… the development of mathematics of the 19th century" (Freudenthal 1973:15).

Certain mathematicians during the interoperational period of algebra paid much attention to the study of the types of numbers which could solve various sorts of equations. Euler's work gave a complete indication of the numerical value of π.

Although various methods of calculating π were found, Beckman (1977:156) claims that none of these gave the value of π faster than Euler's method. Euler reached a new round of the spiral in the conceptual or interoperational level when he posed the question of whether or not π could be the root of an algebraic equation with rational coefficients. Beckman (1977:156) remarked that "By merely asking the question, Euler opened a new chapter in the history of π."

There seemed to be no reason why an irrational number should not be the root of a general polynomial equation of the form $a_n x^n + a_{n-1} x^{n-1} + \ldots\ a_2 x^2 + a_1 x + a_0 = 0$ where a_i are all rational. For example, $\sqrt{2}$ is the solution of an algebraic equation $x^2 - 2 = 0$. By Euler's time, suspicion arose as to whether or not "… there were 'worse' numbers than irrational ones, namely numbers that were not only irrational, but that could not even be roots of an algebraic equation. Such numbers are called 'transcendental'" (Beckman 1977:167). Their existence was not obvious. For example, the equation $\sin x = \frac{1}{2}$ is transcendental because it is not algebraic. When expanded as a power series, it becomes $1 - \frac{1}{2}x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \frac{1}{2}$. $\frac{\pi}{6}$ is a solution but the question arose as to whether a transcendental equation must have a transcendental solution. Furthermore, the transcendental equation $\sin x = 0$ which has, for example π as a solution, also has the solution $x = 0$, which is obviously an algebraic number.

Even the Greeks long before Euler's time were aware of the idea of the existence of irrational numbers or numbers that could not be expressed as a ratio of two integers where $\{\text{Integers}\} = \mathbb{Z} = \{\ldots\ \text{-}2\ ;\ -1\ ;\ 0\ ;\ 1\ ;\ 2\ ;\ \ldots\}$. Euler had posed the question of what kind of number π is. His question of whether it was rational or irrational, algebraic or transcendental, haunted mathematicians for some time afterwards. Thus Euler played a part in the intra or perceptual level of the spiral of transcendental numbers. After investigations had occurred at the conceptual or interoperational level, the existence of transcendental numbers was found at the abstract or transoperational level by Joseph Liouville (1809–1882) when he showed that numbers could be defined (as limits of continued functions) that cannot be the roots of any algebraic equations. Transcendental numbers are of interest because they have many interesting properties which can be derived further up their spiral and the transcendence of π answers the question of whether or not the circle is squarable. In 1794 Legendre had proved the irrationality of π in a rigorous manner and only 88 years later was his conjecture that π cannot be the root of an algebraic equation with rational coefficients and a finite number of terms proved. This was a very important proof that formed part of the overall transoperational or abstract level of algebra..

Jean-le-Rond d'Alembert was born in Paris in 1717 and died there in 1783.He recognised the potential for growth in algebra in his remark: "Algebra is generous; she often gives more than is asked of her" (Eves 1990:438). Johann Heinrich Lambert (1728–1777) was born in Switzerland and was the first to prove that the number π is irrational. Thus the spiral of number systems continued to grow during the interoperational or conceptual level of solving algebraic equations.

Joseph Louis Lagrange (1736–1813) and Euler have been referred to as the greatest mathematicians of the eighteenth century. Piaget and Garcia (1989:150) regard Lagrange as being: "The central figure in the transition from the intraoperational to the interoperational stage". Whereas in the intraoperational stage empirical efforts had been made to solve equations of different degrees, Lagrange instead adopted a more general scope. He considered the nature of the solutions given to equations of the third and fourth degrees and pondered over the reasons for their success. He believed that in this way he would be able to gain ideas of how to solve equations of higher degrees. He managed to show that all the methods involved introducing functions to reduce the original equation. In this way the problem amounted to seeking the relationship between the solutions of the reduced equations and the original ones.

Once Lagrange had reached this idea, he began to consider another very important idea which later led to the theory of groups. He started to consider how many different values a polynomial is able to take when it is permuted in all possible ways. He analysed certain functions of the roots of an equation. For example, if $x_1, x_2, x_3 \text{ and } x_4$ represent the roots of a fourth degree equation, the function $y = x_1x_2 + x_3x_4$ can take on only three different values and this: "… determines the degree of the reduced equation which permits the solution of the equation in question" (Piaget & Garcia 1989:151). He also showed that the approach he had followed failed to work in the case of a quintic. This typified the type of activity being performed at the interoperational or conceptual level.

It was declared by Novy (1973:43) "… the most expressive requirement for a new approach was formulated by Lagrange in his bid for an analysis of the method used a priori." Definite signs of the interoperational level may be seen in the work of Alexandre – Théophile Vandermonde when in 1770 he presented to the Paris Academy a memoir in which, beginning with the well known solution of quadratic and cubic equations, he developed general principles for the solution of equations. The method of Vandermonde and Lagrange led to the solution of the cubic equation. Vandermonde's method did not give a general solution for degrees larger than 4, though it did work in special cases. He succeeded in solving the equation $x^n - 1 = 0$ by first reducing it to an equation of degree 5. Thereafter he solved this quintic equation by introducing his Lagrange resolvents which may be written in the form $L = x_1 + \alpha x_2 + \alpha^2 x_3 + \alpha^3 x_4 + \alpha^4 x_5$ where α is a primitive fifth root of unity and x_1, x_2, x_3, x_4, x_5 are the roots of the quintic equation.

Lagrange clearly did play a significant role in the interoperational stage of algebra. Novy observes how even though Vandermonde "… had not arrived at theorems of such importance in the theory of permutations or at such a clear and wholesome interpretation as Lagrange", he nevertheless stressed the study of permutations in relation to the study of the solution of algebraic equations, introduced certain methods and concepts as well as symbolic means which played an important role in the development of the theory of permutations later on (Novy 1973:40). For example, Vandermonde introduced the idea of a cyclic permutation and used symbols to introduce the composition of permutations of cyclic compositions as well as the resolution of cyclic permutations into smaller cycles.

Eduard Waring was another mathematician who contributed to the further evolution of algebra during the time of Lagrange and Vandermonde. He made a study of the problems of solving equations from various aspects. When studying the functions of the roots, he came to similar conclusions to Lagrange's regarding the solution of any equation of degree n. He claimed to support Viete's idea that approximate methods should be utilised to yield values closest to the roots. An important aspect of Waring's study was the attention he paid to the form of the roots and relationships between roots of equations. In the same way as some other mathematicians of the time, he used substitution to reduce the given equation to one with roots representing powers of the original ones in order to avoid irrationality of roots. He analysed the solvability of the equation $x^n \pm 1 = 0$ for n = *4, 5* and *6*. Later, he stated that if n is of the form $n = m \,.\, r \,.\, s$ where m, r and s are prime numbers then the equation can be reduced to the form $x^m - 1 = 0;\ x^r - 1 = 0;\ x^s - 1 = 0$, but he never considered the solvability for arbitrary n nor for the case $n = 11$. Lagrange also considered the cyclotomic equation of the form $x^n - 1 = 0$ but did not get very far. Vandermonde, Waring and Lagrange all worked in three different countries at approximately the same time (1770). They proceeded from different viewpoints to reach a new approach to the problems of solving the as yet unsolved equations of the time. They began by studying existing methods of solving equations at the intraoperational level and made use of the contemporary state of mathematics.

