Chapter 2
The perceptual level or initial learning level and sublevels in the development of abstract algebra

2.1 Introduction

Piaget and Garcia saw the emergence of group theory and fields in abstract algebra as "… the very source of the notion of structure formulated in the history of algebra" which he describes as "… Ariane's thread which helps us understand the transition from the inter to transoperational stage" (Piaget & Garcia 1989:156). Sawyer, when discussing his own approach to the presentation of abstract algebra, states that he has "… tried to show how modern higher algebra grows out of traditional elementary algebra" (Sawyer 1959:2). Freudenthal is also opposed to the presentation of ready-made mathematics and favours re-invention of mathematical discoveries. He observes: "How such axiomatic systems can arise is marvellously shown by the history of the group concept" (Freudenthal 1973:34).

Freudenthal (1973) saw the history of algebra moving in accordance with levels of thinking. Below, he illustrates this remark by showing how mathematical induction came to develop in history:

> *The complete induction was exercised since antiquity, the 'side-and-diagonal" numbers are a profound application of this principle. The first man who grasped the principle consciously and formulated it was Pascal. The formulation, a noteworthy feat, required quite new linguistic means* (Freudenthal 1973:123).

This suggests a significant attainment of an abstract level or level 3 on the spiral. However, Blaise Pascal (1623-1662) was as yet at the perceptual level of the topic of mathematical induction as it continued to grow and become more accepted after his time. He is said to have been the first to make a reasonable statement concerning the principle of mathematical induction in the "Traité du triangle arithmétique" which was written in about 1654. He began by constructing a triangle in the familiar form and deriving all its properties. However, Victor Katz observes that "Although many of the initial properties he developed should have been proved by mathematical induction, at least from a modern point of view, it is only in Pascal's Twelfth Consequence that the principle is stated and used" (Katz 1996:201). Pascal did describe the reasoning behind the principle but this was formalized for the natural numbers by Dedekind and Peano in the late 19th century. This shows that Pascal was still at a perceptual level as far as formalizing the result was concerned. Pascal did make use of a method of proof that could be described as a generalised example. The fact that this type of proof is preferred by students too gives an indication of the possible relevance of the connection between the historical development and the teaching approach followed with regard to a mathematical topic. Katz (1996:204) points out how

> *These first examples of proof by induction illustrate one major historical pedagogical point – that the general proof of the inductive step may well pose a problem for students and that a careful study of the "proof by generalizable example" must probably precede efforts where we as teachers insist on such a general proof.*

Katz believes that proof by induction is an old technique dating back to at least medieval Islam. Since only in the 18th century mathematicians began to be able to deal with proofs for arbitrary n and in the nineteenth century the modern proof by mathematical induction began to appear, "One cannot expect our students to jump over so many years of history with great ease" (Katz 1996:209). This statement illustrates both the parallelism between the historical development of mathematics and the development of concepts in the minds of students as well as the necessary perceptual, conceptual and abstract levels involved at each round of the spiral of learning.

Hull in NCTM (1989:29), referring to Klein's bio-genetic principle, remarks that "… the pattern of past mathematical discovery should be closely studied in relation to any proposed pattern of individual learning". He refers to what he terms "… the **direct** use of history – where the history itself is an actual part of the subject matter in terms of which teacher and pupil communicate with each other" and "an **indirect** use which may well be thought even more important". The latter refers to "… the teacher's own use of his personal knowledge of history, in trying to formulate the aims and improve the methods of his teaching" (NCTM 1989:27;28). Freudenthal remarked that the gist of the lesson taught to Meno's slave by Socrates was that "The thought – experiment tries to find out how a student could re-invent what he is expected to learn" (Freudenthal 1973:v). In the approach followed here there will not be "… a slavish adherence to the historical order of development in every detail or topic" but what will be borne in mind is that "… the broad general lines along which the race has conducted its creative thought are also the lines along which children can most naturally learn" (NCTM 1989:29).

Tall (1991:162) observed how: “It is interesting for mathematicians to look back at history and note the struggles that gave birth to modern ideas leading to the logical state of the art today”. Many mathematical concepts have their derivations from nature and physical situations in the world. The descriptive or “a posteriori” development of mathematics was known to the ancient Greeks. It arises when an object or structure is known before the definition is formed. In this way the concept precedes and leads to the definition. For example, an isosceles trapezium could be studied in a non-formal manner to investigate its properties. Subsequently, a concept image could be formed, sufficient properties found and links established with other concepts. Similarly in group theory different options could be explored leading to the definition of a group. Definitions could be compared in terms of their ability to lead to easier methods of proof.

Another way in which mathematics has grown may be described as the constructive or “a priori” development of the subject. It arises when something is virtually defined into being by using existing systems. This process begins with an axiom or axioms which change as a result of deductive or experimental procedures. For example, an attempt could be made in the concept of an isosceles triangle to generalise it to other polygons such as hexagons. In the case of group theory a non-commutative or near ring could be investigated by beginning with one example and expanding to others. At times certain definitions could be relaxed such as in the case of left or right inverses of a group.

The descriptive approach is initially necessary to develop an appreciation for the nature of definitions and their functions. Later on the constructive approach becomes more prominent in mathematics. If the original van Hiele approach were to be considered, de Villiers mentioned at a meeting of the supervising panel held in July 2004 that he believes the descriptive approach could be a third to fourth level activity while the constructive method could be a fourth to fifth level one. At the same time Heidema claimed that although the descriptive approach is initially beneficial, students ultimately need to become capable of following a university mathematics textbook on their own. This would most typically imply a constructive approach and would be one which could be followed by students who have passed through several rounds of the spiral.

Cognitive representations associated with pre-conscious and conscious levels of thought are taken into consideration in this study. The first level could be considered as having both a pre-conscious and a conscious aspect to it. Vision seems to be the chief sensory apparatus in mathematics. As the senses analyse the discrete signals going into the brain, the brain in turn integrates or synthesises the sensations into percepts (Gestalten) or unified images of the images acquired. Consequently one unified percept (for example, a triangle) is consciously seen. Westcott (1968:150) regards perception as "… an act of identification or categorisation, that is, an event in which a simple or complex stimulus array is identified as a member of some meaningful category on the basis of characteristics it shares with other members of that category".

Roth and Bruce (1995:13) have done research on "… how we make sense of all the information we receive from the world via our senses". They consider "… the concepts or conceptual category which serve to reorganise our knowledge of the world into manageable chunks" (Roth & Bruce 1995:13). These concepts or conceptual categories represent the common characteristics of objects being grouped together. The simplest concepts, such as the concept of a triangle, lie very close to a percept. However, the more complex ones, such as a torsion-free abelian group, are not. This suggests the spiral idea of concept formation. As one proceeds up the spiral, more and more complex concepts are encountered.

Concepts lie between percepts and linguistic representations or abstractions. The latter involves axioms and definitions. These in turn lead to theorems at different stages of the spiral. The lack of concept acquisition linking percepts to abstractions can often lead to problems of understanding. Moving from level 1 to 2 and level 2 to 3 involves moving from iconic to symbolic, concrete to abstract, picture-like to linguistic or simple to complex. Concepts form the link between iconic representations or percepts and linguistic representations or abstractions. The rise from level 2 to level 3 is important because while a student is still at the level of images or intuitive notions, he/she does not consult definitions which are necessary for further advancement up the spiral.

The word "percept" means the product of being made aware of something by one of the senses and "concept" means an "idea of class of objects; general notion" (Ostler 1969:106). The noun "abstraction" means "withdrawal" and is a derivation of the word "abstract" meaning an "essence" or "summary" (Ostler 1969:3). In Latin "ab" means "from" and "tracto" means "I haul" and so literally abstraction means what has been hauled, pulled or drawn out of something. Since this gives a succinct description of activities associated with the third level, the terms percepts, concepts and abstractions are being used here to describe the three levels of the spiral.

After proceeding from level 1 through 2 and to level 3, the abstractions established at level 3 lead to new percepts, concepts and abstractions and thus the spiral continues upwards in various directions. The spiral may be viewed in many different ways and even small concepts could be identified as being built up in a cycle of the spiral in the above described manner. This suggests the idea of sublevels and appears to be reminiscent of Piaget's theory of three levels termed intra, inter and transoperational levels and sublevels of intra-intra, intra–inter, intra–trans, inter–intra, inter–inter, inter–trans, trans–intra, trans–inter and trans–transoperational levels. These levels could in turn be broken down further into intra–intra–intra, intra–intra–inter, intra–intra–trans operational levels and so on and they bear close resemblance to the spiral level described above. Each intra, inter, trans triple could be associated with one percepts, concepts and abstractions portion of the spiral. However, the spiral seems to explain the growth of learning more easily as it is not necessary to keep on subdividing a level into various sublevels and using such complex terms to describe the levels as those used above. In addition, it also clearly shows how old concepts lead to new ones so that knowledge can continue to grow and branch out in different directions.

Van Hiele (1986) believed that the various thinking levels are inherent in man. In fact, since so many significant people who have considered the development of concepts in history and in learners themselves believe that levels form part of human thought development, applying these to the learning of algebra in general and abstract algebra would seem to be a logical path to follow. Even though not all theories consider three levels of thought, the three main levels of thought of each theory (some of which may contain more than one of their original stages) are considered in Chapters 2, 3 and 4 respectively.

2.2 Intra, inter and transoperational stages of Piaget

All of the three stages in the development of thought described by Piaget and Garcia (1989) termed intraoperational, interoperational and transoperational will initially be discussed here. This is because the sublevels associated with this type of thought would be appropriate for the first as well as subsequent levels of thought. Considering the spiral idea of learning, they form part of all stages of learning.

In geometry, Piaget regards the first stage in the psychogenesis of space as being intrafigural relations. Perpendicular lines may be drawn within one figure but children experience difficulty in drawing a line perpendicular to another one because they see things in isolation and have difficulty in making comparisons. In algebra this stage includes internal elements which do not combine with the other, does not involve any transformations and assumes the existence of invariants. Historically at this stage equations were solved in isolation, each type requiring its own particular method of solution.

Since the forms do not remain in isolation, at same stage, the interoperational stage arises. Here the forms that were previously isolated have correspondences and transformations amongst them together with the invariants required by such transformations. For example, Lagrange (1736–1813) was the mathematician who was the main figure in the transition to reach the interoperational stage in algebra when he adopted a more general scope and examined the nature of the solution of third and fourth degree equations. The third stage, known as the transoperational stage, is characterised by the evolution of structures where the internal relationships correspond to interoperational transformations. In algebra the transoperational stage can be seen as the emergence of a group or a field in the middle of the nineteenth century.

The intra, inter and transoperational periods are not merely ways of classifying stages but, as Piaget and Garcia put it, "… the three notions constitute different, but associated forms of organising knowledge" (Piaget & Garcia 1989:142). They are described by Piaget and Garcia (1989:142) as being the most constructive mechanisms they have ever found in their search for "…. common mechanisms between history and psychogenesis" and Piaget refers to the "…. unavoidable intra, inter, trans sequence" (Piaget & Garcia 1989:29) which occurs at all sublevels.

Thus it would appear that the intra, inter and transoperational stages would feature at each stage of learning in order to promote advancement to the subsequent one. Referring to the spiral idea, the perceptual, conceptual and abstract triple occur throughout the learning process. Piaget claims that whenever a new stage is reached, whatever is surpassed is always integrated into the new structures. Furthermore, he believes that this is common not only to mathematical history and the development of mathematical concepts but to all domains and levels of development. The application of these levels to teaching could certainly make a valuable contribution to the promotion of learning algebra.

The sublevels at each level such as the intra-intra sublevel appear to be significant. They could be further broken down into intra- intra- intra levels and so on ad infinitum. These sublevels seem to be reminiscent of the various processes utilised by Nixon (2002) in the study of the acquisition of the concepts of sequences and series by grade 12 students. Visualisation could be used at an intra sublevel to study isolated forms. Exploring patterns could be employed at an inter sublevel to encourage making comparisons and analysis whilst generalisation would encourage the establishment of structure at a trans sublevel. At each level of learning, promoting visualisation, exploring patterns and generalisation could help to encourage transfer through the intra, inter and trans sublevels to the subsequent learning level. In fact the levels could respectively be termed: intraoperational, perceptual or visualisation level; the interoperational, patterning or conceptual level and the transoperational, abstract or generalisation level.

2.3 The emergence of the perceptual level in the history of abstract algebra

2.3.1 Introduction

The early development of the number systems and symbolism all form part of the intraoperational or perceptual level of abstract algebra. Hence, in establishing these concepts, it would certainly be relevant to trace the outline of their historical background, showing how the inability of a certain operation, such as subtraction of natural numbers to yield natural numbers led to the introduction of integers and the inability to divide integers to yield other integers led to the development of rational numbers. Sfard (1991) comments on the way in which each number system was developed in a cyclic manner to compensate for inadequacies of the previous one. Freudenthal (1973) observed how history moved according to the van Hiele levels.

The van Hiele levels have also been described as being evident in the formation of the concept of an abstract group. The original first two levels of van Hiele are pointed out by Bell (1945) in the following extract:

> *The entire development required about a century. Its progress is typical of the evolution of any major mathematical discipline of the recent period; first the discovery of isolated phenomena [basic level – visual]; then the recognition of certain features common to all [second level-description];* (Land 1990:27).

The initial level is reflected here because although common features begin to be observed, it is only at the following level that further instances are sought and classifications are made. Both this and previously mentioned references give an indication of the relevance of levels both in the historical development of mathematics and the order in which mathematics is learned by students. The abundant illustrations of the levels of thought exemplified by the development of abstract algebraic concepts in history strongly suggest that these levels be taken into consideration in the learning of these topics.

When algebra first became established as an independent discipline, the solution of equations was its central theme. The initial stage could be described as the intraoperational or perceptual level of algebra and began in ancient times. This stage lasted for a very long time as the interoperational stage only began at approximately the middle of the eighteenth century. Methods used at the initial level were trial-and-error or empirical. The fact that each equation was treated as a separate object gives a clear indication that this was the intraoperational or perceptual stage associated with the development of abstract algebra. Up to the end of the overall perceptual level, there seem to have been two independent lines of pursuit which included "the geometrical analysis of Pappus and the arithmetic methods of Diophantus" (Piaget & Garcia 1989:147). However, at an abstract level of the overall perceptual level arose Viete's "new algebra" which "was both geometric and arithmetic" (Piaget & Garcia 1989:147). This necessitated the attainment of a higher level of generalisation than any ever reached by the ancients and so signified an advancement to the overall conceptual level of abstract algebra.

2.3.2 The intraoperational or perceptual level in history

Some of Freudenthal's observations regarding the early or intraoperational stage or perceptual level of mathematics in general and abstract algebra in particular are interesting and will be considered here. Freudenthal observes that no one knows whether man invented writing or arithmetic first, even though the alphabet is two millennia older than our present Hindu-Arabian numerical system. He believes that mathematics is much older than these numerals, whether based on oral counting or counters.

Freudenthal's research has revealed that by the end of the third millennium there existed "… well groomed elementary arithmetic and algebra instead in

Babylonia. It is not our formal algebra of x and y. The unknowns are indicated by the terms “length” and “breadth” (of a rectangle)” (Freudenthal 1973:1). The science in Babylonia was supposed to have been conducted by priests, which actually means the intellectuals of the era. They used to learn useful multiplications and divisions with tables and counters. Astronomy followed as the next science of mankind two millennia later. It was more practical than mathematics because it could be used, for example, to foretell calendars, eclipses, weather and other things besides. Not much of the ancient Egyptian mathematics has been preserved, because their work was done on papyrus rather than on clay tablets. However, enough of it is available to show that “…,. it was a mathematics that quickly and to a great extent surpassed practical needs” (Freudenthal 1973:3).