Paolo Ruffini, who lived from 1765 to 1822, was both a student and an admirer of Lagrange. Several papers were published by Ruffini. In these he claimed to prove that the general equation of degree five cannot be solved by radicals. Ruffini published his first treatise in Bologna in 1798 and thereafter a few years later he published a small treatise followed by further memoirs on the solutions of equations of degrees greater than four in 1802 and 1813. Ruffini adopted Lagrange's methods and considered rational functions of the roots of general equations of degree *n*.

Although Ruffini claimed to have proved that the general quintic cannot be solved by radicals, his proof was not conclusive because it was based on the hypothesis that these radicals could all be expressed as rational functions of the roots. This proof was not well received by his contemporaries and successors and Malfatti criticised his work, stating that it left doubts regarding the insolvability of the quintic equation. Despite this, Piaget and Garcia (1989:157) remarked that

> … the conceptual framework in which he worked places him in an exceptional position during this interoperational period of algebra, very near the following stage which Galois inaugurated .

Ruffini made various valuable contributions during the interoperational period of algebra. He provided definitions for the permutations of variables in a given function and classified them according to genres. He believed that permutations are related to the values of the roots and felt that the class of permutations that do not change the value of a function lacks structure. He was aware of the transformation involved in the change from one permutation to another but did not consider the structure involved in the transformation.

Augustin-Louis Cauchy (1789–1857) was a French mathematician who wrote extensively. He considered functions in a more general way as what he termed "functions of n quantities" (Piaget & Garcia 1989:151). However, these quantities are not considered as the roots of equations but letters representing quantities. Cauchy referred to permutations as the order of the letters. Substitution was the term used for a transition from a permutation A_1 to another A_2 and was denoted as $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. As a result, he was able to define the multiplication of substitutions and the identical substitution and this in turn led to the introduction of the inverse substitution. Using these definitions, he was able to demonstrate a number of theorems. These formed part of the inter-transoperational level of abstract algebra or an abstract level high up the spiral because they are now considered "as immediate antecedents of the general theorems about the groups of substitutions" (Piaget & Garcia 1989:152). However, he did not reach the overall transoperational or abstract level of groups as a whole because he did not reach the point of actually explaining or thematising the structure of a group.

Carl Friederich Gauss (1777-1855) was born in Brunswick, Germany and was another great mathematician responsible for progress up the abstract algebra spiral of learning during the overall interoperational or conceptual stage. In fact, according to Eves (1990), Archimedes, Newton and Gauss have been called the greatest mathematicians of all times. Complex numbers were a very necessary adjunction to real numbers in order to determine the existence of roots of equations and without them some equations would have no roots at all. By the eighteenth century mathematicians strongly suspected that equations always have solutions in the complex numbers. D'Alembert, Euler and Lagrange had all tried to prove this.

At the age of 20, in his doctoral dissertation at the University of Helmstädt, Gauss gave the first completely satisfactory proof of the fundamental theorem of algebra that "for any polynomial $\mathrm{p(z)} = \mathrm{a_0{+}a_1z{+}a_2z^2{+}...{+}a_nz^n}$ with complex coefficients, $n \geq 1$ and $a_n \neq 0$, there is a complex number *r* such that $p(r) = 0$" (Temple 1981:385). In other words, he proved that a polynomial equation of degree $n > 0$ with complex coefficients has at least one complex root. The fact that each type of equation was no longer being studied in isolation made this theorem a true mark of the interoperational or conceptual level of mathematics. Mathematicians such as Newton, Euler, d'Alembert and Lagrange had all made unsuccessful attempts to prove this theorem. In his proof he expressed the variable z in terms of its real and imaginary parts. Thus he replaced z in the equation $f\,(\mathrm{z}) = 0$ by $x + iy$. Separating the real and the imaginary parts of the resulting equation led to two real equations $g(x,y) = 0$ and $h(x,y) = 0$ in the real variables x and y. Gauss then showed that there is at least one real point of intersection $(a;\, b)$ of the Cartesian graphs of $g(x,\, y) = 0$ and

$h(x, y) = 0$. This implies that the equation $f(z) = 0$ has the complex root $a + ib$. Since the proof involved geometric considerations, almost twenty years later in 1816 he published two new proofs and yet another one in 1850 in his attempt to find a proof which was completely algebraic. This was a very important result that formed part of an abstract sublevel of the overall conceptual or interoperational level of algebra.

Another important contribution Gauss made to the theory of equations was the complete solution of the cyclotomic equation $x^m - 1 = 0$ by means of radicals. The equation is called cyclotomic because there is a connection between its solution and the construction of a regular polygon of m sides inscribed inside a given circle. The equation has m complex roots given by $\cos\left(\frac{2\pi k}{m}\right) + i\ \sin\left(\frac{2\pi k}{m}\right)\ k = 0,\ 1,\ 2\ \ldots\ m-1$. Long before Gauss, De Moivre and Euler knew the trigonometric solution. If the complex numbers $a + ib$ are represented by points (*a, b*), the vertices of a regular *n-gon* inscribed in the unit circle is depicted which can be constructed by ruler and compass.

Gauss's greatest publication was his "Disquisitiones Arithmeticae". This was a very important book regarding the modern theory of numbers. The fifth section is concerned with forms and indeterminate equations of the second degree. He studied quadratic forms only in relation to the solution of indeterminate quadratic equations. Nevertheless, as a result of his useful analysis of the properties of binary and ternary quadratic forms, this became his main theme. He not only defined orders with types for these forms but, for the first time in history, defined operations between these forms. For example, he considered the form $ax^2 + 2bxy + cy^2$ where $a,\ b$ and c are integers and called these functions forms of the second degree or just forms. In his " Disquisitiones Arithmeticae" he observed how Lagrange, Euler and Fermat had all discovered results about this form but "… a careful inquiry into the nature of the forms revealed so many new results that we decided it would be worthwhile to review the whole argument from the beginning" (Piaget & Garcia 1989:152). Here Gauss shows the necessity of beginning at the intraoperational or perceptual level and gradually climbing up the spiral. He seems to sense the advent of the overall transoperational or abstract level of algebra when he states: "We have no doubt that many remarkable truths still lie hidden and are a challenge to the talents of others" (Piaget & Garcia 1989:153).

As algebra continued to develop in an "a priori" manner, Gauss proceeded to give several definitions which were very important because they enabled him to improve several important results: He represented the form $ax^2 + 2bxy + cy^2$ by the symbol $(a,\ b,\ c)$ when he was not concerned with the unknowns x *and* y. The properties of the form $(a,\ b,\ c)$ depend in a special way on the nature of the number given by $b^2 - ac$, called the determinant of this form. His new definitions illustrate the "a priori" or constructive historical development of mathematics where mathematics is used to create more mathematics.

The important results arising from these definitions are listed below:

1. *Given any two forms with the same determinant, see if they are equivalent or not, if they are properly so or improperly, or both at the same time. If they have unequal determinants, see if one does not include the other, properly, improperly, or both at once. Finally, find all transformations, both proper and improper, of one to the other.*
2. *Given any form whatsoever, see if a given number can be represented by it. Find all the representations of a given number M by a given form F.* (Piaget & Garcia 1989:154)

Later on Gauss came to what has been considered to be one of the most original points of his work, known as the composition of forms. This was the very first operation ever introduced into a non-numerical domain with properties that cannot immediately be deduced from operations on numbers. One important theorem he arrived at as a result was:

> *If g is a primitive form of the same order and genus as f, and g' is a primitive form of the same order and genus as f', then the form composed of g and g' will be of the same genus as the form composed of f and f'* (Piaget & Garcia 1989:154).

During his teenage years, the gifted mathematician, Niels Henrik Abel (1802-1839) believed that he could solve the quintic equation but soon discovered he was wrong. In 1824 he proved that a solution by radicals is impossible and, at his own expense, published a clear proof of the result in a French pamphlet. Although Abel made use of the results obtained by Lagrange and Cauchy as far as the number of values a function of n variables can assume if the variables are permuted, he introduced a new essential first step in his proof. He also published a memoir dealing with a particular class of equations of all degrees which are soluble by radicals. The cyclotomic equations of the form $x^n - 1 = 0$ belong to this class. Abel also proved a general theorem, forming part of the abstract level of his mathematical spiral. As a result of this theorem, later on in the transoperational or abstract level, when groups were commutative or $ab = ba$ for any members a or b of a group, the group came to be known as abelian.

Abel had reached the peak of the overall conceptual or interoperational level in algebra when he began to work on the new problem which rose out of whether or not a given equation could be solved by radicals. Unfortunately, he died prematurely in 1829 but a brilliant young French mathematician, Evariste Galois was able to lead algebra into the overall abstract or transoperational level.

3.2.3 Sublevels of the interoperational level or rounds of the spiral at the conceptual level

In the eighteenth century, the main work done in algebra concerned the solving of equations. Problems of greater generality were being studied. However, Piaget and Garcia (1989:155) observe how there was "… a long road to travel before a system of transformations can be related to a total structure in which these transformations constitute intrinsic variations." This seemed to be the case at the beginning of the interoperational level or the inter – intra operational level.

As the new instrument of the infinitesimal calculus created by Leibniz and Newton began to be used by mathematicians like Euler, Lagrange and Gauss, the inter – interoperational level within the overall conceptual level of algebra or a new conceptual level was reached. The formulation of algebraic problems of greater generality such as the fundamental theorem of algebra characterised this period. At this stage mathematicians took advantage of the properties of continuous functions and their transformations arising in infinitesimal calculus.

Transformations dominated algebra for a long time before the emergence of abstract algebra at the transoperational or abstract level. There were several mathematicians whose work signified the inter – transoperational level in algebra. Amongst others, these included Lagrange, Ruffini, Cauchy and Gauss who "… are the last representations of the interoperational period in the development of algebra and specifically in the history of the theory of algebraic equations." (Piaget & Garcia 1989:155). Gauss not only transformed functions but tried to find the relations that made them stable. He managed to derive properties that may be regarded as invariants in systems of transformations. Analysis of transformations was made possible by the abstract and general symbolism used in the interoperational period. These became the percepts leading to the conceptual and abstract levels of the "a priori" development of abstract algebra at the abstract level. Advancement from the intraoperational level to such levels as the inter – intra, inter – inter, and inter – transoperational levels all result from

> *'reflective thematization' that is, an exhaustive conceptualisation of progressively constructed mathematical objects, and this even before such representational intuitions were developed into axioms* (Piaget & Garcia 1989:171).

This could be considered as the continual expansion of a spiral. Since it is a result of activity and growth of the learner rather than a physical experience, in teaching one needs to make the effort of trying to encourage the development of reflective thematization. Piaget observes that classification and seriation serve as pre-structures or pre-algebraic systems. In history the intraoperational or perceptual level lasted from antiquity to the middle of the eighteenth century, the interoperational operational level lasted from that time to Galois and then the transoperational level lasted from Galois onwards. However, these levels and sublevels (or many rounds of the spiral) need to be experienced in considerably less time by the learner.

During the course of history, there have been different theories regarding the formation of structures and pre-structures in individuals because it is done in a "largely unconscious" way so that Piaget and Garcia state that "… This may, at first glance, make our triad seem unduly artificial" (Piaget & Garcia 1989:173). However, they believe that it results from internal necessities rather than simply forming a regular sequence and

> *… in addition, each stage encompasses sublevels, which follow the same sequence and for the same reasons. This fact is of fundamental importance* (Piaget & Garcia 1989:173).

This really seems to be a similar idea to the spiral concept of learning where sublevels may be considered as simply forming part of a continuous spiral.

3.2.4 Conclusion

In all of Gauss's studies it can be seen that he was laying the foundation for the theory of groups. He appears to have reached the inter – transoperational level or an abstract level high up on the overall conceptual level of algebra. A quadratic form having the law of composition as defined by Gauss does actually form an abelian group with the unit element being called the principal class. In fact, only one small step was required to move on from the interoperational or conceptual level to the transoperational or abstract level. Historians have expressed their amazement that Gauss and others came so close but never quite reached the stage of defining a group and hence rising up to the transoperational level of algebra. At a perceptual level of the overall abstract level of algebra they needed to operate with results obtained at an abstract level of the overall conceptual level in order to develop the relevant concept. The definition of the concept of a group at the overall abstract level was the next step in the “a priori” historical development of algebra.

3.3 Various theories of intermediate levels of learning

The historical interoperational or conceptual level of algebra lasted for approximately a century. It formed a vital link between the perceptual and abstract levels. In theories regarding human thought development that have been already alluded to up this point, the conceptual level also forms an important link between the perceptual and abstract levels. Characteristics associated with the conceptual level of learning mathematics and algebra in particular, at any round of the spiral, will be considered in detail in subsequent parts of this chapter.

3.3.1 Piaget’s intermediate level of thought

(i) Introduction

The intermediate level of Jean Piaget’s original theory of intellectual development occurs during the concrete operational period lasting from seven to twelve years of age. According to Schwebel and Raph (1974), this is the stage at which a child is able to see things from the perspectives of others, appreciate more than one aspect of a problem and reverse mental actions.

One of the important characteristics that Piaget considered regarding the concrete operational stage is that a child begins to realise that actions carried out on objects can be mentally cancelled out. For example, the action of putting a large truck with a collection of big toys can be undone and it could just as easily be grouped with a mixed set of vehicles. The idea of cancelling out actions is a fundamental one in group theory where inverse elements exist. When the group operation is performed on any element and its inverse, the identity element results.

Reciprocity and annulment are two aspects of a fundamental notion that Piaget calls reversibility. Reversibility is described by Schwebel and Raph (1974:43) as follows:

> *Reversibility implies the construction of a coherent system of operations that, unlike the actions of the earlier period, can be effected mentally and that, instead of contradicting one another or simply being juxtaposed, now reinforces and sustain each other.*

Although reversibility can be considered a very general exploratory principle for changes that take place during Piaget's original intermediate stage, it does not explain how the change comes about. Games involving rules can be followed and various mathematical concepts begin to be understood.

(ii) Mathematics at Piaget's original intermediate level of thought

Piaget made a great deal of references to group structure in his work. In fact, the explanations and reactions to counter-suggestions of a child show:

> *… that a grouplike structure containing an identity element is in the process of being built up. The existence of an identity element is in fact the better known implication of the acquisition of conservation; but it must not be forgotten that this is indissociable from the existence of the operational structure* (Schwebel and Raph 1974:44).

There are several other features which characterise Piaget's original intermediate level. These include conservation of number, variation, length, area, volume and mass. For example, the child learns to distinguish between the permanent and non-permanent qualities of objects or, in other words, begins to acquire the idea of constraints. In addition, he or she learns about the structure of actions or operations. This may be illustrated by the push (x) on a block, its movement (y) where y depends on x or $y = f(x)$.

The development towards the symbolisation process is yet another characteristic of the intermediate level. Symbolism may be regarded as an abstract level lower down on the spiral. This is because it follows as a result of isolated instances being studied, compared and categorised at the conceptual level. This provides an example of how one section of the spiral that forms part of the intermediate level can relate to previous abstract levels, thereby indicating the existence of sublevels of learning. Piaget's work helped to highlight the importance of thoroughly traversing the lower parts of the spiral so that pupils could be provided with a foundation for their subsequent studies. As far as Piaget's experiments are concerned, it may be said that "…. If nothing else, these experiments provide a serious argument against the passive look-and-listen methods of teaching" (Schwebel & Raph 1974:49).