It seems that the Greeks learned Babylonian mathematics and astronomy in about the sixth century BC. Freudenthal claims that the Babylonians knew about the theorem of Pythagoras about two millennia before the Greeks and since it is a type of theorem that is not obvious by sight and needs to be proved, the Babylonians could have proved theorems before the Greeks, contrary to popular belief. Aristotle explained what a deductive system was when he observed that every true science starts with “archaic” principles (Freudenthal 1973:4) and Euclid’s Elements began with definitions, postulates and axioms. The Greeks discovered the irrational numbers as well as the incommensurability of the diagonal and side of a square. Since natural numbers alone did not suffice to explain geometric ratios, these ratios were banished from geometry, irrational numbers were forbidden and real numbers were unknown. Freudenthal remarks that “Geometrical algebra, this impractical product of mathematical dogmatism and practical rigourism, was the disease which killed Greek mathematics” (Freudenthal 1973:5).

Ancient Greeks and Romans used to have such impractical numerals as they made use of counters or an abacus. Hindu-Arabians computed with a dust-plank in dust or fine sand and later used cheap paper when it was available. The Egyptians, Greeks and even mathematicians and astronomers who accepted fractions would display fractions as a sum of fractions with a numerator 1. Only in the Indian period were fractions really accepted and the acceptance of negative numbers was another important development. This was relevant for the development of abstract algebra. The existence of Diophantus, a true algebraist, seems to show that the Babylonian tradition had lasted until the third century AD. Freudenthal claims that algebra was

reinvented in the Arabian world and it was revived through the Indians and the Christian Middle Ages. Descartes changed tradition when he algebraized geometry rather than geometrised algebra. Freudenthal regarded deductivity and the germ of axiomatics as the more modern feature of Greek mathematics. However, Freudenthal warns about the exaggerated ideas of ancient rigour. For he observes that despite, for example, the care taken with the postulate on parallels,
" … Elementary geometry in Euclid in particular shows gaps, and even sham arguments" (Freudenthal 1973:6).

Various historians have traced the origins of algebra to different nations of the world. These include the Assyrians, Babylonians, Egyptians, Hindus as well as the school of Alexandria. However, the Arabs, who concentrated on algebra mainly in connection with astronomy, were the first notable algebraists. They had gone as far as the solution of cubic equations. The word algebra comes from the Arabic word "al-jebr" which means the transposing of negative terms to the other side. This certainly is relevant in the study of equations leading to the development of group theory.

It is generally believed that Diophantus was the first to have formulated arithmetical problems in symbolic terms and to have introduced indeterminate values in order to solve these problems. Then indeterminate values were letters rather than numbers to express unknown quantities in equations. With the exception of a few experts, it would seem that the Greeks made little if any use of algebraic symbols in computation but rather resorted to the abacus.

Smith (1958b:436) claimed that "The earliest solutions of problems involving equations were doubtless by trial". However, in the time of Ahmes of c1550BC, the methods which were used to form these trials seem to have been well simplified, considering his solution of the equation $x+\frac{1}{7}x=19$ below:

Assume 7 as the number

Then to use the form of the text,

Once	*gives*	*7*
$\frac{1}{7}$	*gives*	*I*
I$\frac{1}{7}$	*gives*	*8*

"As many times as 8 must be multiplied to make 19, so many times must 7 be multiplied to give the required result.

Once	*gives*	*8*
Twice	*gives*	*16*
$\frac{1}{2}$	*gives*	*4*
$\frac{1}{4}$	*gives*	*2*
$\frac{1}{8}$	*gives*	*1 | in which he selects the addends making 19 |*
Together 2, $\frac{1}{4}, \frac{1}{8}$	*gives*	*19* (Smith 1958(b):436).

The main contributions that the Arab writers made to the solution of linear equations was "… the definite recognition of the application of the axioms to the transposition of terms and the reduction of an implicit function of x to an explicit one" (Smith 1958b:436). It may seem strange today that in the intraoperational level of group theory, the world found it difficult to solve equations such as $ax + b = 0$. A method known as the "Rule of False Position" (Smith 1958b:437) was used for the solution of the problem by early writers starting from the time of the Egyptians. This rule was very clumsy but was used for many centuries during the intraoperational level of solving equations. This shows how important good symbolism is and how necessary it was to move to the next level in the solution of equations. Another method used at this stage was called the "Rule of Increase and Diminution" (Smith 1958(b):441). The reason for the name was sometimes the error is positive and sometimes it is negative. During the sixteenth century the + and – symbols were used more often in this connection than as symbols of operation.

During the course of history, as the spiral is traversed, the number systems have developed in order to meet the demands of mathematics. Bell (1945:171) claimed that "The extensions of the number system since the sixteenth century are one of the outstanding accessions of all mathematics". The Babylonian and Egyptian fractions were the earliest extensions of the system of natural numbers. These illustrate the method of generating new ones from old ones by means of inversion. For example, in order to find what 12 must be multiplied to produce the result of 7, a new sort of number, the fraction $\frac{7}{12}$ must be inverted. In this case the direct operation is multiplication and the inverse operation is division. These elementary operations as well as the other pairs of elementary inverses including addition and subtraction and also raising to powers and extracting roots were all known to the ancients.

In order to solve linear equations for x which are of the form where $ax+b=0$; $a \neq 0$, the inverse operations of addition and multiplication are necessary. The inverses of the rational operations of multiplication and addition led to the necessity of the invention of common fractions and ultimately negative numbers. Bell (1945:172) claimed that "the operation inverse to powering was in part responsible for the invention of irrationals, including the pure imaginaries and the ordinary complex numbers". However, negative and complex numbers were not accepted mathematically until certain conditions were met. Bell (1945:173) observed that "until a conscious attempt was made to understand negative and complex numbers, and to state rules, however crude, for their use wherever they might occur", they did not have the right to be considered as mathematical entities. When Diophantus first encountered the formal solution of -4 to a linear equation in the third century AD, he rejected it.

Diophantus was a mathematician who worked in Alexandria in approximately 250 BC, although there was uncertainty about this date and his nationality. The most important work he wrote was the "Arithmetica". It involves an analytic treatment of algebraic number theory and concentrates on about 130 problems of considerable variety which all lead to first and second degree equations. He recognised only positive rational answers and in most cases was satisfied with only one answer to a problem (even though many of his problems had an infinite family of solutions). A low down perceptual level within the overall perceptual level seems to be indicated in his work as Eves (1990:180) points out how "striking is the lack of general methods and the repeated application of ingenious devices designed for the needs of each individual problem".

The fact that Diophantus had a strict Greek notion of a number which corresponds to what we now call a positive rational number, forced him into a type activity now termed "diorismos". This means that he stated restrictions for the allowable data for a problem so as to ensure that a positive rational solution did exist. This distinguished his work from the geometric algebra found in Euclid. An example of a diorosmos for a problem is described below:

> *The geometric equivalent of the problem of finding two numbers given their sum and product (applying a rectangle equal to a given area to a line in such a way that the defect is a square) has a solution only when the given area is at most equal to the square on half of the line.*

Although the solution of three term quadratic equations did not seem to be possible for the Egyptians; in 1930 Neugebauer revealed that such equations had been effectively handled by the Babylonians in some of the oldest texts of problems. For example, "… one problem calls for the side of a square if the area less the side is 14,30" (Boyer & Merzbach 1989:37). The solution of this problem is equivalent to solving $x^2 - x = 870$. Their solution amounted to making verbal use of the formula $x = \sqrt{\left(\frac{p}{2}\right)^2 + q} + \frac{p}{2}$ for a root of the equation $x^2 - px = q$. Even after Medieval times $x^2 + px = q,\ x^2 = px + q$ and $x^2 + q = px$ were considered to be distinct types of equations and all three of these are to be found in the Babylonian texts of approximately four thousand years ago.

The fact that the Greeks were able to solve the quadratic equation by using geometric methods seems to indicate that they were at a perceptual level low down within the overall perceptual level of solving equations. In his "Elements" Euclid (c300BC) gives such geometric problems as the following: "To cut a given straight line so that the rectangle contained by the whole and one of the segments shall be equal to the square on the remaining segment" (Smith 1958b:444). The algebraic representation of this problem would be the equation $a(a-x) = x^2$ or $x^2 + ax = a^2$.

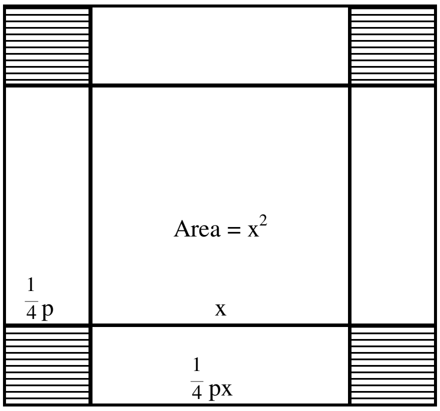
During the first third of the seventh century it is said that Brahmagupta first stated the rules of signs for multiplication but he discarded negative roots of quadratic equations. In the ninth century the rule of signs became common in India and at the same time both positive and negative roots of quadratic equations were given without rejecting negative ones. However, as far as the Europeans were concerned, in the early thirteenth century Fibonacci rejected negative roots but did regard a negative number in a problem concerning money as a loss rather than a gain. In the middle of the sixteenth century M. Stifel, who was a very good German algebraist of the time, stated that negative numbers were absurd. In 1572 Bombelli gave an indication that he understood the rules of addition in the case of $m - n$ where m and n are positive integers. At approximately this time Vieta rejected negative roots.

The present method of classifying equations according to their degrees is modern. In c1100 Omar Khayyam attempted a systematic classification in algebra but it was not the modern one. The degree of an equation in a variable is the numerical value of the highest power of the variable appearing in the equation.

Simple equations of the first three degrees were of the type:

$r = x,\ r = x^2,\ r = x^3,\ ax = x^2,\ ax = x^3, ax^2 = x^3.$

There were twelve forms of compound equations classified as trinomials such as $x^2 + bx = c,\ x^2 + c = bx,$ and so on whilst $x^3 + bx^2 + cx = d$ is an example of a compound quadrinomial. The many different forms were due to the fact that negative values were not allowed for coefficients b and c. Al-Khowârizmi of c825 used two different methods of solving quadratic equations of the form $x^2 + px = q,$ both of which were based on Greek models. One involved constructing a square as shown whilst the other involved adding $\frac{1}{4}p^2$ to both sides as is the case today.



$$x^2 + px = q$$
$$x^2 + px + (\frac{p}{2})^2 = q + (\frac{p}{2})^2$$
$$x^2 + px + \frac{p^2}{2} = q + \frac{p^2}{4}$$

(Smith 1958b:446)

Menoe of 350BC is said to be responsible for the oldest form of the cubic equation $x^3 = k$. Nevertheless two thousand years before this time Babylonians had worked out tables of cubes. It is said that Menaechmus solved the cubic by finding the intersection of two conics. The Greek Archimedes presented a problem of cutting a sphere by a plane so that the two segments have a given ratio. This reduced to the proportion $\frac{c-x}{b} = \frac{c^2}{x^2}$

$or\ x^3 + c^2b = cx^2.$

This problem was solved by Archimedes during the third century BC by finding the intersection of two conics $x^2 = \frac{a^2}{c}y$ (a parabola) and $y(c-x) = bc$ (a hyperbola).

Diophantus solved a single cubic equation $x^3 + x = 4x + 4$ which arises in connection with the following problem: “To find a right-angled triangle such that the area added to the hypotenuse gives a square, while the perimeter is a cube” (Smith

1958b:455). His method is not given, but in modern language the solution of the corresponding equation $x(x^2+1)=4(x^2+1)$ is clearly 4.

Although nothing more is known about the cubic equation among the Greeks, during the 9th century the Arabs and Persians took up the problem of Archimedes. Almâhânî, when commenting on Archimedes, emphasised the equation $x^3+a^2b=cx^2$ so much that it became Almâhânî's equation. It was one of his contemporaries Tâbit ibn Qorra of about 870 AD who considered special cases of cubic equations but was unable to make a contribution to the general algebraic theory. This gives an indication of a perceptual level or sublevel within the overall perceptual level of the solution of equations.

It is evident that polynomial equations do in fact have a rather lengthy history. A Babylonian tablet dating back as far as c1600 BC presents a problem which amounts to the solution of quadratic equations. The tablets reveal that the Babylonians did possess methods of solving them although they had no algebraic notation with which to express their solutions. The Greeks appear to have had difficulty in resolving many geometrical problems because of the absence of a science of algebra as this would have helped them to formulate problems in terms of operations. Furthermore, it seems that the ancient Greeks solved equations by means of geometrical constructions but there was no sign of algebraic formulation until at least 100 AD. They also had methods which they applied to cubic equations such as the problem of Archimedes which involved points of intersection of conics. At this time they were still at an intra-intra sublevel or a perceptual level low down on the spiral of abstract algebra.

The origin of abstract algebra can also be seen in the introduction of a symbol for zero which (according to Smith 1958b) was done by the Hindus at a controversial date some time before 800 AD. This was regarded as one of the greatest practical inventions of all time and was greatly appreciated by the merchants and traders of medieval Europe. In approximately 1225 Fibonacci discussed some special Diophantine systems of the second degree, including examples such as $x^2+5=y^2,\ x^2-5=z^2$, which are more difficult than they appear. Fibonacci did not seem to have considered finding all solutions rather than some. In this way his work was on a lower level than Euclid's integer solution of $x^2+y^2=z^2$. The failure to be aware of the generality of a problem characterises the intraoperational level of thought. Bell (1945:115) claims that "Euclid was the one exception in about two thousand dreary years of debased theory of numbers; even Diophantus, as we have seen, was content with special cases". The symbols + and – appeared first in a book printed in 1489. Algebraic solutions of the cubic remained unknown and Pacioli in 1494 ended his *Summa di Arithmetica* with the remark that finding solutions to "…the equations $x^3 + mx = n$ and $x^3 + n = mx$ was as impossible at the existing state of knowledge as squaring the circle" (Stewart 1998: xiii).

At Bologna the Renaissance mathematicians discovered that the three basic types to which the solution of a cubic could be reduced were:

$x^3 + px = q;\ x^3 = px + q;\ x^3 + q = px$.

The reason why they distinguished between these cases was that they did not recognise negative numbers, an important characteristic leading to the formulation of groups and fields. Bortolotti (1925) in Stewart (1998) claims that Scipio del Ferro solved all of these types. Once news of the solutions had spread, others became encouraged to try as well. In 1535 the solutions were re-discovered by Niccolo Fontana and Girolamo Cardano persuaded him to tell him about his method, promising him not to reveal his secret. However, in 1545 Fontana's solution appeared in Cardano's "Ars Magna" in which he claimed that the solution was his own work. Tartaglia (c1499–1557) and Cardan (1501–1576) solved the cubic equation for the general case. The "Ars Magna" also contained a method discovered by Ludovico Ferrari (1522–1565) of solving the quartic equation by reducing it to a cubic. One striking property of all formulae discovered was that they all could be built up from the coefficients by repeated addition, subtraction, multiplication, division and extraction of roots. These types of expressions became known as radical expressions. This gives an indication of the intra – inter operational level, intra – inter sublevel or perhaps some conceptual level low down on the spiral leading to the establishment of group theory. At this point all equations of degree four or less had been solved and so naturally the question of how the quintic could be solved by means of radicals began to be asked.

Francis Viete (1540–1603) was the first to use letters not only for unknowns but also for known quantities. He used the vowels A, E, I, O. U to denote unknowns and the consonants B, C, D …. to denote known quantities. Viete presented rules for solving equations. The one involved the transferral of terms from one side to another and was called "antithesis", corresponding to what the Arabic algebraists called aljabr. Another of his rules involved dividing all terms on both sides of the equation by the same thing. Here the origin of an inverse element under the operation of multiplication in group theory can be identified. Viete still appeared to be in the intraoperational stage or percepts level because he was dealing with specific types of problems. However, he was advancing towards the intra – trans operational stage or some abstract level low down on the spiral of group theory as he was establishing rules pertaining to the structure of equations. He was thus at the perceptual level of the overall spiral of abstract algebra and beginning to advance to the conceptual level.