(iii) Piaget's later intermediate learning level termed the interoperational level

At the intermediate level of Piaget's later theory of levels, there is a shift from analysing objects "…. to one concerning relations of transformations between objects." Piaget and Garcia (1989:x). As the learner proceeds from one level to the next, such as from the intraoperational level to the interoperational level, there is an "… integration of elements going back all the way to the initial phases." (Piaget & Garcia 1989:2). In the process, there is a projection to a higher level from a lower level and reflection causes a reconstruction and reorganisation, within a larger system, of what has been transferred as a result of the projection.

Piaget notes how there is a danger that when one considers a system of knowledge in its completed state such as when it has become axiomatic, "… one might get the impression that such systemised knowledge can be reduced to a series of statements" Piaget and Garcia (1989:10). However, before knowledge becomes formalised, Piaget speaks of comparative tools, built on correspondences and transformational operations, forming a necessary part of the whole process. Ancient Greeks used to study transformations in themselves and equations involve a series of correspondences. However, in ancient times, the people did not have a conscious grasp of transformations. Correspondences did not attain the interoperational level until the advancement of algebra and infinitesimal calculus in the seventeenth century. Empiricism or experience based on observation and experience rather than theory leads to a rise from the intraoperational level to the interoperational level but not from the interoperational level to the transoperational level.

At the interoperational level, it may be possible to explain how something happens but not why it happens. It may be possible to discover and utilise something that works without actually being aware of the theory behind it. Piaget and Garcia (1989:24,25) claim that "…. All through history, scientists have used cognitive structures without ever becoming conscious of them." They give as an example Aristotle, who did use the logic of relations but did not include them in the instruction of his own work on logic and observe that "… Thus, there is a long distance between the spontaneous, unconscious use of structures, and their becoming conscious" (Piaget & Garcia 1989:25). For this reason it may be possible to reverse the order of history when presenting a new topic. Group theory structure has been found to be prevalent in nature and many of its applications have been discovered long after its initial formulation. It is thus possible and could be effective to study one or some of its recent applications and in this way deduce the axioms of a group, thereby reversing the order of history. Nevertheless, the learner could in this manner still pass through the intra, inter and transoperational levels referred to here as the perceptual, conceptual and abstract levels of the spiral.

Whereas isolated forms are identified with intraoperational stages, correspondences and transformations amongst these forms as well as invariants needed by such transformations characterise the interoperational level. For many years equations were studied in isolated forms but the advent of infinitesimal calculus created by Leibniz and Newton led to algebraic problems of greater generality, including the fundamental theorem of algebra. In the interoperational level Lagrange observed that all methods of solving equations involved introducing functions to reduce the original equation. The seeds of ideas that finally led to the theory of groups at the transoperational level were the observations involving the total number of different values that could be assumed by a polynomial when permuted in all possible ways.

(iv) Sublevels of the interoperational level or rounds of the spiral at the conceptual level

According to Piaget's adapted model of learning development, the preoperational, concrete operational and hypothetical deductive operations stages constitute the three main stages of learning. The interoperational stage is characterised by the ability to make a deduction either from an initial operation, once it has been understood, to deduce other things that are implicated by it or to coordinate it with other similar ones. Although this does lead to the development of new systems, there is a limitation to what Piaget and Garcia (1989:176) term a "step–by–step approach". This means that at this level it is impossible to go beyond "natural embeddings" (Piaget & Garcia 1989:170) or to directly combine objects which are far removed from another into a single class or else to go "… through a complex set of intermediate embeddings" or "… stay with very general classes" (Piaget & Garcia 1989:176).

As learners pass through the sublevels of the interoperational stage (or rounds of the spiral found in the overall conceptual level), one of the simple general transformations is negation which is very important in the groupings of classes and forms an important basis for group theory. Reciprocities form another important general transformation which, together with negation comprise of the two possible forms of reversibility. These ideas dominate the interoperational or conceptual level and as the learner progresses through the inter sublevels (or spiral rounds forming part of the overall conceptual level), they come to appreciate reversibility or the common characteristic of all operations and their compositions.

The fact that conceptual stages are evident in the learning of mathematics during the overall perceptual, conceptual and abstract levels of its acquisition, gives an indication of the existence of sublevels of learning. In history, a parallel development can be seen between the inter sublevels or conceptual spiral rounds described here and the progress followed in the algebra of equations developed from Viete to Galois.

(v) Conclusion

Piaget and Garcia abbreviate the intra, inter and transoperational levels by means of Ia, Ir and T and observe how, to whatever content they are being applied, "… constructions that we designate as Ia, Ir and T include development changes from Ia to Ir and T respectively" (Piaget & Garcia 1989:182). Both in the history or the learning of mathematics, they observe how in the first essential phase or perceptual level particular cases are analysed but not related to each other, in the second phase or conceptual level they are compared, correspondences are formed and translations are constructed whilst in the final phase or abstract level, once it has become possible for these transformations to be mastered and generalised, new synthesis can take place.

The Ia, Ir and T triad is important at every level of learning. For this reason Piaget describes the levels as being functional rather than structural. Consequently they are common to all levels of development including the interoperational one for they describe developmental changes in general. Students entering the interoperational level at first analyse specific cases, then establish differences, correspondences and transformations and finally generalise these results at the inter – transoperational level. This same pattern is evident in the growing rounds of the spiral during the overall conceptual phase of an algebraic topic and the corresponding overall conceptual level in the history of algebra.

3.3.2 Freudenthal's intermediate level of thought

(i) Introduction

Freudenthal was the chairman of the Dutch Educational Committee for Mathematics from 1954 and in 1967 became the president of ICMI, the International

Commission on Mathematics Instruction. He always used to express his ideas about teaching mathematics with great enthusiasm . However, one problem he experienced in this regard was: “Freudenthal struggled his whole mathematics educational life with the contradicting interests of the retention of insight and the training of skills, or so it seemed” (J.D. de Lange quoted in Streefland 1993c:148).

Freudenthal believed in learning levels but felt that his levels differed from those of van Hiele’s because his “… levels are relative rather than absolute” (Streefland 1993b:37). He did mention three levels in learning a topic like mathematical induction. This would seem to suggest a correlation between Piaget’s triad of levels and sublevels and Freudenthal’s conception of mathematical levels. It does also appear to reflect the perceptual, conceptual and abstract levels of the spiral. The fact that his view of levels is not an absolute one also seems to further the idea of a spiral of learning, where different parts of the spiral are covered in the process of learning a particular topic.

(ii) Freudenthal’s intermediate level of learning

When discussing three levels of learning mathematics, Freudenthal refers to the intermediate level in the following manner: “On the next level it is made conscious as an organising principle and can become the subject matter of reflection.” (Freudenthal 1973:122–123). He stresses the importance of the first level preceding the second one. By the time the intermediate level is reached, the learner should have encountered sufficiently compelling examples and experiences on the first level to reflect upon and encourage him/her to organise the subject matter. This corresponds to Piaget’s interoperational level of learning or the level of concepts which is a mediating stage between iconic representations, which are close to perceptual and linguistic representations or abstractions where concepts become structured.

Freudenthal does, in fact, regard mathematical activity as “… an activity of organising fields of experience” (Freudenthal 1973:123). As learners progress from the first level to the second, the experiences of the first level become reorganised and form the subject matter at the second level. He believes that the connection between one level and the next is extremely logical and that the levels involved could be determined by pedagogical experience.

Freudenthal stresses how important it is for the teacher to promote the advancement of students from one level to the next. Situations can be organised in order that the students can become stimulated to rise to the subsequent level. This is important because "… Even if the pupil masters the operations on a lower level, it is no use pressing him to pass to the next as long as the need, which is the mother of re-invention, is not felt" (Freudenthal 1973:124).

Whenever a new topic is being taught, it is so important to allow the student to pass from the first level to the second. At certain parts of the spiral, particularly at early phases of topics, the "a posteriori" or descriptive development of mathematics is relevant. In this connection Freudenthal (1973:124) complains how "… almost all courses start with a mathematically organised matter. The student is deprived of the finest opportunity that exists in mathematics to learn to mathematize a non-mathematical subject matter."

As students continue to climb up the algebraic spiral, they find mathematics being built out of mathematics, illustrating the "a priori" or constructive historical development of mathematics. Here mathematical entities such as symbols, definitions or even theorems form the objects of analysis at the perceptual level and become organised at the second or conceptual level.

Freudenthal gives the geometric illustration of how defining a parallelogram too early when teaching the topic would mean that "… a level is passed by, and the student is deprived of the opportunity to invent that definition" (Freudenthal 1973:124). He continues that this mode of teaching would be so wrong that "… it is utterly improbable that at this stage the student understands the meaning and the aim of a formal definition or even would be able to understand it" (Freudenthal 1973:124). This highlights the relevance of the conceptual stage as a link between the perceptual and abstract levels at any part of the learning spiral.

In the case of the parallelogram, the student could be provided with a variety of parallelograms and given the opportunity to discover many properties and the connections between these properties. Logical organisation could follow as a result until the student "… finally discovers one property among them from which all others can be derived" (Freudenthal 1973:125). Different students may come up with different results but in this way they really come to appreciate the meaning of a definition, having passed through the necessary conceptual level of that particular round of the spiral.

Freudenthal stresses the importance of teaching students by encouraging the operational matter of the lower level to become the object of analysis on the higher level. This causes the pupil to learn "organising by mathematical means" and "to mathematize his spontaneous activities" and he stresses that "… it would be desirable that he should be taught this way" (Freudenthal 1973:125). If a child is taught above his or her own level, rules or algorithms need to be provided without using any sense attached to the actions. Freudenthal argues that although a child may possibly be able to seem to achieve some mastery of the topic by means of being taught some application patterns as well as the use of mnemonics, the knowledge would not be long term, particularly relevant and could eventually lead to serious problems.

(iii) Freudenthal's intermediate level of learning in algebra

Freudenthal stresses the importance of having the second or conceptual level in algebra come after the bottom or perceptual level. For he observes how "… On the bottom level, the child operates with the concepts by manipulation but he does not know what he is doing" (Freudenthal 1973:128). However, he strongly believes that in order to tell what is being done in algebra or some other branch of mathematics, it is necessary for progress to the second level to take place.

People have actually tried to teach children the beginning of such topics as group theory even at the primary school level. However, if the children remain on the first level with no chance of reaching the second, then Freudenthal does not believe that any justice is being done to the topic. He complains

> *To call what a child is performing at this level set theory, group theory, or linear algebra, is the same as to claim that a child who is singing is learning music theory, that a child who is tinkering is doing mechanics, that a child who is looking at the sky is doing astronomy, and that a child who is speaking is doing linguistics* (Freudenthal 1973:128).

In mathematics it is particularly necessary to rise to higher levels in order to appreciate and understand what has occurred at the bottom level.

The bottom level of any topic in mathematics should serve as an introduction to something leading up to the second and third levels and beyond to further rounds of the spiral. However, Freudenthal cautions that "… the mathematics that these precursory exercises are to prepare for should follow them closely or, in any case, such that the intensive stream of preparations does not ooze away" (Freudenthal 1973:129). Freudenthal does not believe that it is right to direct the mathematical activities of a child to some first level activity unless it is possible or probable to follow this up by giving the child the opportunity of reflecting and organising what he has learnt on the first level, thereby resulting in a rise to the conceptual level.

Freudenthal strongly believes in reinvention as a method of teaching mathematics. However, even though stimulation on the bottom level is very necessary, “It is meaningful as long as it takes place under conditions where it is preparatory rather than an unessential game” (Freudenthal 1973:130). This highlights the need to rise to the second level and engage in organising and reflecting in order to reinvent the structure involved at the third level.

(iv) Sublevels of Freudenthal’s intermediate or conceptual level

One of the criticisms which Freudenthal had levelled at the van Hiele’s was that their levels were too rigid. He saw mathematics as growing continuously in various directions. He observed “Throughout history, tradition and renovation alternate with each other” (Freudenthal 1973:13). This certainly suggests the idea of sublevels within levels or a spiral growing in various directions within what could be the overall intraoperational or perceptual level, the interoperational or conceptual level or else the transoperational or abstract level.

Even though the ancient Greeks were operating within the overall perceptual level of algebra, they were in touch with an abstract level and Freudenthal (1973:5) remarks: “Deductivity and the germ of axiomatics are in our view the most striking, and in fact the most modern, feature of Greek mathematics”. The Greeks did not invent algebra as they did not possess adequate notation and symbolism. Skemp (1971) has observed that the historical development of algebra was delayed until the development of an appropriate symbolic representation which both emphasised structure and assisted in the carrying out of certain procedures. At the conceptual level, limit procedures and infinitesimal methods became popular and led to calculus. However, Freudenthal claims that even though it was only in the nineteenth century that rigour was recaptured and “… people understood the essence of Greek mathematics”, he believed that “… perhaps each link was historically indispensable” (Freudenthal 1973:6). This suggests the importance of the development of sublevels and lower rounds of the spiral in the growth of mathematics.

Freudenthal acknowledged how other concepts in history led up to the structure of a group at the dawn of the abstract level. For example, Freudenthal (1973:34) claimed that “The projective, affine, equiform, and other mappings were discovered as well as their compositions and the fact that they form groups” were discovered a long time before the term group was defined and before people could know or tell what a group was. This shows how the idea of a group followed a spiral development in history.

The growth of the concept of a field can likewise be traced to a spiral of levels. For centuries people were familiar with the system of rational numbers and the four operations of addition, subtraction, multiplication and division which are governed by certain basic laws. As the spiral was traversed, real and complex numbers evolved and algebraic equations were solved within the system of complex numbers. Smaller number systems in which the few basic operations could be studied were investigated by Dedekind and the general field concept was established in 1910.

The importance of sublevels of levels or parts of spirals which lead to one another are all important in the growth of mathematics. However, Freudenthal stressed the relevance of following the path of growth of a topic in mathematics because: "A mathematician who did not follow step by step the continuous reorganisation of mathematics could get trouble if suddenly he were to jump over a whole chain in its evolution" (Freudenthal 1973:45). Freudenthal felt that the teaching matter should be analysed in order to determine how it should be presented. He strongly believed that mathematics should be taught as a reinvention but felt that often the implementation was too narrow and superficial. However, he held that the van Hiele's interpretation was rooted more deeply. Considering levels and sublevels in the teaching of mathematics is very important because "Mathematical activity is, as we will see later, an activity of organising fields of experience" (Freudenthal 1973:123).

(v) Conclusion

Freudenthal clearly believes in the existence of levels and sublevels in learning mathematics. He does not see levels as rigid but acknowledges how important it is to follow the essential steps involved in the formation of concepts. The principle of reinvention is very relevant to him. He acknowledges how some of the many parts of the spiral covered in history are important. He refers to both the descriptive and constructive historical developments of mathematics in his comment:

> *Besides its direct relation to reality mathematics can boast numerous indirect ones, as applied and appliable mathematics* (Freudenthal 1978:7).

As far as teaching mathematics is concerned, he feels strongly that

> *… pupils should learn mathematizing, too, and certainly on the lowest level where it applies to unmathematical matter, to guarantee the applicability of mathematics, but not much less on the next level where mathematical matter is organised at least locally* (Freudenthal 1973:134).