In 1593 the Dutch mathematician Adrianus Romanus proposed the solution of a certain equation of degree 45. This proposition was extended to all mathematicians and the French King Henry IV did not believe that anyone in France would be able to solve the problem. Viete, however, could see that the equation would in fact be solved by the chord subtending an arc of 8 degrees in a circle of radius 1 so that the solution could be found by dividing the circle into 45 equal parts.

Diophantus and Viete differed in the way in which they made use of mathematical symbols. In his “Arithmetic”, Diophantus made use of various signs and abbreviations referring specifically to unknown quantities in equations. These have usually been considered as forming part of an algebraic symbolism. The idea that Viete reached the intra - transoperational level in group theory (or the abstract level low down on the abstract algebra spiral) seems to be supported by the following claim made by Piaget and Garcia (1989:146):

> *Viete took up a methodology characteristic of Greek thought, but he gave it a scope and a depth which enabled him to reorganise Diophantus’ work on a very different level.*

Piaget agreed with Klein that Viete should be regarded as the founder of algebra. Viete’s “new” algebra was both geometric and arithmetic and could only be attained by reaching a higher level of generalisation than the ancient mathematicians ever had (the intra – transoperational level or an abstract level within the perceptual phase of group theory). The crucial distinction which enabled him to establish algebra as a new discipline lay in “… his jump from the concept of “arithmos” to that of general symbols” (Piaget & Garcia 1989:148). Whereas arithmos refers directly to things or symbols, the symbols or letters he used referred to “… the property of ‘being a number’” (Piaget & Garcia 1989:148). Fleming and Varberg (1989) claim that if algebra is considered as the art of using symbols then Viete could indeed be regarded as the father of algebra. This is because the French lawyer Viete was the first person:

> *… to use letters systematically and purposefully both to represent unknowns and as symbols for arbitrary numbers. Arithmetic dealt with numbers, he said, but algebra was a method for operating on the forms of things* (Fleming & Varberg 1989:12).

Simon Stevin was born in 1548 in Brugge in Holland and made decimal fractions, which had been used by the Chinese and Arabs, popular in the west. Soon after Stevin, the decimal point was introduced and decimal notation came to be used for all real numbers, whether rational or irrational. As previously mentioned Stevin did not accept negative numbers and in addition he did not make a distinction between rational and irrational numbers but saw real numbers as forming a continuum. Here the intraoperational stage (or perceptual level) of the emergence of the concept of a field is evident. Van der Waerden (1985:69) remarks: "Thus, with the stroke, the classical restriction of "numbers" to integers (Euclid) or to rational fractions (Diophantos) was eliminated."

Pierre de Fermat (1601–1665), was a French lawyer who did brilliant work in number theory. Fermat and Descartes (1596–1650) simultaneously and independently invented analytic geometry. Although this was not a very difficult invention, it was of fundamental importance in the development of both algebra and geometry, allowing algebraic methods to be used for geometric problems and geometric methods to be used for algebraic problems. René Descartes was also responsible for introducing new algebraic notation. Later Newton (1642–1727) worked on the theory of equations and the binomial theorem and it was at about this time that negatives, so important in the study of groups and fields, came to be accepted as proper numbers. In addition, complex numbers were used but were not perfectly understood until a later stage. However, complex numbers did already seem to feature implicitly in the already developed technique for solving cubic and quartic equations.

2.3.3 Sublevels of the intraoperational level in the history of algebra

The intraoperational stage or perceptual level of abstract algebra as a whole is evident historically in the solving of many different types of equations, each with its own method of solution. However, this stage itself could be broken down into sublevels, each consisting of three stages too. Each type of equation would have an intra or perceptual level when different examples are solved in isolation. At the inter or conceptual level, solutions of different equations of a particular degree could be compared. Then at the trans or abstract level a structure consisting of a specific method of solving any equation of a particular degree could be established. However, for example, the intra – intra level of group theory could also be broken down into sublevels because it consists of studying isolated solutions of equations of a particular degree but this presupposes that such equations have been established using algebraic methods.

The German GHF Nesselman (1842) in Bell (1945) considered three historical phases of algebra which he called rhetorical, syncopated and symbolic. This provides an example of how at the initial stages algebra can be divided into sublevels. At the earliest rhetorical stage (which could also be called a many times intraoperational level or a very low down perceptual level of the spiral of abstract algebra), the entire statement and solution of an algebraic problem were completely verbal. This hardly even appears to be algebra at all "… unless it be that similar problems and their solutions reappeared in a later phase scantily clothed in at least the suggestion of a symbolism" (Bell 1945:123). During the middle syncopated or a conceptual level low down on the spiral of abstract algebra, abbreviations were substituted for the more frequently occurring concepts and operations which gave an element of truth to the idea that "mathematics is a shorthand" (Bell 1945:123). The third symbolic phase or an abstract level low down on the spiral of abstract algebra, algebra is fully symbolized both with respect to its operations and its concepts. Bell (1945:124) remarks that the symbolic phase "… also does much more than this." This is certainly the case when, for example, the intra – intra – trans level raises to the intra - intra; intra – inter and intra – trans levels and continues to proceed to further higher levels as well. Since the levels could be seen to be broken down even further into intra – intra – intra – intra sublevels and so on, a spiral of learning seems to give an excellent illustration of the level idea, without having to give new names for subdividing each previous level.

The first clear advance in mathematics since the time of the Greeks can be seen in the solution of cubic equations in the early sixteenth century. This represents the attainment of an abstract level within the overall perceptual level of algebra. Stillwell (2002:92) remarks how this advancement

> *… revealed the power of algebra which the Greeks had not been able to harness, power that was soon to clear a new path to geometry, which was virtually a royal road (analytic geometry and calculus)*

Viete (1540-1605) made a major contribution to algebra in the sixteenth century when he "… helped emancipate algebra from the geometric style of proof by introducing letters for unknowns and using plus and minus signs to facilitate manipulation" (Stillwell 2002:93). He did, however, manage to strengthen links with geometry at a higher level by the way in which he managed to relate algebra to trigonometry. For example, "in his solution of the cubic by circular functions" he was able to show that "… solving the cubic is equivalent to trisecting an arbitrary angle" (Stillwell 2002:93).

The Greeks had used curves to study algebra rather than the other way around. Rather than seeing an equation as an entity in its own right, they saw it as a property of a curve that could be studied after the curve had been geometrically constructed. Stillwell (2002:104) remarks how "What was lacking in Greek mathematics was both the inclination and the technique to manipulate equations to obtain information about curves".

The solution of equations and the improvement of notation in the sixteenth century made analytic geometry feasible in the sixteenth century. This is because a new level was reached at which equations, and consequently curves, could be considered in some generality. Coordinate geometry was discovered independently by Fermat (1629) and Descartes (1637). Descartes "… treated many higher-degree curves and clearly understood the power of algebraic methods in geometry" (Stillwell 2002:106). However, Descartes did define what are now called algebraic curves, geometric, showing that he was still attached to the Greek idea of curves being the product of geometric constructions. Viete (c1590) had been the first algebraist after the time of Ferrari to make an advancement in the solution of the biquadratic of the type $x^4+2gx^2+bx=c$ but "Descartes (1637) next took up the question and succeeded in effecting a simple solution of problems of the type $x^4+px^2+qx+r=0$" (Smith 1958b:469). This marked an advancement to an abstract level high up in the perceptual level in the evolution of abstract algebra and led to the beginning of the conceptual level.

The general rule of each stage having the three intra, inter and transoperational stages (or percepts, concepts and abstractions) may similarly be seen in the higher levels of thought development. This is evident in the evolution of the structure of a group, which may also be broken down into sublevels. Piaget and Garcia (1989:168) describe "group structure" as being "in itself transoperational in character”. Since the first groups introduced by Galois concerned only permutations already given and not constructed by the group itself, considering the matter from the point of view of the group, the relations involved were in fact intraoperational or formed part of the perceptual level of a round of the spiral. This would be the case even though, if the entire progress of groups were considered, it would be included in the overall transoperational or abstract level. Thereafter with Klein came the group of transformations which Piaget and Garcia (1989:168) observe “… play a constitutive role, such as the projective transformations which represent in themselves components of the group.” The structure represented here is of an interoperational type. At the transoperational stage, the concept of abstract group was elaborated and referred to any class at all, such as that operating on vector spaces.

Although Galois introduced the transoperational stage of the solution of quadratic equations when he introduced group theory, he was still at the interoperational stage as far as fields were concerned because he had to consider two laws of operations on a group and combine them to form the structure of a field. In studying the topic of fields, there would be an intra, inter and trans stage (or perceptual, conceptual and abstract levels) in the establishment of the structure of a field and then further inter and trans operational levels (or concepts and abstractions) would follow as various types of fields were studied at the perceptual level. Consequently general results regarding fields in general would be established. Thus different aspects of algebra could be considered as forming parts of a variety of levels and sublevels, depending upon which aspect of the subject is being studied. In the presentation of a topic to learners, the various levels and sublevels or the order of the spiral involved in the particular body of knowledge should be considered and presented in a sequential order similar to the actual or probable historical development of the theory in order to facilitate and enhance understanding. Lower down the spiral there is a greater tendency towards the descriptive historical growth of mathematics but higher up there is often a more constructive development evident in the historical progress of the subject.

2.3.4 Conclusion

Symbolism and the use of signs not belonging to everyday language to indicate variables is an algebraic idea. Old narratives used to mention “think of a number” problems and in Diophantus’ work the word number became progressively more of a computation symbol. Algebra as it is known today started at the end of the 16th century when Viete indicated not only the unknowns but also the indeterminates by means of literal symbols. Freudenthal remarked how "It is an algebraic idea to demand the unrestricted possibility of all operations and to fulfil it by introducing new objects" (Freudenthal 1973:10). This is an important factor that led to the development of abstract algebra.

Piaget and Garcia (1989:149) note how after the time of Viete "…. The study of algebra was limited to equations". For a long time many attempts were made to find a formula for solving equations of degree higher than four. During this period, systems of linear equations were successfully solved and algebraic solutions to certain specific problems proposed by geometry or mechanics were found. However, the fact that each problem requires its own method of solution or a particular procedure, led Piaget and Garcia (1989:150) to note that: "This indicates that we are dealing here with a period corresponding to the one we have characterized as intraoperational." If the development of groups were to be considered as a whole, then this stage could be considered the stage of percepts where isolated instances and various types of examples are being studied in order to lead to the next level of concepts.

2.4 The first level of various theories of stages of learning mathematics

Just as in history an initial level termed the intraoperational or perceptual level has been observed, so too in various theories of learning mathematics can an intraoperational or perceptual level be identified. Even at the first level of learning children do experience great conceptual difficulties. Research in this area has been undertaken by people such as Kathleen Hart and Dietmar Küchemann. In arithmetic the use of the symbol "=" in the statement $3+4=7$ seems to be different from its use in the algebra where, for example, in the equation, $3x+4x=7x$, it could be associated with an equivalence sign. Research done by Hart and Küchemann has shown that most children have either incorrect views of letter symbols as concrete objects or else very limited views of perceiving them as standing only for specific unknowns. De Villiers (personal communication, 2005) points out the following distinctions that both Hart and Küchemann make between the following meanings of letter symbols for children:

(1) letter as concrete object, e.g. "a" as an abbreviation for apple, etc.

(2) letter as unknown, e.g. as in $x+3=10$.

(3) Letter as generalised number, e.g. as in identity; $2x+3x=5x$.

(4) Letter as variable, e.g. as in $y=2x+3$.

The difficulties which students experience in actually understanding algebra highlights the importance of levels of learning in the teaching and learning of algebraic concepts. The three levels of percepts, concepts and abstractions in the development of concepts that have been identified as belonging to the spiral of learning are relevant because they concern:

(1) the successful levels (stages) in historical or psychological development;

(2) the different phases within each of these levels, since these require a regular sequence of sublevels for each new construction;

(3) the way in which previous acquisitions are reinterpreted from the perspective of the newly attained stage (Piaget & Garcia 1989:167).

Consequently, the provision of activity to encourage appropriate sequencing in order to pass through the relevant levels would seem to be relevant.

2.4.1 Jean Piaget's initial level of thought

(i) Introduction

Jean Piaget (1886–1980) was a famous Genevan who was born in Switzerland and spent most of his life studying human intellectual development. Since he was both a zoologist and an epistemologist, both of his fields of interest seemed to "… converge in the study of human intellectual development" (Donaldson 1978:130). After becoming involved in Freud's techniques and Alfred Binet's intelligence tests, he became interested in the mistakes being made by children. Donaldson (1978:130) claimed that Piaget believed that "… there is a normal course: a sequence which we all follow, though we go at varying speeds and some go further than others". His work has been very important for education because he has deepened people's understanding of the way in which children think. His original theory is particularly relevant to the initial level of learning.

(ii) Piaget's original first level of development

Piaget's initial theory of the stages of intellectual development consists of four stages. These have also been regarded as three stages where the first stage is called sensorimotor intelligence, the second pre-operational intelligence and the third is subdivided into concrete as well as formal operational intelligence. However, in the development of mathematical concepts in mathematics education, there appear to be three significant stages. Zevenbergen (1993) refers to these as the preoperational, concrete operational and formal operational stages. Consequently, here the sensorimotor and pre-operational stages will be regarded as forming part of the first level of mathematical development.

The sensorimotor stage

According to Schwebel and Raph (1974), the first of the original stages of Piaget is called the sensorimotor stage which extends from approximately 0 to 2 years of age. Before language appears, the initial growth of intelligence can be detected in sensorimotor activities of children. As a small baby grows, he/she repeats actions in many different sequences, employs more complex perceptual motor co-ordinations and discovers properties through his or her various senses. Even at this stage the growth from the level of percepts to concepts and then abstractions is evident.

During the first stage of intellectual development, as a result of his/her bodily actions, the child begins to form objective knowledge of the spatial, temporal and causal characteristics of the world around him. He/she explores experiments and builds up his knowledge by means of assimilating information from the environment and by accommodating his/her actions. The child comes to know the world through his/her bodily senses. He/she constructs practical concepts which Piaget terms sensorimotor schemes. At this stage he/she coordinates senses and perceptions with both movement and actions but has only a limited capacity of anticipating consequences of actions. During this stage Piaget recognised that the reflexes develop through a series of six sub stages "… into organised behaviour patterns which can be used intentionally" (Donaldson 1978:136).

The pre-operational period

According to Piaget's theory, the significant role of activity becomes less evident and "…. more subject to superfluous interpretations" (Schwebel & Raph 1974:15) as children grow older. From about two to seven years of age, a child's representation reaches a higher level as he/she is able to use signifiers to represent objects or events. Language, symbolic play and inventions are the important features of pre-operational thought. It is a stage at which a child is egocentric, attributes human qualities to inanimate objects, is unable to conserve, tends to centre on only one feature to the exclusion of others and is unable to mentally reverse actions. Both free play and symbolic play are important features of learning. There are two sub stages, namely preconception and perceptual or intuitive thought that are associated with this stage of development. At the perceptual stage the child begins to use language and mental images and, although he/she does generalise, it is illogical. At the perceptual or intuitive thought stages, the child relies on intuition instead of judgement or reasoning. Although his/her ideas are connected, they are neither logical nor consistent.

(iii) Piaget's reference to groups

Although a group is a mathematical structure, Piaget believed that

> *… it can be used to specify the nature of some of the fundamental structures of human intelligence, ranging from the first organization of that intelligence on a practical level to its final organization on a highly abstract symbolic plane (Donaldson 1978:133).*

In any group there has to be a set of elements and an operation carried out on the elements such that the following four conditions are satisfied:

(1) Composition: If the operation is carried out on any two elements, the result is also an element

(2) Associativity: The order in which two successive operations is carried out does not matter

(3) Identity: Among the elements there is always one, and only one, identity element"

(4) Reversibility: Every element has another element called its inverse. When an element is combined with its inverse the result is the identity element

(Donaldson 1978:136-137).