3.3.3 Van Hiele's intermediate level of thought

(i) Introduction

The van Hiele's research was conducted over a three year period to try and explain why children were having problems with learning geometry. It seemed that the level of thought required to master certain concepts was beyond the capabilities of the majority of their pupils. More focus appeared to be needed at lower and intermediate levels. Fuys, Geddes and Tischler (1988) claim that the van Hieles found that the developmental levels followed by children learning geometry seem to correspond to Piagetian theory. Teachers need to ask their students why and give explanations about what they have observed at the lower level in order to encourage development at the intermediate level.

(ii) Van Hiele's second level of learning

Part of van Hiele's third original level could be considered as representing the conceptual level here. Some but not all of the second or descriptive level of his later three level model will be considered as the conceptual level.

Original 3rd level – Level 2 – informal deduction

At this level, students are able to understand the relationship between the properties of different figures. For example, they are able to reason that because a square has all the properties of a rectangle, it must be a rectangle too. However, here precise definitions are regarded as forming part of the abstract level of the spiral. As they move up the spiral, at the conceptual level of parts of the spiral, they are able to recognise the concept of class inclusion as they can begin to appreciate, for example, that a square is a rectangle, which is a parallelogram which is in turn a quadrilateral. They become capable of "if then" thinking, begin to be able to provide logical geometric arguments, reason about properties of classes of figures and interrelate properties of figures and classes of figures. However, neither at the informal deduction level nor the conceptual level are students as yet able to construct proofs. This is an activity which would become possible at an abstract level at a higher position on the spiral.

Later 2nd level – pre-operational

At the second level geometrical shapes are recognised by their parts and considered to have parts. At this level they study networks of relationships and consider how to order the properties of geometrical shapes. Pupils no longer merely study objects in isolation and so they are able to find interrelationships of properties both within and amongst figures. They are able to reason based on the evidence which they have obtained experimentally. However, at van Hiele's later second level, Mariotti (1997:223) claims that "Pupils are able to logically connect shapes and their properties by definitions". Formal definitions are not considered as forming part of the overall conceptual level here. Instead, since they require reaching a new level of abstraction, only intuitive definitions could be considered as being included here whilst formal definitions will be associated with the abstract level. Further up the spiral they should then be able to logically connect shapes as well as properties as described above at a conceptual level at which definitions have already been established.

(iii) Van Hiele's second level of thought in algebra

Pierre van Hiele (1986) declares his levels of learning apply to all forms of mathematical understanding. However, particularly in algebra, it can be seen that "… the consequence of teaching at too high a level is a - poorly functioning - mastery of algorithms" (van Hiele 1959:22). Furthermore, since the pupil has not been taught to develop the algorithm for him or herself, it is necessary for a new one to be taught in every situation.

The unfortunate result of not observing levels in the teaching of algebra is that the child becomes accustomed to the idea of learning subject matter with little or no understanding of it. Pupils develop the unsound habit of not expecting to be able to discover anything in algebra for themselves. In addition, they become indifferent to the subject and do not expect that they would be able to express any opinion in that regard. Consequently there is a danger that when the conceptual level of an algebraic topic is being taught without any attempt made to first pass through the perceptual level, the child "… accustoms itself to absorbing subject matter without possessing a personal relation to it" (van Hiele 1959:22).

Van Hiele and others have argued that learning content and developing the powers of deductive reasoning are the two main goals of teaching mathematics. However, Seymour Schuster (1971) believes that not only geometry but algebra too could be used to accomplish this goal. MacLean (1957:100) agrees that "… traditionally high school geometry is said to be the subject where logic can best be learned. Algebra would be a better place".

At the first level students could have noticed all the properties of $\mathbb{Z}$ under addition, $\mathbb{Q}$ under multiplication and the set of vectors under the operation of addition. However, these would have been studied in isolation. But at the second or conceptual level comparisons could be made regarding the similarity of these concepts in order to lead to an intuitive notion of a group. Only at the third or abstract level would the axioms of a group be formally established.

Van Hiele clearly believes that his theory applies to the teaching of algebra. Although the spiral theory is being followed here and van Hiele's levels are not being rigidly adhered to, they are relevant and his ideas could certainly be taken into consideration in the teaching of secondary school algebra and the concepts of abstract algebra to students at university level.

(iv) Sublevels of the van Hiele levels

Van Hiele was most concerned about how to promote thinking from one level to the next. In the following passage, by saying that a person first needs to rise to a higher level of thought within a field before directing their actions towards a specific goal, he seems to definitely suggest the idea of sublevels or parts of a spiral of learning:

> *Only when a reasonable orientation in the field is possible, when thus an adequate number of symbols have received a sign character, in consequence of which the field presents itself as a totality, can a person be said to have attained the higher level of thought. Then for the first time is he capable of varying his actions directed to a certain goal* (van Hiele 1959:15).

Van Hiele suggests three stages directed towards promoting development from one level to another. These stages seem to closely resemble the perceptual, conceptual and abstract levels or Piaget's intra, inter and transoperational levels. Van Hiele (1959:15) describes these stages in the following manner:

> *a first stage, during which the symbols of the field in question must be developed; a second stage, during which the properties and relationships must be explained, whereby the person becomes aware of a network of relations; and a third stage, that of which the person learns to orientate himself in that network of relations with the help of the symbols.*

As these stages keep being repeated, the student is able to keep on climbing up the spiral of learning. In order to facilitate the student's advancement from one round of the spiral to the next, the teacher or lecturer would need to be aware of student's sublevels and provide the appropriate learning experience at suitable stages of advancement up the spiral of learning of a topic such as abstract algebra.

(v) Conclusion

Although van Hiele's theories regarding the second level of learning are not identical to those of others, there are considerable similarities to be found. These all suggest that appropriate foundations be laid down to promote advancement up the spiral of learning. Van Hiele's ideas definitely seem to suggest the existence of sublevels in the learning process. These may be regarded as the parts of the spiral that need to be traversed in order to reach up to the desired levels above. Thus level two or the conceptual level continually needs to be revisited in the journey up the spiral. Many different parts of the spiral of abstract algebra all form part of the overall conceptual level in question.

3.3.4 Land

(i) Introduction

As has already been mentioned, both Land's descriptive and informal theoretical levels seem to relate to the perceptual level. However, the informal theoretical level or Level 2 seems to involve elements of both the conceptual and abstract levels and so only part of it will be considered here. The rest of her third level as well as her fourth level will form part of the abstract level although the former could be regarded as being situated lower down the spiral than the latter.

(ii) The second or conceptual level of Land's theory

There are certain characteristics of Land's which seem to relate to the level of concepts or Piaget's interoperational level. These include several attributes of her third level or the theoretical informal level. In her study involving exponential and logarithmic functions at Level 2, she regards the relationships between the properties of exponents and logarithms as well as the relationship between exponential and logarithmic functions as the object. However, accurate and concise definitions would in this case be considered as forming part of the abstract level. Derivation of the mathematical formula $A = (1 + \frac{i}{m})^{nm}$, explaining the relationship between $f(x) = 10^x$ and $g(x) = log\ x$ and solving equations such as $\log_6(x-5) + \log_6 x = 2$ could form part of the conceptual level. But these are higher up the spiral once the definitions and graphs of the relevant functions have been reached by passing through the appropriate perceptual, conceptual and abstract levels of the spiral.

(iii) Sublevels of Land's levels

Since Land dealt with properties of exponential and logarithmic functions and their graphs, there are clearly sublevels associated with the various topics taught. She herself acknowledges this in the implications of her research in the comment: "These students have succeeded in passing through high school algebra (some three years) untouched by mathematical understanding; they have succeeded by learning rote" (Land 1990:171).