During this stage Piaget recognised that the reflexes develop through a series of six sub stages "… into organised behaviour patterns which can be used intentionally" (Donaldson 1978:134). However, at the concrete operational level Piaget found the group structure does not correspond to the structure of the mind. For example, when a class is added to another, a new class does not result. For this reason Donaldson (1978:137) claims that Piaget: "introduces the notion of a 'grouping'. A grouping is a kind of variant of a group, specially adapted to take account of the structures of classification, seriation and the like". Despite the difference between Piaget's reference to a group and a grouping, the reversibility condition always applies and once thought has become operational it is possible to mentally perform the reversibility act.

(iv) Mathematics at Piaget's original first level of thought (pre-operational stage)

By the time a child begins school mathematics, according to Zevenbergen (1993), he/she is usually at the preoperational stages of thought or specifically, the perceptual or intuitive sub-stage of thought. At this level in mathematics, Piaget advocates plenty of free play and the use of concrete materials.

Before the 1960's there was a strong emphasis on rote learning in mathematics. However, thereafter Piaget's notions of free play and concrete material began to assume importance in education, particularly mathematics education. One of the biggest changes in mathematics education in infant grades took place with Piaget's proposal that young children need to construct meaning for themselves through direct interaction with the environment. As a result, teachers began to change their learning environments and provide a variety of concrete experiences in order to encourage their pupils to engage in activities of play. This stage is important because of its contribution to the organisation of mental actions that will follow in the second level of thought. Thus Piaget saw the importance of the level of percepts which needs to precede concepts and consequently the provision of activities to encourage students to pass through the perceptual level would seem to be beneficial.

Piaget regards grouping as being the principle from which stems classification, seriation, number, innovation and space. The three types of grouping he considers are: those that relate to identity or equivalence; the logical system of classes in which two classes may be included in the other or may overlap or be mutually exclusive; those groupings that refer to relationships between parts and wholes of concrete objects or collections of objects. The ability of classification emerges out of a child's experience with both noting and acting upon resemblances and differences. He gradually begins to gain control of logical quantities such as one, some and all. Here the emergence from percepts to concepts can be seen at a place somewhere low down on the spiral. Seriation is another fundamental development at this stage. It relates to a child's ability to arrange objects in increasing or decreasing size. Once a child develops numeric concepts, he begins to grasp that a given number of objects, when compared with an identical number of objects, contains the same amount despite possible different spatial arrangements. In conservation the child is able to compensate internally for an external change such as the appearance of the same volume of a liquid in different shaped containers. As far as space is concerned, a child at this level is able to know where one object is in relation to another. For example, he knows the relationship between the way from his house to his school and from his school to his house.

Piaget feels that if mathematical entities do exists independently of the pupil as suggested by Platonism, then teachers could act as transmitters of knowledge while their pupils passively sit back and listen. However, he believes that there are logico-mathematical structures which develop gradually and naturally in children's minds and that the structures used in modern mathematics are much closer to these than those used in traditional mathematics. Thus he suggests that pupils should become actively involved in the learning process. This helps to ensure that they do pass through the necessary perceptual, conceptual and abstract parts of the spiral.

Piaget recognises socio-physico-logico-mathematical knowledge. He does not believe that logic arises out of language. His view contrasts strongly with that of the Soviet psychologist Vygotsky who basically believed that language is a prerequisite for logical thinking. Instead Piaget believes that logic is caused by actions. Once actions have been repeated several times, they become generalised and form assimilation schemes, which are very similar to the laws of logic. For this reason, pupils, especially at a young age, should experiment with objects in order to pass through the important perceptual stage necessary to help form arithmetic, algebraic and geometric concepts. Piaget recognises two types of experience resulting from activity. These he calls physical experience and logico-mathematical experience which are not the same. For the former involves such activities as the weighing and measuring of objects while the latter (of which few people are aware) involves not the physical properties of objects but the coordination of actions carried out upon them. For example, a child playing with ten pebbles is able to discover that their sum remained the same, no matter how they are ordered. Any objects could be used to serve this purpose and it is conservation that gives rise to logico-mathematical experience. Thus actions and logico-mathematical experience help to bring about the development of deductive thought. This is so partly because mental or intellectual processes, which intervene in the processes of deductive reasoning at a later stage, stem from actions and partly because as co-ordinations of actions and logico-mathematical experiences are internalising themselves, they give rise to a special type of abstraction which corresponds to mathematical and logical abstraction. Unlike

Aristotelian abstraction, which is derived from the physical properties of objects, logico-mathematical abstraction could be called “reflective abstraction” for the following two related reasons: This abstraction reflects what was on a lower plane to a higher plane of thought; reflective abstraction reconstructs at a higher level everything that was drawn from the coordination of actions. This suggests movement from the perceptual levels up the spiral.

Piaget holds the belief that “… speed of movement through the periods of development is influenced by the social and cultural environment (although the order of stages remains unaffected)” (Donaldson 1978:141). This depends on the ability to assimilate what is offered by the environment (although the order of the stages remains unaffected). Piaget also emphasised the importance of discussion to assist the development of thought. In particular, the exchange of ideas is relevant for “… strengthening the awareness of the existence of other points of view” (Donaldson 1978:141).

According to Piaget, teachers should provide children with suitable materials to enable them to become actively involved in the learning process. This would give them a firm basis on which to build their operational thought and ensures they pass through the perceptual level in their learning of concepts. Classroom activity is essential for the teaching of arithmetic to primary school children, especially grade one pupils, many of whom are still in the pre-operational stage and for this reason are incapable of understanding the concepts and symbolism involved in counting, adding and subtracting. At this stage these concepts are higher up the scale for them and still need to be reached by passing through the relevant rounds of the spiral. This indicates that the perceptual level needs to precede the conceptual and abstract levels at any stage of the spiral. Even at secondary and tertiary level, visualisation or forming mental pictures of concepts is very important in order to ensure that the perceptual level is covered.

Van Hiele (1959) indicates how the teaching of stereometry shows us that “…. various spatial concepts only come into being after a period of touching and doing. But it also appears that omission can be made good at a later stage” (van Hiele 1959:14). He adds that although sensorimotor development is often encountered by very young children, “…The stages and periods described by Piaget are not essentially connected with a particular age, but are characteristic for very many learning processes irrespective of the age at which they take place" (van Hiele 1959:14).

Consequently, it would seem that Piaget's first level of development associated with the level of percepts would be applicable even in the learning of such topics as group theory and fields. The historical origins of abstract algebra as well as introductions to topics covered can all be linked to the initial stages of Piaget's theory. Thus it would seem appropriate that a student learning group theory be introduced to it in such a manner that he/she is encouraged to pass through the initial stages of thought in order to gain better understanding and insight regarding the concepts involved. As one goes up the spiral, objects used at the perceptual level may be of an abstract mathematical nature. Nevertheless, observing their characteristics helps to build up concepts and subsequent abstractions. Both the descriptive and constructive development of mathematics in history could be illustrated in the learning of group theory. As the student advances upwards, it eventually is not always necessary to provide perceptual activities first since he/she could begin to form his/her own concept image from the concept definition or axioms provided.

(v) The intraoperational level or Piaget's later initial learning level

As already been mentioned, later on in his life, Piaget developed the notions of intra, inter and transoperational relations. The first level, known as the intraoperational level, is the stage at which relations appear in forms that may be isolated. In geometry, Piaget speaks of intrafigural relations. Here properties of individual figures are studied but no consideration is given to space or to transformations of these figures within a space. In algebra, Piaget refers to intraoperational relations as those which "… include, as their name indicates, internal elements, which, however, do not include any transformations, which presuppose the existence of invariants" (Piaget & Garcia 1989:141).

Piaget and Garcia (1989:174) recognise that their extensive research has led them to recognise three successive stages in the development of operations in children. These they termed preoperational, concrete operational and the stage of hypothetical deductive operations. They believe that these three stages represent the age levels "four-to-five, seven-to-ten and eleven-to-twelve and beyond" which "respectively, correspond to our sequence intra, inter and trans" (Piaget & Garcia 1989:174).

Despite the great contribution made to education by Piaget, Donaldson (1978:9) claims that as a result of research that she and others had performed she felt it became necessary to reject some of his teachings. She points out that “No theory in science is final, and no-one is more fully aware of this than Piaget himself” (Donaldson 1978:9). Some of her findings include the following ideas: children are more able to appreciate another person’s point of view than Piaget has suggested; in Piaget’s experiments the children’s responses often did not correspond to the questions they were asked; children are not as limited in their ability to reason deductively as Piaget seemed to claim. However, the intra, inter and transoperational levels are not bound to children’s ages or fixed stages of development.

The intra level applies in one sense to very young children but could be applied to the introductory stage of the learning of any concept. This is because
“The intra is characterised by the discovery of some operational actions and the analysis of its internal properties or immediate consequences” (Piaget & Garcia 1989:174). Two limitations associated with this level are a lack of “organised grouping” and ‘a presence of errors in the internal analysis of the operation” (Piaget & Garcia 1989:174). An example of the intraoperational level associated with young children is illustrated by an experiment in which pebbles are placed in two containers where one has two or three more than the other. Children in the intraoperational level believe that if one more pebble is continually added simultaneously to each container then eventually they will have the same number of pebbles. Historically during the intraoperational level of the solution of equations, only solutions for specific equations were sought. Trial and error methods were employed and “… Each equation was treated as a separate object” (Piaget & Garcia 1989:166).

Piaget introduced the abbreviations Ia, Ir and T for the intra, inter and transoperational levels and notes they must always occur in a sequence in the order Ia, Ir and T as there are developmental phases associated with changes from Ia to Ir and Ir to T. At the first stage Ia, which Piaget terms “preliminary and necessary”, particular cases which are not as yet related to one another are analysed. After comparisons have been made,

> *Differences and correspondences are found between them which lead to the construction of transformations (Ir). Once these transformations are mastered and generalised, new synthesis becomes possible (T)* (Piaget & Garcia 1989:183).

The levels Ia, Ir and T are necessary for the formation of concepts at different stages of development and are necessary for the introduction of any new topic and hence are relevant for both teaching and learning. They may be associated with the levels of percepts, concepts and abstractions and "… include, at each level, general psychogenetic or historical processes" (Piaget & Garcia 1989:182).

(vi) Sublevels of the first thinking level

Piaget devoted his life to studying the development of thought processes and adopted "…. a historical-critical method based in turn upon a psychogenetic method" (Piaget & Garcia 1989:i). As time progressed, he developed and adapted his perspective and his theory of the intra, inter and transoperational levels and sublevels in the construction of knowledge was revealed in the last book he wrote. This final work of his seems to have been truly inspirational and justifies the following assessment given of him by Barbel Inhelder in its forward: "His vision of the growth of knowledge in the child, refined and deepened by the historical study of scientific thought, continued to renew it, even to the present work" (Piaget & Garcia 1989:I). Rolando Garcia, who was a student of Carnap and Reichenbach, assisted Piaget in his last book and in the process was led to envisage the evolution of scientific thought in a new way.

Concepts are often taught as though there is no relation between the way in which they are formed in their elementary stages and the way in which they evolve at higher levels. This idea suggests that learning is a linear process, in which each stage replaces the previous one. However, this disregards the sequential nature of the construction of knowledge where "… each stage is at once the result of possibilities opened up by the preceding stage and a necessary condition for the following one" (Piaget & Garcia 1989:I). In this way knowledge can be seen to be built up in such a way that at each level it is reorganised and integrated in such a way that each level has elements going all the way back to initial stages. As learners proceed from one level to the next, reflective abstraction causes them both to project onto a higher level what is derived from a lower level and also to reconstruct what is transferred by the projection within a larger system. This process is constantly renewed as development takes place at each level. Consequently, each level in a reorganised structure still contains links with the most primitive elements of the topic. This shows both the importance of building up a topic from its basic original historical form and passing through the various levels and sublevels in the construction of knowledge. There is a danger that when a system of knowledge is seen in its completed state, such as when it has become axiomatic, then the impression could be created that the knowledge could be reduced to a series of statements. However, in order to reflect about a suitable build-up of its historical emergence, the topic needs to be sub-divided into levels and sublevels. This would certainly be necessary in the learning of abstract algebra, where a spiral of learning levels can be detected in its emergence and growth.

Piaget explains how since Greek geometry had no algebra, it remained at the intrafigural level, implying that it was "… subordinated to exogenous sources." (Piaget & Garcia 1989:137). Later on he remarks that "… Algebra is in itself a system of forms which generate their own contents" (Piaget & Garcia 1989:137). This suggests that at the intraoperational level these sources or forms be studied in order to encourage progression to the interoperational level. In fact, when on the intraoperational level of a topic, it is necessary to consider the sublevels some of which could be referred to as the intra – intra, intra – inter and intra – trans operational levels or previous rounds of the spiral which could go very far down the spiral. In this way, old content is transferred from previous levels and integrated at higher levels to promote further growth of knowledge.

Piaget and Garcia (1989:3) note how "reflective construction" and "constructive generalization" repeat themselves indefinitely at each successive level. This suggests that it is vital to provide the necessary tools to promote construction, generalisation and advancement to the subsequent level. Piaget (1973:17) refers to laying a "thinking foundation". Furthermore, a notable parallel can be found between the way in which concepts develop in learners and the history of scientific thought where sequential stages are found. Historically, the stages in the formation of the theory of groups and fields can be seen to follow sequential steps, which may be likened to the growth of a spiral in various directions.

(vii) Freudenthal's criticism of Piaget's theories

Freudenthal has made several criticisms regarding the Piaget school's investigations on the development of concepts in mathematics. He claimed that Piaget had a tendency of borrowing mathematical terminology and applying it with quite divergent meanings. He felt that Piaget revealed misunderstanding of mathematical solutions such as cardinal number, ordinal number, mapping, transformation group and the content of a cylinder as its diameter and height are altered. Furthermore, he questioned the verbal elements in Piaget's experiments. He felt that misunderstandings could have arisen from misunderstanding of language rather than notions. He complained that at times, instead of being given a simple task, children were given a combination of several tasks. He also noticed a disparity between the description and execution of an experiment.

However, these criticisms seem to be aimed at his experimental work with children and are not related to the work performed in his later life, such as his concept of intra, inter and transoperational stages. Freudenthal (1973:662) does also give some praise to Piaget, stating "… I would like to stress the wealth of ideas in his work, his originality, not to say genius". Piaget did after all initiate the idea of levels of learning and this has certainly led to further developments and improvements in mathematical education.

(viii) Van Hiele's criticism of Piaget

Van Hiele (1959) claimed that Piaget in his preface to "Didactique psychologique" clearly stated that he did not intend to work out the didactical consequences of his theory as he was a psychologist with insufficient experience in matters of teaching. Van Hiele continued that the pupil, the teacher, the subject matter and their interrelations are the main features in a didactical situation and didactics could never be considered merely as the application of psychology. Piaget stated that the development of a child's mathematical thinking went hand in hand with the development of his ability for logical thinking, but did add that thinking encompasses more than logic alone. But van Hiele remarks: "To most teachers of mathematics it seems very odd that Piaget keeps trying to prove that the ability to work with numbers or figures can be acquired only on the basis of logic" (van Hiele 1959:3).