The fact that students lack so much background in algebra shows how important the spiral learning approach is. Right from the beginning students should be encouraged to pass through the perceptual, conceptual and abstract levels as they progress up the spiral of algebra. Land observed that years of successful learning by rote had actually had a "mentally crippling affect" (Land 1990:172) on some of the final ten students she interviewed in her study. In fact, there was a particular student who had managed to achieve a B in pre-calculus but struggled to think even on the predescriptive level on exponential and logarithmic functions. This person appeared to be "… aware of the mental cul-de-sac in which years of successful learning-by-rote had left her" (Land 1990:172).

The idea of levels of thought in algebra seems to be strongly suggested by Land's study. This would help the students perceive algebra as something that grows and develops rather than being merely static. As the sublevels or levels of the spiral are covered, students should be encouraged to see algebraic concepts "… as something continually revised and enriched in the light of new knowledge" (Land 1990:122).

(iv) Conclusion

Land's research clearly shows the importance of establishing sound understanding of mathematical concepts right from the bottom of the spiral of learning algebra. Without this pre-knowledge, it would be very difficult to help students rise to the conceptual level higher up on the spiral when so much of the necessary groundwork is missing. Since abstract algebra is continually built up by passing through rounds of the spiral, it would certainly be necessary to lay a solid foundation so that progress up the spiral would not be hampered.

3.3.5 Nixon

(i) Introduction

In the research done by Nixon (2000) both the descriptive and theoretical informal levels seem to relate to the perceptual level. However, in this study the levels are viewed in a slightly different manner since the rest of the theoretical informal level as well as the theoretical formal level are also regarded as forming part of the abstract level but at a higher position up the spiral.

(ii) The intermediate or conceptual level

In the topic of sequences and series, the following objects and characteristics would seem to be appropriate for the level of concepts or Piaget's interoperational level:

2nd level: the descriptive level

At this level properties of sequences and series are established in an inductive manner and formulae were discovered by experimentation. A change of formula would give an indication of a change in the sequence without reference to pictures or particular examples. This does suggest a conceptual level lower down the spiral of learning.

3rd level: the informal theoretical level

At this level, as in the case of the conceptual level, objects are regarded as statements showing relationships between the properties of sequences and series. Characteristics would include the following features: recognising interrelationships that exist between different types of sequences and series; showing an understanding of statements which relate properties of sequences and series; being able to solve equations by manipulation of symbols; formulating statements which show interrelationship of symbols. Several of these characteristics rely on sound algebraic knowledge from the past. However, the characteristic of defining words accurately and precisely would not be considered to form part of the conceptual level but rather the abstract level. Following the derivation of a formula and following a deductive argument would form part of the conceptual level but at a higher part of the spiral once the necessary foundations had been laid.

(iii) Sublevels

At each level students involved in the study were encouraged to progress from one level to another by being encouraged to visualise, explore patterns and make generalisations. These could be regarded as the sublevels or parts of the spiral involving percepts, concepts and abstractions which helped students pass from one level to the next. In this way it would appear that they passed through several rounds of the spiral and the object at each level seemed to represent one round of the spiral.

As was the case in Land's research, students' pre-knowledge was vital and a lack in any such area would hamper progress up the spiral. If all the necessary pre-knowledge had to be covered, then it would have been necessary to start off very low down on the spiral indeed.

The process of exploring patterns seems to be strongly associated with the conceptual level at any stage of the spiral. Low down the spiral it could mean finding patterns in nature following the descriptive historical development of mathematics. Higher up the spiral, following the constructive historical development of mathematics, it would mean studying different types of groups or fields to find relationships at the conceptual level which could be proved at the abstract level.

In the study of sequences and series, exploring patterns did seem to help students to think, investigate, understand and operate on the conceptual level. Students became very enthusiastic about searching for patterns and relationships whenever they began something new. Two students' responses to the question:
"Did exploring patterns help you to do the examples on worksheet 2.5 and 3.4?" were:

> *Yes! We have become used to be looking for patterns and detecting sequences by exploring patterns*.
>
> and
>
> *Yes, it helps you to be more aware of the types of patterns that can occur and helps you recognise other patterns and how they occur* (Nixon 2002:120).

These responses show how developing an awareness of patterns and encouraging an investigation of these seems to help promote an understanding of concepts and the establishment of more patterns at the conceptual level. Although specially devised diagnostic items were not used in this case, students certainly became very positive about the concepts they were being taught and demonstrated an ability to utilise them independently in various contexts. In teaching abstract algebra, numerous examples of patterns could be found both in a descriptive and a constructive approach in order to establish concepts right from low down extending to high up the spiral of learning of this topic.

(iv) Conclusion

The importance of considering learning levels became clear in Nixon's study. The type of activity particularly relevant to the conceptual level seems to be exploring patterns. This leads to an appreciation of transformations and interrelationships, all of which are characteristic features of the conceptual level. Once an appreciation of discovering such relationships has been established at an initial conceptual level, students can become hungry to continue to explore for similar findings as they climb the spiral.

3.4 Concept understanding scheme

3.4.1 Introduction

The concept understanding scheme of people such as Vinner, Tall and Dreyfus illustrate a distinction between what they describe as concept image, concept definition and concept usage. In the spiral described here, a different type of distinction is drawn. Concept image, concept definition and concept usage are not regarded as belonging to levels one, two and three respectively but may be subdivided in a different manner in order to be associated with the spiral scheme. As has been mentioned, concept image and concept usage could both be associated with the first or perceptual level. Concept image and concept usage also form part of the second or conceptual level. This is because the concept needs to be firmly established and students need to have a good intuitive appreciation of it before they can be introduced to its formal definition at the abstract level.

3.4.2 Concept image and concept usage

In the early nineteen nineties, Pierce conducted research regarding the acquisition of concepts and understanding of proof in mathematics. He reported that although it might be true that some students were guilty of a lack of diligence, many of their problems appear to have been of a cognitive nature. Some of the problems which seem to relate to the conceptual level were: "The students had little intuitive understanding of the concepts" and "The students' concept images were inadequate for doing the proofs" (Moore 1994:287).

Naturally definitions would need to follow concept images in order to establish proofs but it would seem that the underlying meaning of definition was lacking, preventing students from being able to prove results from them. This indicates the importance of a very firm foundation being established at the conceptual level so that students have a very clear concept image before encountering a definition. If students could become involved in the establishment of a definition then this would certainly help promote their understanding of it.

The concept image may be referred to as "the cognitive structure in an individual's mind associated with the concept" (Moore 1994:282). Moore elaborates further on the meaning of a concept image and the description would seem to show the division of concept image into two parts as is being considered here. For he remarks: "… the term *concept image* refers to the set of all mental pictures that one associates with the concept, together with all the properties characterizing them" (Moore 1994:252). Here the mental pictures seem to relate more to the level of percepts. The characterising properties form part of the perceptual level when studied in isolation but once they are compared and relationships are sought, level 2 or the conceptual level is reached.

It would seem that part of the problem of not being able to rise to level 3 stems from the possibility that, either appropriate mental pictures have not been established at the perceptual level or, even though students may have formed some sort of mental picture of a concept, time has not been taken to establish properties and consider relationships and inter-relationships. Since students do not have a thorough understanding of a concept, they are not able to proceed naturally to the definition. The gaps in their build-up to the definitions means that they have problems in understanding, stating and using them.

The role of concept images and concept usage evidently does play a relevant part at the conceptual level. Students involved in the research project by Pierce observed that they needed to have two types of examples: "(a) illustrations of definitions or concepts and (b) worked problems" (Moore 1994:257). This request of the students shows that concept usage also has a part to play at the conceptual level. In the history of equations, different types of method of solutions could be studied and compared at level two.