Van Hiele felt that Piaget's theory of child development gave "… every reason to believe that intellectual maturity is biologically conditioned" (van Hiele 1959:4). Furthermore, he argues that if the process of mathematical thinking develops parallel to that of logical thinking, then it should be possible to rely on logical structures that, according to Piaget, already exist in a child if he has already reached the corresponding stage of maturity. This would imply that no problems should exist with the deductive development of mathematics at high school because a child would have reached the relevant stage of maturity. Van Hiele feels that this does not explain why children who have reached the stage of logic are satisfied with learning rules by heart and how the mental activity of implicit replication of algorithms should be classified. He observes that Piaget's concept of logical thinking is different from the one encountered in formal logics and adds: "We must assume that Piaget judges a child to be thinking logically when it reasons on the basis of an organised structure, of an available network of relations" (van Hiele 1959:9). He furthermore believes that Piaget's work shows how the child develops in the direction of Piaget's concept of the world rather than the genesis of thought itself.

All of the criticism levelled by van Hiele at this stage relates to Piaget's early work on child development. Piaget's three original stages of the genesis of thought that van Hiele detects include: firstly there is no concept concerning the subject in question, secondly the child shows clearly that he knows something about the concept but does not make use of it, finally the child knows reasonably well how to operate with the concept. Van Hiele complained that these stages were too vague and bound to a particular period of a child's development. However, van Hiele concludes that Piaget's stages are relevant in the progress from one level of thought to another and are not associated with a particular age. In van Hiele's words "… the stages Piaget has described must not be regarded as belonging to a particular period of life nor do they belong exclusively to the intuitive thinking described by him." and "In the second place, we must point out that the three stages can occur several times within a learning process, which concerns one object" (van Hiele 1959:9). This suggests the spiral of learning growing and developing in various directions.

Thus van Hiele and Piaget both had the same idea of three sublevels in a learning level or three stages in the process from one thinking level to the next. Here these sublevels have been given the names of percepts, concepts and abstractions. Van Hiele also praised Piaget for the contribution he had made in the sensorimotor sphere. According to stereometry, certain concepts only come into being after a reasonable period of touching and doing. It has been found that even if this might have been omitted from a person's life, it can be rectified at a later stage. Van Hiele felt that Piaget did not realise that the object of thought is different at various levels. But his criticism seems to be aimed towards Piaget's earlier theory because Piaget's views on levels towards the end of his life were neither biological nor static but had great relevance. Van Hiele acknowledges the importance of Piaget's work when he writes:

> *The stages and periods described by Piaget are not essentially connected with a particular age, but are characteristic for very many learning processes irrespective of the age at which they take place* (van Hiele 1959:14).

(ix) Conclusion

Jean Piaget began his investigations after the first World War. But his findings were not applied until the early 1960's in the Western World. Up until that time students were taught in very much of a rote fashion. However, when the launching of Sputnik led to the first man going to space, the USSR appeared to the Western World to be the more advanced nation. Educators in the United States began trying to improve mathematics and science learning in school in the hopes of producing students with mathematics and science orientations who were able to challenge the Russians. The Russian method of teaching had been strongly influenced by Vygotsky, who himself had been influenced by Piaget. In order to modernise mathematics, there followed the zealous introduction of the new maths. De Villiers in his correspondence with the writer in March, 2005 expressed the belief that this was not necessarily based on any of Piaget's work but it was driven by some mathematicians' intent on "modernising" mathematics with no or little consideration for psychological aspects of learning.

Many prominent mathematicians like Jean Deudonne from the French Bourbaki School of thought were heavily involved in the teaching of mathematics in a ready made axiomatised form. Bourbaki and another group of largely French and other Western mathematicians had the intention of formalising and rigorously systemising mathematics. They belonged to what is known as the "formalist" school of thought to which Lakatos (1976) was greatly opposed and which Freudenthal strongly criticized:

> *Should axiomatics be taught in schools? If it is taught in the form it has been in the majority of projects in the last few years, I say "no". Prefabricated axiomatics is no more a teaching matter in school instruction than prefabricated mathematics in general. But what is judged to be essential in axiomatics by the adult mathematician, I mean axiomatising, may be a teaching matter. After local organisation the pupil should also learn organising globally and finally cutting the ontological bonds. But to do so, he must be acquainted with the domain that is to be organised, and the bonds that are to be cut should exist and should be vigorous. This is an exacting demand. The pupil who has never been self reliant in organising, will overlook connections only if the number of links is small, and the bonds with reality usually are little cultivated and weak. Of course, all these demands can be dismissed if the student is confronted with a ready made axiomatic system from which he may obediently draw a few conclusions. He can be drilled in this art, but this can hardly contribute to*

understanding axiomatics. The only result would be to enlarge again the stock of denatured school mathematics (Freudenthal 1973:451).

Herscovics and Bergeron (1981) suggest a four level model in the construction of conceptual schemes in mathematics. These levels include: intuitive understanding involving pre-concepts and visual perception; initial conceptualisation; abstraction and then formalisation. Herscovics in Steffe et al (1996:357) notes the similarities between his levels and Piaget's three levels of practical understanding and conceptualisation followed by consciousness and verbalisations. In the spiral of learning theory formalisation could be regarded as an abstract level higher up than what they refer to as abstraction in this context

The psychologist David Ausubel has a learning theory which shares some similarities with the theory of Piaget. However, unlike Piaget's approach it tends to be teacher dominated. Ausubel et al (1978:368) observes that: "Of all the possible conditions of learning that affect cognitive structure, it is self-evident that none can be more significant than organization of the material". The use of "advance organizers is the central strategy of the advance organiser model. These are described by Bell as follows:

Advance Organizers for topics in any discipline are introductory materials that are presented to students at a higher level of generality, abstraction, and inclusiveness than subsequent learning tasks (Bell 1978:232).

The actual advance organizer can have a variety of forms and can be introduced in many different ways. They are not merely outlines or summaries but should be more abstract, general and inclusive than the actual content of material they are organizing. Ausubel considers two types of organizers:

Expository organizers are formulated to provide the learner with a mental structure to which he or she can relate unfamiliar material that will follow the organizer. Expository organizers are used to introduce material that is unfamiliar to students. Comparative organizers also help students integrate new concepts and principles with concepts and principles that they have previously learned in the same subject. Comparative organizers also help students discriminate between familiar and unfamiliar ideas that are essentially different but which may be confused (Bell 1978:234).

Ausubel's model is based on his theory of meaningful verbal learning. He believes that academic disciplines can be uniquely structured into hierarchies of concepts, principles, facts and skills and that:

> *... the objective of educational system should be to identify and organize these information structures within each discipline and impart the structures in a meaningful way to students* (Bell 1978:231).

These structures seem to be reminiscent of Piaget's intra, inter and transoperational levels. When properly structured and received by students, this method can assist students to develop mental structures to help them understand new learning material and integrate it with other material from the past. This they can do as they advance further up the spiral after successfully traversing the perceptual, conceptual and abstract levels of the relevant lower rounds.

Piaget has been one of the most influential people in education, especially regarding mathematics and science. His stages of development have become so entrenched in mathematics that Zevenbergen (1993) notes that it is difficult to imagine it as being otherwise. During the middle period of his work, Piaget's investigation involved the development of stage theories. Later on his life, he returned to his earlier work involving how learning occurs. His intra, inter and transoperational levels and sublevels developed at this time as well as the parallels drawn between the history, teaching and learning of mathematics are all of major importance. He observed that the development of knowledge occurs in stages rather than as the accumulation of new facts. He stressed how these stages "… represent characteristic cognitive levels such that at each stage there is a reorganisation of previously acquired knowledge" (Piaget and Garcia 1989:141). The spiral theory is related to his ideas and forms a crucial part of the proposed general theory of introducing algebraic topics including group theory in Chapter 5.

2.4.2 Hans Freudenthal

(i) Introduction

Hans Freudenthal (1905 – 1990) was a mathematician who made a great contribution to mathematics in the fields of Homotopy Theory and Lie groups in group theory. He believed in mathematics for all and made a great contribution of ideas and conceptual tools for mathematics education.

His interest in mathematics education was aroused when he taught his two young sons arithmetic at primary school level. As a result, he wrote his didactics of arithmetic in 1942 and during the 1960's became a member of the Modernisation Committee of the Mathematics Curriculum in the Netherlands. His name has always been associated with the Wiskobas project. This began in 1968 and led to the development of realistic education at primary schools in the Netherlands, where it is currently the main approach followed in the teaching of mathematics. He was also the founder of what is called realistic mathematics education.

Freudenthal claimed that in no other field of education besides mathematics "… is the distance between useless aim and aimless use so great" (Freudenthal 1973:64). He has expressed the view that mathematics involves growing, developing, looking for and solving problems as well as organising subject matter. He stressed the point that "… the pupil himself should re-invent mathematics" (Freudenthal 1973:118). However, Freudenthal (1973:118) notes that there are mathematicians who deny students of that right with the attitude "Quod licet Jovi, non licet bovi" meaning "I, Jupiter, organized the world mathematically for the student, why should he, the ox, start anew?"

Freudenthal points out the danger of teaching the results of centuries of mathematical activity as a rigid system, no matter how rational and beautiful it might seem. He believes that the sources of insight be kept open for the learner of mathematics and that they be kept aware of the origin and development of the mathematics they are studying. He remarks: "Today, I believe, most people would agree that no teaching matter should be imposed upon the student as a ready – made project" (Freudenthal 1973:118). Like Piaget and van Hiele, he believes in learning levels where on the higher levels, the contents or activities of the lower level become the object of analysis. Consequently, any theory of learning, such as a theory regarding the learning of abstract algebra, should "…. include phases of directed invention" (Freudenthal 1973:118).

(ii) Freudenthal's first level of learning

Freudenthal sees mathematical activity as organising fields of experiences where each time "… the means of organization of the lower level become a subject matter on the higher level" (Freudenthal 1973:123). This suggests that concepts established at the abstract level are utilised as percepts for the next climb of the spiral. He regards levels as being of such major importance that he observes that "The learning process is structured by levels" (Freudenthal 1973:125). This suggests the existence of a spiral of learning levels advancing from percepts to concepts and then abstractions over and over again.

Freudenthal stresses the significance of the first level or percepts in the learning process. In fact, he commented that passing over the bottom level "… is one of the mistakes of traditional mathematics instruction" (Freudenthal 1973:127). However, he observes how at times this activity might appear to non-mathematical people as being irrelevant. He chooses to call this level the pre-mathematical level and regards it as a very important one. For at this level children can be playing a game without even realising that they are doing mathematics. He describes this bottom level as being "indispensable and transitory" (Freudenthal 1973:136) as it is a stepping stone to the second level.

As an example of bottom level activity, he mentions Dienes' experiments involving six to twelve year-olds handling quadratic equations, finite groups, isomorphisms and modules. He was impressed by this experiment, showing the children revealing more ability than usually revealed in traditional education. However, the experiments involved children operating only with some special groups which could only be defined at a higher level. He remarks that by "… no reasonable criterion is it group theory" as this might make one think "… that one has fulfilled one's duty towards group theory" (Freudenthal 1973:68). Thus although the attempts to teach group theory at the first level were genuine, he concludes that "I think they really proved it was possible though neither at primary level nor in the lower grades of the secondary school" (Freudenthal 1973:68). At these lower grades of schooling, students would only be able to operate with some basic first level concepts of groups and fields but they could only rise to a higher level at a much later stage.

(iii) Sublevels of Freudenthal's first level of thought

As an example of his first level or pre-mathematics level, Freudenthal gave as an illustration the learning of complete mathematical induction. Initially, at the perceptual level, students should be provided with examples which compel them to invent complete induction. As a result of these examples, they come to recognise the common principle. Non-examples should be provided too, to help students appreciate the concept. The examples provided should be non-trivial or at least non-trivial looking types.

For example

$$1^2 = 1$$
$$2^2 = 1+3$$
$$3^2 = 1+3+5 \quad ,$$
$$n^2 = \sum_{k=1}^{n}(2k-1)$$

would serve as an example of the intra - intraoperational or initial perceptual level. Further examples and non-examples would be suitable at the intra-inter-operational or conceptual level. Binomial coefficients and the binomial theorem would be examples suggesting structure at the intra-transoperational or abstract level. However, Freudenthal complained that textbooks often present the binomial theorem as a consequence of complete induction which forms a "vicious circle" (Freudenthal 1973:122) in mathematical invention. Furthermore, to deduce complete induction from Peano's axioms and then apply it to various examples would be what Freudenthal would call "… a striking example of the antididactic inversion" (Freudenthal 1973:122). Freudenthal believes that a principle like mathematical induction cannot be formulated unless it has been experienced and the binomial theorem is "… the decisive experience that leads to this principle" (Freudenthal 1973:122).

In a similar way, abstract algebra should not be presented by means of initially stating axioms without first investigating the previous levels involving symbolism, number systems and operations as well as the solution of equations that build up to the structure. These various levels and sublevels need to be considered in the development of a theory of learning the topics of algebra at all levels. Once a student has passed through several rounds of the spiral, it could become possible to adopt a constructive approach in which definitions and axioms are initially provided and the student is able to construct his/her own concept image from these.

(iv) Conclusion

Freudenthal observed how Socrates did not believe that true knowledge is really invented but rather that acquiring knowledge involves re-discovering what a person knew when his or her "… soul stayed in the realm of the idea" (Freudenthal 1973:102). Now re-discovering would mean learning from the history of mankind. Some time after Socrates followed Comenius and his teaching theory was modernised by Freudenthal's statement that "The best way to learn an activity is to perform it", (Freudenthal 1973:110) as opposed to Comenius' original comment "The best way to teach an activity is to show it" (Freudenthal 1973:110). Here Freudenthal stresses that the emphasis should be shifted from the teacher to the learner.

Freudenthal strongly believed in showing interest in the bottom or perceptual level of mathematics but was adamant that activities should be relevant and followed up so that "… the intensive stream of preparations does not ooze away" (Freudenthal 1973:129). He could not see the point of directing the child to train its mathematical abilities on the bottom or perceptual level unless he or she is given the opportunity of progressing to the next level (concepts) where it is possible "…. to reflect on its bottom level mathematical activities" (Freudenthal 1973:129). This is certainly a relevant consideration in the learning of abstract algebra. For example, a game could be used as an introductory perceptual level activity but the student should be given the opportunity of considering its relevance on the second or conceptual level of thought. This base or perceptual level type of activity would seem to be extremely important as, unless a student is able to reflect on his or her own activity, "… the higher level remains inaccessible" (Freudenthal 1973:130).

2.4.3 Pierre and Dina van Hiele

(i) Introduction

Pierre and Dina van Hiele were students of Freudenthal and as young teachers, considered their own experiences, compared them with others and reflected on their own actions. This enabled them to learn how to teach geometry and so they transferred this idea to the learning process of pupils learning mathematics and discovered similar levels (1959). This was an important discovery that has had a strong impact on the learning of mathematics, particularly geometry.

According to Kipfinger (1990:4), the van Hiele's research "… had its roots in Piaget's work." The van Hiele's found that many secondary students experience difficulty in high school geometry as a result of a lack of understanding stemming as far back as the early and middle grades. This highlights the importance of the establishment of concepts at the bottom or perceptual level. If this background is lacking towards the bottom of the spiral, it is not possible to continue climbing it in an effective manner.

Pierre van Hiele considers students to be important in the learning of mathematics. He has a Gestaltist type of approach and he believes that structure has four main properties: it may be extended; it may be seen as part of a finer structure; it may be seen as part of a more inclusive structure; it may be isomorphic to another structure. Van Hiele uses the human skeleton to illustrate the four properties of structure that he describes. The first property may be seen when we become acquainted with the skeleton by looking at it and it may be extended when people realise that they have one themselves. The second property is apparent when a finer structure is obtained by naming the parts of the skeleton. The third property comes into being when people begin to study the skeletons of animals and compare them to human skeletons. The fourth property or the rule of isomorphism is also used when the skeleton of animals are compared, using the same names for bones and using global isomorphisms.

The idea of objects is a further significant aspect of van Hiele's theory. Objects are the elements of a set called a category which has a set of relations between the objects satisfying certain postulates called morphisms. According to Hoffer (1983) each van Hiele level is a category with different objects at each level. At the first level or level 0 the objects are the base elements of the study. These are followed by the following objects for levels 1, 2, 3 and 4 respectively: properties which analyse base elements; statements that give the properties; partial ordering of statements that relate the properties; properties that analyse the partial ordering. If the van Hiele model is used for designing learning experiments in other structured subjects besides geometry then Hoffer (1983) believed that the objects as perceived by the student and suggested by the topic should be determined at each level.

Another important aspect of van Hiele's theory is insight. He realises that the learning of facts could not be the purpose of teaching mathematics but rather the development of insight. He feels that there is a relationship between insight and structure and writes "… I learned that insight might be understood as the result of a perception of structure" (van Hiele 1986:4,5). He believes that someone has insight when they "… act in a new situation adequately and with intention" (van Hiele 1986:24). He claims that in order to obtain this insight in the teaching of geometry, the levels of learning should be applied. He feels that all of these levels, except for the basic one, possess a structure and that a student obtains insight where he or she perceives the structure, is able to continue it, explore it in detail, expand it and envisage a greater more encompassing structure. He observed that it indicates that once a student has reached this point at a particular level, then he or she would be ready to move on to the next level.

Considering the intra, inter and transoperational sublevels of Piaget or the levels of percepts, concepts and abstractions, the first and second properties of structure could be related to the intra level of Piaget (or percepts), where objects are studied in isolation. The third property of structure seems to relate to the interoperational level (or concepts) where objects are compared and generalisation takes place. The fourth property of structure appears to relate to the transoperational level where structures are established.

In their original work the van Hiele's hypothesized the existence of five levels of thought in geometry, namely visualisation, analysis, informed deduction, formal deduction and analysis. But later three of these were emphasised. These include the visual, descriptive and theoretical levels as it was considered that these ones adequately described the learning process. It is the visual level which is the lowest level of the van Hiele's later theory or the levels of visualisation and part of analysis of the five level theory that will be considered in this chapter. This level will be considered to be repeated at the bottom of each part of the spiral and hence will form a very relevant part of progress up the learning spiral.

(ii) Van Hiele's first level of thought

Van Hiele's original five levels of thought are subdivided into three parts in order to observe the three main stages of thought development. His later three stage model is also taken into consideration here. A comparison is drawn between van Hiele's levels and Piaget's intra, inter and transoperational stages as well as the perceptual, conceptual and abstract levels. Since Piaget's first level or intraoperational level deals with isolated forms, van Hiele's visual level of his later model or his first and some of his second original level would seem to form part of the first of three main levels of thought and are studied here.

Original 1st level – Level 0 – visualisation

The original first level of thought of the van Hiele's is called the level of visualisation. At this level children may recognise figures such as a square by their appearance but they may not be able to perceive their properties, such as possessing four right angles. They are able to identify many figures and recognise a figure by its shape as a whole. However, irrelevant attributes, such as the orientation, can cause them to be unable to recognise a shape any longer. It is possible for children to represent figures as visual images and see objects, such as triangles, as classes of figures. Geometric figures are recognised, compared and operated with according to their appearances but no attention is paid to geometric properties or characteristic features of classes of figures.

Original 2nd level – Level 1 – analysis

In the overall intraoperational stage of algebra properties were recognised and analysed for isolated types of equations. In this sense the second level of van Hiele seems to form part of Piaget's intraoperational level though it would not be situated at the bottom round of the spiral. The intraoperational or perceptual level is suggested because it is possible for children at this level to recognise and characterise shapes by their properties and to begin to analyse geometric figures. Figures are no longer seen as mere visual Gestalts but possessing a collection of properties. They become aware not only of the properties of figures, but develop concepts of figures resulting from the internalisation of attributes they have observed. For example, they could say that a pentagon has five sides or a parallelogram is a four sided figure which has both pairs of opposite sides and angles equal. Furthermore, the intraoperational or perceptual level is suggested because objects are studied in isolation so they do not as yet know that a rectangle is a special type of parallelogram. However, although the original second level can be seen in the overall intraoperational level of algebra, it seems that it may be subdivided into sublevels leading up to the recognition of properties of isolated geometric figures at a previous abstract level.

Later 1st level – visual

At this level geometric figures are recognised by their physical appearance rather than their properties or components. Pupils may recognise similar figures at this stage but since they just see them as similar, they are unable to explain why they are similar. This is because the first visual level is based on non-verbal thoughts and observations. In algebra the first visual level seems to coincide with Piaget's intraoperational level or perceptual level and forms the lowest part of any section of the learning spiral.

(iii) Van Hiele's first level of thought in algebra

Van Hiele (1986:52) believed that his theory of levels of thought had application in algebra too. Land (1990:33) observed that "The van Hiele model seems to provide such a blueprint for the teaching and learning of algebra as well as geometry."

Van Hiele (1989) wrote a letter to Land (Land 1990), indicating that he did not believe that algebra had the same visual level as geometry. He remarked:

> *What you call visual is quite another thing than the visual level of geometry. What you call visual cannot be called like that in the psychological sense. Algebra has no visual level.*

However, he did conclude that there was a basic level in algebra as substantiated by the following quotation from a letter he wrote to Land at a later stage:

> *Before your first column you add one in which the descriptive level has not been attained* (Land 1990:131).

At this level objects can be understood by merely being observed.

Algebra need not be taught as a rigid frozen beautiful system but could be taught by taking the various learning levels into account. Van Hiele also observed how studying the similar properties of various number systems or other objects in isolation at the first stage could lead to a comparison at the second level and ultimately the group concept at level three of the introduction to group theory. He states:

> *A great conformity exists in the set of integers for addition, the set of rational numbers (without 0) (Q/{0}) for multiplication, and the set of vectors for addition. This conformity makes the 'group' concept useful. Therefore, in the set of integers* $(\mathbb{Z})$*, for the operation 'addition', the four following properties apply:*
>
> 1. *Addition is possible for all elements of* $\mathbb{Z}$*.*
> 2. *There exists a neutral element for addition.*
> 3. *For every element of* $\mathbb{Z}$ *there exists an inverse element for addition.*
> 4. *Addition in* $\mathbb{Z}$ *is associative.*
>
> *One can also notice that a similar structure exists in the set* $\mathbb{Q}$ *for the operation 'multiplication'. A similar structure also exists in the set of vectors for the operation 'addition'.* (Land 1990:34,35).

Studying these various sets of objects in isolation would comprise of the first level of development in the establishment of the concept of a group. Once the structure of a group has been established at level three, it would form part of level one when studying various types of groups in order to formulate further results. Depending on the way in which the process of learning groups is approached, when looking at the larger picture, establishing the concept of a group could be regarded as comprising of sublevels of the intra level of the study of groups as a whole or various rounds lower down on the spiral of abstract algebra. This is because although the establishment of group axioms forms part of the overall transoperational or abstract level of algebra, many further definitions, theorems and applications flow out of these as the abstract algebra spiral is traversed.

(iv) Sublevels of the van Hiele levels

As has been mentioned in 2.5.1, van Hiele believed that all the levels except for the base level possess a structure. Here van Hiele's visual level of his later model or the first and part of the second stage of his original level are considered as constituting the first level. At the base or intra-intra level it would seem that learners perceive the structure while at the intra – inter – level (or concepts level lower down on the spiral) they would be able to continue it and explain it in detail. This would lead to the intra- transoperational level or an abstract level at which the learner would have the ability to envisage a greater and more encompassing structure. Once a pupil reaches this point, he or she should then be ready to move up to the inter – intra level or to a new round of the spiral.

Commenting on Piaget's early work with children, van Hiele (1959:12) recognised that "… The children's reactions as well as many other situations discussed show fairly obviously the existence of three stages." These stages include: a first stage when they clearly show that they have no understanding of the problem at all; a second stage where they do realise what it is all about but allow themselves to be misled by external circumstances; a third stage where they operate with complete certainty within the system provided. Van Hiele remarks that he would most strongly wish to combat the idea that "… that these three stages are bound to a particular period in the child's development" (van Hiele 1959:1). Piaget's intra, inter and transoperational levels indicate that he did come to recognise three stages of development in learning in general Van Hiele applauds Piaget's contribution to sensorimotor development but emphasises that although sensorimotor development is often encountered by very young children, it is not exclusive to children and its omission may be rectified at a later age. He remarks that "… The stages and periods described by Piaget are not essentially connected with a particular age, but are characteristic for very many learning processes irrespective of the age at which they take place" (van Hiele 1959:14).

Van Hiele too recognises three specific sublevels in the learning process. He mentions three stages or sublevels in the development from one level to the subsequent one as follows:

> *a first stage, during which symbols of the field in question must be developed; a second stage, during which the properties and relationships must be explained, whereby the person becomes aware of the existence of the network of relations; and*

a third stage, that at which the person learns to orientate himself in that network of relations with the help of the symbols (van Hiele 1959:15).

These stages could be likened to the three levels of one part of the spiral.

There is a notable correspondence between Piaget and van Hiele's sublevels. At the first stage a learner is unacquainted with symbols and so reacts as though he has no knowledge of the topic. At the second sublevel, although the student might be acquainted with all or most of the relations, he does not have any possibility of checking his answers because he lacks orientation in the relevant field of symbols. However, at the third level he is able to answer the questions put to him with confidence because he has mastered the relevant network of relations.

The correspondence between Piaget's intra, inter and transoperational sublevels and van Hiele's three stages of advancement from one learning level to another seem to indicate that these sublevels should be given serious attention in the learning process. Van Hiele stresses the point that these stages can occur at any time in life. He also hints of not only sublevels but sublevels of sublevels and so on in the comment "… the three stages can occur several times within a learning process, which contains one object" (van Hiele 1959:16). Once again this suggests the idea of a spiral of learning growing and developing in various directions.

(v) Some opinions expressed regarding the van Hiele's levels

Freudenthal was influenced by the van Hieles and their work regarding learning levels and their significance. He conceded that the van Hieles deserved all the merit for the levels of development but observed that his own idea of levels differed from theirs because it was "… relative rather than absolute" (Streefland 1991:37). The spiral idea also gives an indication of many different sublevels or projections of a spiral rather than one distinct set of levels. Freudenthal, however, did agree with van Hiele that the learning process is structured by levels and that the activity of the lower level becomes the object of analysis of the higher level. He also agreed with van Hiele that there are jumps or discontinuities in the learning curve.

John Pegg (1985) mentioned that the van Hiele's have been criticised for the discontinuous nature of their levels, their seemingly oversimplified one-dimensional nature and the fact that they cannot be used to explain diversities in student behaviour. Pegg and Currie (1998:37) refer to the SOLO taxonomy and describes the difference between the theories in the following manner.

The van Hiele levels are a series of signposts of cognitive growth reached through a teaching / learning process as opposed to some biological maturation. SOLO,

> *however, is particularly applicable to judging the quality of instructional dependant tasks.*

SOLO means the "Structure of the Observed Learning Outcome". It may be seen as a shift from Piaget and van Hiele's ideas because it describes responses rather than people and evaluates the quality of tasks following instruction. It provides a means of measuring a student's attainment at a particular time and a particular place. Unlike van Hiele's theory, it provides a way of drawing conclusions as far as the student's overall level of thought is concerned. Pegg and Currie (1998) have made use of SOLO in order to provide a broader description of levels in the van Hiele theory.

Pegg's description of van Hiele's first two of the original five levels of thought (which he terms levels 1 and 2) strongly suggests that they both belong to Piaget's intraoperational level.

> *Level 1:* *Figures are identified according to their overall appearance.*
>
> *Level 2:* *Figures are identified in terms of properties which are seen as independent of one another* (Pegg & Currie 1998:335).

Here the first two van Hiele levels (termed levels 0 and 1 in the rest of the text) clearly represent a stage when objects are studied as isolated forms. However, the interoperational or conceptual level is also alluded to in the mention of properties of the objects of study. Pegg and Currie (1998:338) remark that at the onset of the original second van Hiele level

> *… the characteristic of thinking in terms of independent properties can be interpreted within SOLO as an aspect of a broader thinking category in which concepts are addressed in isolation. These concepts need to have an obvious visual basis and individual closures (answers) must have a strong real-world referent for students.*

Although this would seem to be the case in the introduction of topics low down on the spiral where the descriptive historical development of mathematics is apparent, eventually a strong real world reference no longer becomes possible. However, this does not mean that there is no first level higher up the spiral where mathematics is used to create more mathematics in the constructive historical development.

At the van Hiele's later visual level or at the beginning of the original second level or Level 1 of van Hiele, Pegg remarks how the ability to work with pronumerals gives students the chance to stop needing to calculate each step of a problem which is characteristic of Level 1. It also prepares them for relationships between different concepts at van Hiele's original third level or the onset of his latter descriptive level or the interoperational level. Pegg claims that although van Hiele did not pursue the broadening of his original second level of thought, support for these ideas can be found in van Hiele's own writing when he states "At the second level, calculation deals with relations between concrete numbers: $4 \times 3 = 12,\ 6 + 8 = 14$" and that "… at the second level there is a focus on actual numbers, single concepts are involved, and working memory demand is relatively light" (Pegg & Currie 1998:3–339).

Freudenthal claims that much of mathematics involves organising subject matter and this is something which van Hiele's model promotes. Van Hiele himself believes that his theory of levels has application in algebra and other topics as well. According to Han (1986:19):

> *A strength of the van Hiele theory is the detailed description of levels which can be directly employed in designing curriculum and instruction, while Piaget's theory is too general and Gagne's is too subjective to provide a specific guide for the design of curriculum and instruction.*

(vi) Conclusion

The van Hiele theory possesses significant properties: A student should pass through all the levels in a fixed sequence; whatever was the object of perception at level $(n-1)$ becomes the object of thought at level n; knowledge acquired at level $(n-1)$ needs to be organised or reinterpreted at level n in order to perceive a new structure with its own linguistic symbols; two people reasoning at different levels cannot understand each other.

Van Hiele, Freudenthal and Piaget all seem to have similar ideas regarding the sequential nature of the learning process and the existence of levels and sublevels of learning. Hoffer (1983) observes how the van Hiele model gives "… prescription for instruction not only in geometry but in most structural disciplines" and "… provides us with a blueprint for future work. The main task is to interpret each task that we want students to learn in terms of the model" (Land 1990:36). In this study an attempt is made to apply the spiral theory to facilitate the teaching of algebra.

2.4.4 Judith Land

(i) Introduction

In 1990 Judith Land completed her doctoral dissertation on the "Appropriateness of the van Hiele model for Describing Students' Cognitive Processes on Algebra Tasks as typified by College Students' Learning of Functions" at Boston University. She investigated the feasibility of applying the van Hiele model to the teaching of algebra and chose the algebraic topic of functions to conduct her research.

(ii) The first algebraic level of Land's theory

As has already been mentioned, van Hiele did not believe that algebra had the same visual level as geometry but did acknowledge a basic level and described it by saying that "…everything at the basic level can be learned by simply pointing to and giving the name". He believed that this level was of sufficient importance to be included in the development of algebraic concepts and added:

> *…. Before your first column you add one in which the descriptive level has not been attained. You can try to answer the question: What problems can children be given before they have attained the descriptive level, which can be understood by pointing to and which may lead to graphs* (Land 1990:131–132).

The first of three levels called the intraoperational or perceptual level seems to incorporate both Level 0 and some of Level 1 of Land's theory. These are called the pre-descriptive and descriptive levels and are described below:

1st level – Level 0 – pre-descriptive level

As has already been mentioned, this is not a true descriptive level but one which van Hiele believed should definitely be included in algebraic topics. In Land's work on functions, the base elements were exponential and logarithmic functions, which formed the objects of the study. An example of a pre-descriptive level actively employed by Land would be the graphing of exponential functions such as $y = 2^x$ by substitution as this would involve a skill brought to the topic by students. This shows that pupils have already climbed up the spiral of functions as an assumption is made that amongst other things they already know how to sketch graphs and operate with exponents.