The importance of considering stages of learning concepts is emphasised by the concept understanding scheme. It seems that establishing and comparing properties and paying attention to concept usage at the conceptual level would help students to successfully attain this level and prepare them for level 3 or the level of abstractions.

3.4.3 Sublevels of the conceptual level involving concept images and concept usage

Although informal defining may belong to these levels too, actual concept images seem to form part of the perceptual and conceptual levels. Formal concept definition is restricted to the abstract level whilst concept usage may play a part at any of these levels. This is because concept usage may help to form mental images, illustrate comparisons and relationships and lead to or result from consequences of the abstract level.

In teaching a subject like abstract algebra, many rounds of the spiral could be traversed. Whether it is when initially establishing the concept of a group or higher up the spiral, it would first be necessary to establish mental pictures and properties and also possibly use previously attained concepts at the perceptual level. Further concept images need to be established at the conceptual level by drawing comparisons and possibly considering further concept usage. At the abstract level, concept usage helps to establish concept definition which in turn generates further concept usage leading to new concept images and so the process continues.

The findings of a British committee which was established to make inquiries regarding the teaching of mathematics in schools, under the chairmanship of WH Cockcroft, are relevant. It began to meet for the first time on 25 September 1978. The importance of properly laying the foundations at the early rounds of the spiral is alluded to in the observation that there is a fundamental need “to obtain a sufficient understanding of certain topics before being able to proceed to others which depend on them” (Cockcroft 1983:128). The importance of sound establishment of concepts is further emphasised by Cobb et al (1991:5) in their remark that “mathematical learning is not a process of internalizing carefully packaged knowledge but is instead a matter of reorganizing activity, where activity is interpreted broadly to include conceptual activity or thought”.

Carolyn Kieran observes how the definition of learning has undergone change. At first Watson and Skinner were regarded as the “hallmarks of what learning is” followed by “Tyler and Gagné coming along in the mid ‘60’s and turning that into an educational definition of learning” which as an important transformation because it “moved a psychological kind of principle, and it was a definite philosophical principle, into education” (Kieran 1994:587). She favours Piaget’s notion of learning “because it’s adaptive and not caused by the outside world” (Kieran 1994:587). She notes how in the past learning tended to be separated from understanding whereas now it is not believed that learning without at least some level of understanding is possible. In the 70’s researchers seemed to view learning in such a way that it included “understanding, retention, and transfer and to equate understanding with knowing, applying, and analyzing” (Kieran 1994:593). As a result of Richard Skemp’s book written in the 1970’s, where relational understanding was defined as “knowing what to do and why” as opposed to instrumental understanding involving “rules without reason” (Kieran 1974:595), distinctions began to be drawn between conceptual and procedural knowledge. Thus in the 70’s and 80’s a shift in focus can be seen from achievement scores to the processes and strategies employed by students. This has led to the “now-very-widespread belief among mathematics educators that learning is a constructive process” (Kieran 1994:598).

Korthagen & Langenwerf (1995:1014) refer to three levels or sublevels of learning: “Image formation, schematization, and theory building are three fundamentally different phases in the process of grasping a subject”. These stages also seem to relate to the perceptual, conceptual and abstract levels of the spiral. Firstly, they believe that the formalisation of images is based on examples and it is the process whereby a concept or phenomenon is formed by being experienced. However, at the conceptual level the students stop taking things for granted and some problem or curiosity prompts them to start looking for more. They begin to experience “… the need to recognise and name some kind of regularity in what they experience or observe”. Korthagen and Langenwerf (1995:1016). A spiral of learning is suggested with the statement “This schema may comprise all sorts of detailed subschemata, and may itself be part of one or more larger schemata”. As the spiral is climbed, previous experiences continue to be very significant as they provide “… students with support as they venture further a field” (Korthagen & Langenwerf 1994:1019). Just as at the conceptual level in the spiral theory, the schemata help students to give explanation and justification for what they are doing. This is an important aspect of mathematics illustrated by quotations from the following three people:

Mathematics arises from the attempt to organise and explain the phenomena of our environment and science (Bell 1993b:4).

Mathematics is ….an activity of organising fields of experience (Freudenthal 1973:123).

Mathematics concerns the properties of the operations by which the individual orders, organises and controls his environment (Peel in Korthagen & Langenwerf 1995:1021).

Concentrating on the aspects of concept image, definition and usage helps to encourage passage through the levels of percepts, concepts and abstractions. However, as has been already mentioned, they seem to form part of one or various learning levels and need to be dealt with adequately, timeously and appropriately in order to be beneficial to the learner studying a topic like abstract algebra.

3.4.4 Conclusion

Concept images and concept usage both seem to be relevant at the concept level. Pierce, in his research regarding the concept understanding scheme, reported how students found illustrations and examples of concept usage helpful as far as primarily promoting their understanding is concerned. For he observed:

> *It was through the examples particularly those of the first type presented by the professor and by the textbook that the students were able to build their concept images and, subsequently their understanding of the definitions and notation* (Moore 1994:25).

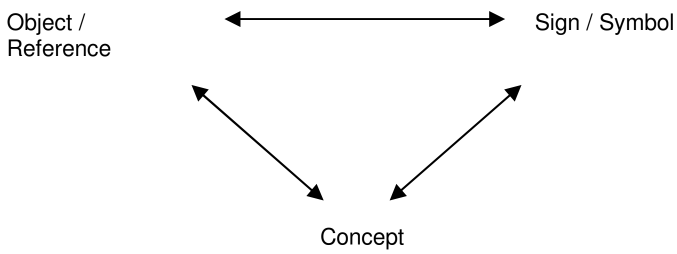
Ideally these concept images and usages should lead to interrelationships between properties and pave the way for an easily accessible pathway to definitions at the abstract level.

3.5 Chapter consolidation

After the level of percepts or the intraoperational level described by Piaget, there is clearly another level at which properties are compared, patterns are studied and interrelationships are established. This level, called the conceptual level here, is named the interoperational level by Piaget. The word “inter” in Latin means among or between and suggests that comparisons are being made between properties of different objects. Historically, in the case of the solutions of algebraic equations, during the interoperational level, different types of solutions of equations were compared and this led, amongst other things, to the Fundamental Theorem of

Algebra. The fact that this theorem was proved at the conceptual level gives a clear indication of the existence of sublevels or spirals of learning.

The historical development of mathematics is significant and what Sierpinska and Kilpatrick (1998:574) refer to as "…an important source of improvement of our understanding of school mathematical knowledge". They agree with Steinberg (1989) regarding the way in which mathematical concepts develop in history as illustrated by the triangle below:



Sierpinska and Kilpatrick (1998:515)

Low down on the spiral, knowledge may be concretely embedded in specific places or events but this is certainly not always the case. Sierpinska and Kilpatrick (1998:515) acknowledge both the descriptive and constructive historical developments of mathematics when they observe that

although scientific mathematical knowledge has been constructed in context - specific ways during earlier historical phases, since the beginning of the century at the latest, the epistemological status of mathematics has changed to viewing it as an abstract and formal knowledge structure.

At no matter what stage of the spiral a concept may be formed, it is often beneficial if it is preceded by a perceptual level and followed by an abstract level. After progressing through several rounds of the spiral, the students may be able to form their own concept images independently but it is very important that they have the opportunity of establishing concepts for themselves. Furthermore, it seems to be preferable if mathematical concepts are built up as a result of a variety of experiences in order to avoid association rather than abstraction occurring. This is so important because

… when a child has effectively formed a concept from his own experiences, he has really created something that was not there before, and this something is built into his personality in the psychological sense in the same way as essential substances in his food are built into his body (Dienes 1968:17).