2nd level – Level 1 – descriptive level

At this level the objects would be the properties of exponential and logarithmic functions which could be established inductively. This would form part of the overall intraoperational level of algebra where objects are studied in isolation. However, sublevels of the spiral would have had to have led to the establishment of properties in the first place. Land believed that at this stage elementary symbol manipulation and formulation of expressions as well as interpretation of symbols is possible. For example, learners can appreciate the effect of changing the values of a, b and c in $f(x) = a^{x+b} + c$, they could draw such graphs as

$f(x) = 3^x, f(x) = (\frac{1}{3})^x$ *or* $f(x) = 3^{-x}, f(x) = 3^x - 2$ *and* $f(x) = 3^{x-2}$ without plotting points.

At the pre-descriptive and descriptive levels various exponential graphs are studied in isolation but they are only compared with logarithmic graphs at the third level (Level 2) or the theoretical informal level. Since it is only at Level 2 that the relationship between the exponential and logarithmic functions can be explained, it would seem that Land's pre-descriptive and descriptive levels primarily form part of Piaget's intraoperational level.

(iii) Sublevels of Land's levels

Since the overall intraoperational level of Piaget, in the history of algebra, when applied to Land's theory, seems to incorporate to a large extent both Land's pre-descriptive and descriptive levels, sublevels evidently do exist in Land's theory of the development of algebraic concepts. At the pre-descriptive level, in order to plot graphs such as $f(x) = 3^x$, students first need to substitute to obtain values, plot and join points and then recognise the resulting curve. In order to be able to draw various exponential graphs without plotting them, several examples of a variety of types of functions need to be sketched in order to become familiar with the resulting curves. Consequently, all the steps involved in arriving at these conclusions would evidently involve various sublevels such as the intra, inter and transoperational sublevels of Piaget. The steps involved in learning to plot a graph such as $y = 3^x$ would also form part of one spiral where actually being able to sketch the graph would form the relevant abstract level.

(iv) Conclusion

As a result of her study, Land was convinced of the existence of levels in the study of algebraic functions. She described the level of language at the pre-descriptive level as being a "looks like" one while "… students thinking at level one spoke in a language expressive of properties of exponential and logarithmic functions but not of the relationships between these properties" (Land 1990:174). This certainly suggests that the pre-descriptive and to some extent the descriptive levels form part of what Piaget terms the intraoperational level. As has already been suggested, the fact that studying properties of exponential functions is the object of study gives an indication that pupils are already at a point sufficiently high up on the spiral to consider the mathematical objects of exponential functions as their source of reference at the visual, intraoperational or perceptual level.

2.4.5 Glenda Nixon

(i) Introduction

In 2000 Glenda Nixon conducted a study on the van Hiele levels in the teaching of the topic of sequences and series. A literature research was undertaken and a series of lessons on sequences and series was presented to six higher grade mathematics matriculation students (three boys and three girls) from a local secondary school.

In the lessons on sequences and series, the four previously mentioned learning levels of Land adapted from van Hiele, known as the pre-descriptive, descriptive, theoretical informal and theoretical formal levels were considered. Lessons were designed in such a way as to take students through these levels as they progressed through the various topics. Each new subsection of work was taught by following through these levels. The roles of visualisation, exploring patterns and generalisation were investigated in the advancement from one level to the next. At the end of the series of lessons, proof by mathematical induction was included as a fourth level or Level 3 activity. However, proof was taught after passing through all the lower levels that led up to it.

(ii) The initial level

As was the case in Land's study, the initial or intraoperational level includes the pre-descriptive level as well as part of the descriptive level.

The pre-descriptive level

In agreement with the ideas of Land (1990), the objects at level 0 were sequences and series and the characteristics included: being able to recognise a sequence or series in various situations; recognising a particular kind of sequence or series in different contexts; associating the appropriate name with a sequence, series or particular type of sequence or series. At this level students were provided with a variety of visual representations and patterns to encourage them to rise to the descriptive level. The pre-descriptive level was considered for each new topic taught.

The descriptive level

Once again, part of the descriptive level here could be associated with Piaget's overall intraoperational level of algebra. At the descriptive level the objects were the properties of sequences and series which may be established in an inductive manner. The characteristics included: recognising and being able to accurately state properties of sequences and series; discovering formulae as a result of experimentation; formulating expressions which involve symbols, performing simple manipulation of symbols; recognising that a change in formula indicates a change in the sequence without having to be provided with pictures or particular examples. At this stage each type of sequence or series was studied in isolation and so this largely represents the intraoperational stage or the perceptual level. However, several rounds of the spiral were evidently needed to reach this stage in the topics studied.

(iii) Sublevels

Here sublevels were considered and were integrated into the learning process. At each level visualisation, exploring patterns and generalisation were encouraged to promote progress from one level to the next. Visualisation could be compared with the intra sublevel or the level of percepts where students are encouraged to recognise a new concept in various contexts. Exploring patterns could be associated with the inter- sublevel or the level of concepts where students are encouraged to make comparisons and recognise inter-relationships between objects. Generalisation could be regarded as forming part of the transoperational sublevel or the level of abstractions at which structures are established. The fact that knowledge acquired at previous stages did often form part of the initial level of a topic strongly suggests the idea of a spiral of learning.

(iv) Visualisation

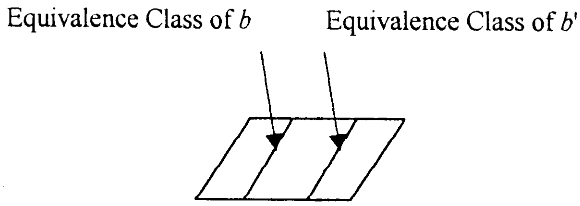
Visual cognition forms a very important aspect of life. In fact, “Recognising and reasoning about the visual environment is something that people do extraordinarily well” (Pinker 1985:1). However, de Villiers (2005) in his correspondence with the writer observed that Walter Whiteley, a geometer, would disagree as he claims most people are visually “naïve” and easily “misled”. A further problem that can arise is that “The wrong application of audio-visual teaching aids usually adds up to using them for too long” (Duminy & Sőhnge 1990;204) so that as a result “the child’s thinking stays in the observational level for too long” (Duminy & Sőhnge 1990:202).

In the 1880’s Galton (1822-1911) is said to have drawn attention to the fact that some people have “strong visual imagery” while others “thought mainly in words” (Skemp 1971:88). Both visual and verbal imagery is used in mathematics.

Skemp claims that algebraic symbols, which have great clarity and power, have more to do with verbal symbols than diagrams or geometric diagrams. He describes visual symbols as being "harder to communicate, more individual" and verbal ones as being "easier to communicate, more collective" (Skemp 1971:92).

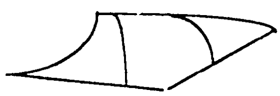
Visual understanding involves both the representation of information regarding the visual world before a person "…remembering or reasoning about shapes that are not currently before us but must be retrieved from memory or constructed from a description" (Pinker 1985:3). Skemp (1971) points out that at the intuitive level, as a result of our senses, people are aware of data from the external environment but not of the mental processes involved in the activity. However, "…At the reflective level, the intervening mental activities become the object of introspective awareness" (Gatgatsis & Patronis 1990:30). This seems to reflect what happens at the perceptual level, where people are aware of external data and at the conceptual level, where they begin to reflect on and process them.

It has been found that "… in many cases, mathematicians create *"mental images"*, which are present in their work and help them, and which may become the object of conscious reflection" (Gatgatsis & Patronis 1990:31). Furthermore, it has been found that some mathematicians use these images consciously whilst others do not. It may be difficult to communicate these images and some mathematicians try to do so to their university students. For example, the following diagram was presented to students to show that if b and b' are in different equivalence classes then $f(b)$ and $f(b')$ are different.



(Nardi 2000:175)

However, a student asked why the lines are all straight so that the lecturer re-drew the picture as follows.



(Nardi 2000:176)

It seems that mental imagery is often personal and not always easy to communicate. There is also a danger of students concentrating on irrelevant attributes of the images used as illustrated by the previous example. Gatgatsis and Patronis (1990:32;33) identify three levels of awareness in the development of concepts in a mathematical problem. At the beginning the knowledge is fragmentary. This is followed by partial reorganisation of knowledge at the intermediate level. At the third level the solution can be seen globally through elaborate mental images.

Visual imagery can be used very successfully to reveal algebraic concepts. For example, $\{0; 1; 2; 3\}$ is a group under $\oplus \bmod 4$ and $\{1; 2; 4; 3\}$ is a group under $\otimes \bmod 5$. Their tables are depicted below. The third table with entries rewritten in powers of 2 clearly shows that the two groups are isomorphic to each other.

$\oplus$ mod 4	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\otimes$ mod 5	1	2	4	3
1	1	2	4	3
2	2	4	3	1
4	4	3	1	2
3	3	1	2	4

$\otimes$ mod 5	2^0	2^1	2^2	2^3
2^0	2^0	2^1	2^2	2^3
2^1	2^1	2^2	2^3	2^0
2^2	2^2	2^3	2^0	2^1
2^3	2^3	2^0	2^1	2^2

(Skemp 1971:82)

Visualisation is clearly extremely important in mathematics because it promotes the establishment of concepts. Fuys, Geddes and Tischler have described the pre-descriptive level as being "… analogous to a ground floor of a building – it represents the type of thinking that all students will initially bring to a new subject" (Land 1990:132). In the lessons on sequences and series, at the initial level the sort of examples provided were simple visual ones which could be understood by being pointed out as mentioned by van Hiele (1989) in his correspondence with Land (1990).

In Nixon's study (2002), the students responded positively to the use of visual examples of sequences and series and it seemed to help them to pass from some particular perceptual level or pre-descriptive level to some conceptual level or the descriptive level. This could be more easily described as advancing from the level of percepts to concepts low down on the spiral of learning sequences and series. They were given encouragement to think about relevant aspects of visual illustrations. One student remarked that "… making a mind picture is only making the problem easier and I don't think it helped me to establish a rule" (Nixon 2002:107). This seems to suggest that visual illustrations help promote progress from the intra or perceptual to the inter or conceptual sublevels but not straight from the perceptual sublevel to the abstract sublevel.

Students were extremely enthusiastic about visual approaches leading to the establishment of the summation formulae for arithmetic and geometric series as well as the concept of mathematical induction. They generally seemed to find that visual illustrations provided them a good base to work from and gave them understanding and insight when they reached the interoperational or conceptual sublevel.

(v) Exploring patterns

In mathematics the word "pattern" is usually used in connection with "… a search for order, so regularity is more likely than not" (Orton 1999:vii). Biggs and Shaw (1985:1) regard mathematics as being a "… a search for patterns and relationships". This suggests activities associated with the conceptual or interoperational level. Sawyer (1955:12) shows the importance he attaches to patterns in mathematics in his comment "mathematics is the classification and study of all possible patterns" (Orton 1999:vii).

There are many wonderful patterns to be found in mathematics and studying these at the inter sublevel or conceptual level of the learning spiral can lead to the establishment of rules at the trans sublevel or abstract level of the overall perceptual level involving both the pre-descriptive and descriptive levels. Concrete or pictorial representations are often perceived to be more simple than symbolic ones. Patterns are relevant because they can be explored, extended, created and generalised.

After being provided with numerous patterns at a perceptual sublevel of the descriptive level, students in Nixon's study responded that this had helped them to see what was happening in a sequence, recognise the pattern and provided insight to establish a formula for the *nth* term of a sequence. In response to the question of whether the study or patterns helped them to understand concepts learnt, one student replied "Yes, they gave us a good basic understanding" (Nixon 2002:124). After several lessons they claimed that they had become accustomed to looking for patterns and it had contributed to their understanding of relevant concepts and the establishment of rules. Thus their perception was that patterns had encouraged them to rise to the abstract level.

(vi) Generalisations

Generalisation may be regarded as going beyond what is explicitly given. After the establishment of patterns, the goal is to move beyond to a stage of generalising the results of findings. Orton (1999) believes that it is possible to introduce concepts in algebra through generalisation of results from number patterns. However, sufficient attention needs to be given to both visual and algebraic or verbal systems in parallel in order to give learners the opportunity of moving from one system to another. This could probably result from considering two different parts of the spiral simultaneously.

According to Skemp (1971:215) “Algebra is concerned with statements involving variables of any kind”. Thus it is necessary to know what kind of variable is being spoken about. The first kind of algebra developed and the algebra “still probably the most widely used” is “the algebra of numerical variables (Skemp 1971:215). Many children find the change from arithmetic to algebra to be very difficult and the fact that some teachers say let “a” stand for apples and “b” stand for bananas in order to show that unlike terms may not be added can really confuse pupils. Herscovivcs (Steffe et al 1996:358) claims that the mastery of mathematical notation “… can be viewed as another level of understanding”. Symbolism is very important in elementary mathematics because it helps to detach a concept from its concrete embodiments” (Herscovics in Steffe et al 1996:358).

Some of the functions of symbols which are listed by Skemp (1971:64) include communicating; recording of knowledge; communication of new concepts; the making of multiple classification straightforward; explanations; making reflective activity possible; helping to reveal structure; causing routine manipulations to become automatic, recovering and understanding information; creating mental activity.

After being encouraged to generalise and rise from Level 0 to Level 1 as well as Level 1 to Level 2 in the topic of sequences and series, certain students gave the following responses to questions posed to them:

Do you think it is useful to find the general term of an arithmetic sequence? Explain.

C: Yes because it saves you time when having to work out a large value.

J: Yes so that we can easily work out the value of for example the 100th term.

K: Yes, helps you to find any other answer you need.

L: Yes, then it is possible to predict any term in the sequence.

N: Yes, it allows calculations to be done for any term.

T: Yes, you can now determine the sequence by using a calculation instead of having to count.

Is it necessary to establish rules in mathematics? Explain.

C: *Yes, it makes it easier to work out problems.*

J: *Yes! It simplifies future working out.*

K: *Yes, makes it easier to work out answers.*

L: *Yes, makes life easier.*

N: *Yes, it makes life easier so that we won't have to waste time.*

T: *Yes, we can deduce formulae and shortcuts to understanding maths.*

(Nixon 2002:128).

Students were asked numerous questions related to the effect of stressing generalisation at the pre-descriptive and descriptive levels. Their responses gave an indication of the positive effects they had perceived of promoting generalisation in order to move from the inter sublevel to the trans sublevel. It would appear that they felt ready to rise up from either the predescriptive to the descriptive level or the descriptive to the theoretical informal level. This suggests how, even low down on the spiral of learning sequences and series, generalisation plays an important part. This gives an indication of how percepts, concepts and abstractions are all relevant right from the initial stages of the learning spiral.

(vii) Conclusion

The study of teaching sequences and series seems to reveal not only the importance of levels of learning, but also the existence of sublevels at each level as well. Since sublevels too can be further broken down, it would be easier to describe the growth of learning in terms of a continuously expanding spiral, where each part contains the three levels of percepts, concepts and abstractions. The provision of activities of visualisation, exploring patterns and generalisation seemed to be successful in helping students to pass through the intra, inter and trans sublevels of both the pre-descriptive and descriptive levels or the level of percepts, concepts and abstractions low down on the scale. It seemed to make them feel confident that they were ready to move on to the subsequent level or part of the spiral. This does not mean that they have to follow this path. After the experience of travelling through several rounds of the spiral, they can become capable of establishing their own images from concept definitions.

2.4.6 Concept understanding scheme

(i) Introduction

Vinner in Tall (1991:68), referring to his original terms of concept image and concept definition described in his book written with Tall in 1981, remarks

> *The concept image is something non-verbal associated in our mind with the concept name. It can be a visual representation of the concept in case the concept has visual representations; it also can be a collection of impressions or experiences. The visual representations, the mental pictures, the impressions and the experiences associated with the concept name can be translated into verbal forms. But it is important to remember that these verbal forms were not the first thing evoked in our memory. They came into being only at a later stage.*

In this study, the concept image is considered to span the first two levels before the concept definition is reached at the third level. For first the percepts at the initial level provide stimuli involving visualisation or various types of experiences. Then at the second level these percepts are reflected upon and a concept image is formed. Since a concept image only comes about after a variety of experiences have been undergone and some mental analysis has taken place to form a mental picture or concept image, this is regarded as the conceptual level. Once this concept image can be translated into a linguistic form or definition, the third level or level of abstraction can finally be considered to have been reached.

Vinner, Tall and Dreyfus developed a concept understanding scheme described in (Moore 1998) in which a distinction is drawn between what they describe as the concept image, concept definition and the concept usage. The concept image is the cognitive structure in an individual's mind associated with the concept. They regard this as being different from the definition of the concept or concept definition and the concept usage, which refers to the way in which the concept is used in examples and proofs. The concept image relates to the intraoperational level or level of percepts. However, concept usage can form part of the level of percepts where concepts defined at a previous level are used to establish and define new concepts in the progress up the spiral. Concept image also forms part of level 2 which is called the level of concepts here as this is a level at which concepts are further established until they are defined at the level of abstraction. At the conceptual level, concepts need to become disassociated from the original image so that a general concept can be formulated in order to lead to concept definition and proof at the next stage.

(ii) Concept image and usage

As has been mentioned in the introduction, the concept image is the cognitive structure connected with a concept that exists in a person's mind. It relates to the set of all mental pictures that a person associates with the concept as well as all properties that characterise them. The concept image is derived from experiences that an individual has had with the concept. These include various types of examples or illustrations, diagrams, graphs, symbols or other types of encounters which might have taken place with the concept.

Moore (1994:255) described a professor teaching the equivalence class concept to a group of students. He defined the concept using set notation, described it in words without using set notation and also used a family metaphor to illustrate the concept. The authors of the textbook in question gave the following suggestion involving thinking of an equivalence class as a box: "The set $[x]$ is called the *equivalence class* of x (but "$box - x$" is shorter and suggests the right way to think about $[x]$; it is a set of all, a box so to speak filled with x and all its relatives." When the concept that the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 3$ maps $\mathbb{R}$ onto $\mathbb{R}$ was taught, the students were provided with the following concept image: "You want to show that f maps $\mathbb{R}$ onto $\mathbb{R}$. What does this mean? Informally it means that everything in $\mathbb{R}$ gets hit." (Moore 1994:255). This provided students with a very appropriate image of the concept of onto.

Concept image is very important in establishing mathematical intuition. It helps to prevent concepts from being too abstract for students. Encouraging the formation of concept images helps learners both to find or create mental pictures of concepts and attach meaning to them. An informal understanding of a concept at level 2 is a necessary prerequisite for understanding the written form of a definition as well as proofs at level 3.

Anna Sfard (1991:13) observed that historically "… the development of the notion of number was a cyclic process, in which approximately the same sequence of events could be observed time and again". The first stage she termed the "preconceptual stage, at which mathematicians were getting used to certain operations on the already known numbers or, as in the case of counting on concrete objects". This supports the idea that concept usage forms part of the first or perceptual level of learning. She regards the preconceptual stages as the first stage followed by a second stage involving a predominantly operational approach leading to the structural phase at the third phase. Referring to the preconceptual stage, she continues to add that

> *… at this point, the routine manipulations were treated as they were: as processes, and nothing else (there was no need for new objects, since all the computations were restricted to those procedures which produce the previously accepted numbers)* (Sfard 1991:13 – 14).

(iii) The relevance of concept images for the formation of concepts, definitions and proof

Building up concept images at the perceptual and conceptual levels is extremely important but is inadequate for further use in mathematics unless they are followed by concept definitions in order to establish the language required to express mathematical ideas. Moore (1994) described a student who believed that her concept image served as an adequate definition and felt that the notation involved in learning an adequate definition was a burden rather than essential to the concept. This changed the perception of the concept and prevented her from adequately progressing in the topic. This does suggest that percepts or concept image and usage require an intermediate stage of forming general notions of concepts in order to successfully lead to concept definition and proof. The notion of definition needs to proceed from an intuitive and informal one to a formal one at the abstract level.

Although concept images established through examples and informal approaches are often helpful in describing a proof, they are inadequate when it comes to writing out a correct proof. Concept images alone cannot express mathematical ideas and do not supply individual steps for a proof. Furthermore, unlike concept definitions, concept images fail to reveal the logical structure of a proof. It seems that concepts need to be established after passing through the perceptual level and then defined at the abstract level. Then concept definitions and further concept usage can form part of the perceptual level leading to proof at the abstract level higher up the spiral.

(iv) Concept images and usage in abstract algebra

In order to promote an understanding of group theory, it seems that attention needs to be given to the formation of suitable concept images. Once the concept of a group has been established, the development of number systems as, for example, how $\mathbb{Z}$ is introduced after $\mathbb{N}$ to ensure the existence of an inverse element under addition with $\mathbb{Q}$ following subsequently to ensure the existence of an inverse element under multiplication, can be studied. This helps to lead to the establishment of the concept of a field.

Various methods could be used to create concept images in group theory and these are further explored in chapter 5. Martin Gardiner (1966) has drawn a comparison between the concept of a mathematical group and the grin of a Cheshire cat:

> *The body of the cat (algebra as traditionally thought) vanishes, leaving only an abstract grin. A grin implies something amusing. Perhaps we can make group theory less mysterious if we do not take it too seriously.* (Huetinck 1996:342).

A game called “It’s a SNAP” has been invented that could be used to introduce group theory. Its purpose is to give students the opportunity of playing with the concepts of group theory and developing an understanding of modern algebra. Linda Huetinck points out that in order to play the game, the two fundamental concepts of an operation and a nonnumeric set are required. These may be introduced by means of considering the transformations of an equilateral triangle with vertices A, B and C. Here reorientation is defined as the operation and the triangle vertices, known as configurations, form the nonnumeric set. Students could be encouraged to find the number of new orientations that can occur for the vertices (labelled A, B and C) for an equilateral triangle, stressing that the new orientation always begins with the initial configuration.
This idea is illustrated in the diagram below:

$$\begin{array}{ccc} A & C & B \\ {}_{C}\Delta_{B} & {}_{B}\Delta_{A} & {}_{A}\Delta_{C} \end{array}$$

$$\begin{array}{ccc} A & C & B \\ {}_{B}\Delta_{C} & {}_{A}\Delta_{B} & {}_{C}\Delta_{A} \end{array}$$

Possible orientations of an equilateral triangle (Huetinck 1996:342).

Once students have discussed and tested their ideas, they would become ready for the game of “It’s a SNAP” which will be described in Chapter 5. The game simply involves moving rubber bands around pegs on a board and recording results on a table. This game would provide an excellent means of establishing a concept image for groups, promotes class discussion and leads to the establishment of generalities. Recording results in group tables would serve to help students develop concept images of groups, abelian groups and other ideas as well. It provides a successful and quick way of visualising a group and the performance of group operations. There could be a danger that students come to see a group only as a table, so they need to go beyond this image and form a more general picture of a group as they reach the conceptual level.

Initially the main interest in groups, which begin in the second half of the nineteenth century, arose from their connection to the solutions of equations in algebra. However, evidence of groups and fields is abundant in several areas. For example, the concepts of symmetry and group theory were necessary in the development of the schematic families of quarks and leptons, the fundamental building blocks of matter. The opportunities of providing excellent concept images and showing the prevalence of group theory in nature is illustrated in the following comment by Durbin (1973:151): “The concept of group theory arises naturally in other places in mathematics, such as in geometry, and it is also useful in physics, chemistry and crystallography”. Thus the group concept could be introduced by not necessarily following its original historical path of development.

(v) Sublevels of the perceptual level involving concept images and usage

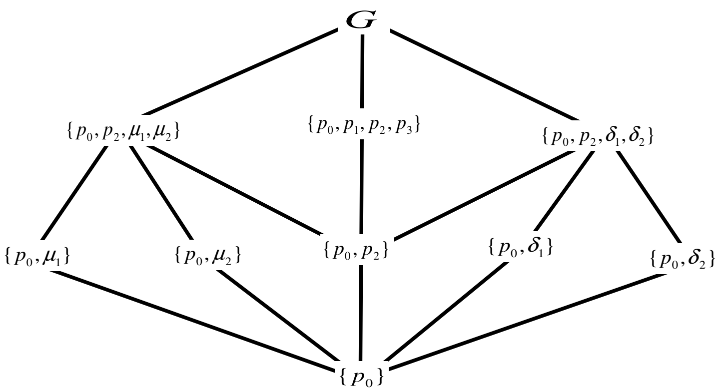
As new topics are studied in abstract algebra, previously learnt concepts may serve as concept images in order to learn new concepts. In this way the completed cycle of percepts, concepts and abstractions can lead to new percepts, concepts and abstractions further up the spiral.

Several examples could be used to illustrate the idea of concept usage leading to percepts at a higher level. For example, once students have established the concept definition of a group and used it in various contexts, they would begin to use group tables as concept images to find out how many group tables there are of a particular order. The tables could also be useful in establishing lattice diagrams showing the number of subgroups of a particular order that might exist for a particular group.

For example, from the group table of D_4 below, the lattice diagram may be derived as follows:

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_2	δ_1	μ_1	μ_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_1	δ_2	μ_2	μ_1
μ_1	μ_1	δ_1	μ_2	δ_2	ρ_0	ρ_2	ρ_1	ρ_3
μ_2	μ_2	δ_2	μ_1	δ_1	ρ_2	ρ_0	ρ_3	ρ_1
δ_1	δ_1	μ_2	δ_2	μ_1	ρ_3	ρ_1	ρ_0	ρ_2
δ_2	δ_2	μ_1	δ_1	μ_2	ρ_1	ρ_3	ρ_2	ρ_0

Group table for D_4. Fraleigh (1977:41).



Lattice diagram for D_4. Fraleigh (1977:42).

This could lead to further concepts, definitions, proofs and applications of subgroups of groups. In addition, studying groups under different operations could lead to the establishment of the concept of a field.

(vi) Conclusion

Concept images and usage are clearly necessary for the formation of concepts leading to definitions and further usage in proofs and applications. These could come from a variety of sources including examples, diagrams, graphs, games, verbal explanations or symbolic forms. As the student moves on to higher levels, the concept images become less concrete and are based on concepts acquired at previous levels. Moore (1994: 262) observes:

It seems that this reliance on concept images for understanding definitions and notation may diminish as the students move beyond this transition point in their

> *learning of mathematics and become more comfortable with standard notation, mathematical grammar and syntax, and the logical structure of proofs.*

This can be seen in the historical development of mathematics. For initially there was the “a posteriori” or descriptive development that involves real world experiences leading to mathematics. But then, in addition, the “a priori” or constructive development involving mathematics constructed from other mathematics emerged and still continues today.

2.5 Chapter Consolidation

From 1974 to 1979 research concerning “Concepts in Secondary Mathematics and Science” (CSMS) was conducted by the Social Science Research Council based at Chelsea College, University of London. At the time it was felt that although much research had been done concerning younger children about “the many experiments and “check-ups” of Piaget”, more attention needed to be paid to secondary school children. In this research, Matthews (1979) in the forward of Hart (1982) refers to a “hierarchy” or “concept tree” from which “authors and teachers could determine proper orders of topics and levels appropriate to the various children”. Hart reported that the results of the CSMS research had far reaching consequences for mathematics teaching at a secondary level. She remarked that: “The overwhelming impression obtained is that mathematics is a very difficult subject for most children” (Hart 1982:209). It seemed that a spiral approach to the curriculum was being followed as the same points had to continually be reiterated. However, the re-teaching is often treated as a revision exercise or time of “patching up” (Hart 1982:209) without approaching the topic anew and bringing in more concrete referents to illustrate the basis on which the mathematics is built. This would seem to suggest that the relevant levels at each round of the spiral were not being taken into account, thereby prohibiting further advancement up the spiral.

In the CSMS research conducted in the 1980’s, Küchemann found that in the algebra test, children of ages 13 to 15 were at what he termed Piaget’s original stage of concrete operations. He observed that “… the use of letters as objects totally conflicts with the eventual aim of using letters to represent numbers of objects” (Hart 1982:119). However, he did concede that it could in fact be that very type of conflict that was necessary to help students become aware of the need to reorganize their thinking and move to the next level. The algebra levels he refers to as the “the Piagetian sub-stages” listed in Hart (1982:117) are as follows:

Level 1 *Below late concrete*
Level 2 *Late-concrete*
Level 3 *Early-formal*
Level 4 *Late-formal* (Hart 1982:117)

In this context Below late concrete and Late-concrete could be regarded as both forming part of the perceptual level with Early-formal and Late-formal being connected to the conceptual and abstract levels respectively. However, the fact that each topic can be broken down into very many small parts, each of which could represent a round of the spiral, suggests that each of these levels is not absolute and could be continually re-visited within the study of a single topic.

Investigating the research of many people who have made in depth studies of levels in mathematics and the historical development of mathematics, it seems to be clear that levels of learning are useful in describing human cognition and that the first level of learning should not be overlooked as it forms an extremely important part of the learning process. It helps create the base or foundation of study and provides learners with intuition and insight regarding their topic of study. This is extremely relevant in the abstract algebra topics of groups and fields where students often find the need for concept images and usage to introduce and help them visualise what they are learning. This stage also forms an important part of the early development of algebra. The perceptual level needs to be continually revisited as the learning spiral is traversed.

Concepts may be introduced in various ways. For example, Alwyn Olivier observes how, contrary to popular belief, children who have not acquired the concept of a function can be introduced to it through the medium of a calculator. For by entering a number and finding that by then pressing the square root key, out pops the corresponding square root, they are indirectly being introduced to the concept of a function machine. Consequently at the perceptual level foundations are being laid for the concept of a function.

At each perceptual level of the spiral, students need to acquire an intuitive appreciation for concepts and be provided with examples, diagrams, pictures, explanations or illustrations that help them visualise or form a mental picture of concepts being introduced. This is certainly the way in which those topics evolved in history and the links between the historical development and the evolution of concepts in the minds of students seem to be extremely significant. At each of the three main learning levels, termed intraoperational, interoperational and transoperational by Piaget, sublevels seem to exist and have been recognised by various researchers in the field, including Piaget, Freudenthal and van Hiele. Using Piaget's terminology, these have been termed intra – intra, intra – inter and intra – transoperational, inter – intra, inter – inter and inter – transoperational as well as trans – intra, trans – inter and trans – transoperational. Depending on the object of study and from what point of view it is being observed, these sub-stages could be considered as being broken down even further into more subdivisions. For this reason the spiral illustration is utilised here when showing how percepts, concepts and abstractions can continue to grow in all directions. Nixon (2002) encouraged visualisation, exploring patterns and generalisation to promote advancement from one level of learning to another. This seems to correspond to passing through the intra, inter and transoperational sublevels or the perceptual, conceptual and abstract levels and is utilised in Chapter 5 in the establishment of a general theory of learning and teaching algebra and culminating in abstract algebra.

Group theory arises or has applications in many situations and fields of study in life. For example, it appears in computers, art, nature, physics and medicine. Abstract algebra has arisen from continued historical research in algebra and forms the transoperational level in the history of algebra. Its historical development may be described as being of both an "a posteriori" or descriptive and an "a priori" or constructive nature. Yet it has been found to have many unanticipated applications in various spheres of life. This seems to suggest that it is possible to find concept images or visual illustrations to introduce abstract algebra and show its relevance. For as, Durbin (1973:150) has commented: "Abstract mathematical objects are so much a part of our culture that we often are not aware when we are using them".