CHAPTER 1
Introduction and Overview

1.1 Background to the study

Piaget and Garcia (1989) observe that various stages involved in the construction of different forms of knowledge are sequential and that the same sequential order is evident in history. As a result they note that

> *It is not surprising that a piece of knowledge cannot be dissociated from its historical context and that, consequently, the history of a concept gives some indication as to its epistemic significance* (Piaget & Garcia 1989:7).

They believe that there are three stages in the development of algebra involving solving equations or the evolution of group theory. These correspond to three stages in the levels of learning of concepts and include: the intraoperational stage which involves isolated forms; the interoperational stages concerning correspondences and transformations amongst forms; the transoperational stage characterised by the evolution of structures of forms. In the intraoperational stage of history, equations were solved in isolated forms with each one having its own method of solution. During the second half of the eighteenth century there was a new development when mathematicians formulated problems of greater generality and the fundamental theorem of algebra characterised the interoperational stage of algebra. Transformations dominated algebra for a long time until the emergence of the first algebraic structure, the group, which heralded the transoperational stage in the theory of algebraic equations.

As it became clear that many sets of elements possess the same properties as numbers do, properties common to such classes of elements that do not define a specific domain but a structure common to many domains were studied. When Richard Dedekind (1831-1916) investigated this type of structure, he named it a field. Evariste Galois (1811-1832), who died very prematurely, had introduced the transoperational stage in the form of a group structure but remained at the interoperational stage regarding the idea of a field. Dedekind was thus the one who successfully identified and thematised the structure of algebraic fields.

Burn (1996) (in Nardi (2000:169)) points out that "… historically permutations and symmetry are the fundamental concepts of group theory". However, according to Burn in Nardi (2000:169), "… the way in which group theory has been presented in a decontextualised manner over the years has prevented learners from forming a meaningful construction of these concepts". Dubinsky and others (Dubinsky et al `1994:268) express the point of view that group theory poses a serious educational problem. This is because it makes it necessary for students to go beyond copying behaviour patterns, come to grips with abstract concepts, work with significant principles in mathematics and also learn to write proofs. The stages of thought in the development of the concepts of group theory and the study of fields seem to play a significant role in this regard.

Although many pupils are introduced to definitions and proofs in high school geometry, they often have difficulty in adapting to definitions, axiomatisation and proof in abstract algebra courses at a higher level. Very often students are merely presented with definitions without being made aware of their origin or relevance. Consequently, they struggle to evaluate their correctness when stating and making use of definitions and theorems.

In recent years there has been a decline in popularity in the teaching of Euclidean geometry to high school students. South Africa inherited its geometry from England at a time when it was adopting a very conservative approach and followed Euclid very closely. However, subsequent changes effected in Great Britain did not influence South Africa. Schuster (1973) in Land (1990:19) concluded that the goal of teaching the formal structure of geometry in the United States of America for most of the twentieth century had failed. Students need to develop a strong geometric intuition but when Euclidean geometry is poorly taught, not enough is done to help develop this type of intuition. It appears that even if Euclidean geometry is understood at a high school level, there is a lack of transfer of ideas to other branches of mathematics at a university level. Moore (1994) observes how students at this level are required to write proofs in mathematics with no general perspective of proof or methods of proof.

It seems that adopting a learning level approach to the acquisition of knowledge in abstract algebra and algebra in general would be most appropriate. Piaget and Garcia's three levels seem to resemble Hans Freudenthal's idea (1973:123) of working with the subject matter at the first level, becoming conscious of it as an organising principle and reflecting on it at the second level and then finally establishing a linguistic or symbolic pattern at the third level. In 1959 Pierre and Dina van Hiele also devised a theory of learning levels relevant to geometry. Initially it involved five stages but later on Pierre van Hiele (1986:47,51,53) reduced the number of levels to three. The original five levels were adapted and four of these were utilised by Judith Land in 1990 for the teaching of the algebraic concept of functions. In agreement with van Hiele's suggestion, she began with a pre-descriptive level or level 0 at which students were encouraged to recognise initial concepts in various contexts. This was followed by a descriptive level or level 1 where students were able to identify properties of concepts, an informal theoretical level or level 2 at which definitions were understood and a formal theoretical level or level 3 at which it became possible for proofs to be performed. It was felt that the fifth level of van Hiele did not apply to students at high school or early university level. In 2000 Glenda Nixon (2002) also adapted the van Hiele levels to be applied to the teaching of sequences and series to matriculation students. The four algebraic levels proposed by Land were utilised for this purpose. The implications of these four levels as well as the original fifth level of van Hiele appear to be very relevant for the study of group theory and fields as well. They could be appropriate even at the third year university level when students are often being introduced to groups and fields for the first time.

The idea of conceptual scaffolding in introductory algebra has been used by people such as Alwyn Olivier of the University of Stellenbosch. In his correspondence with the writer, Professor Michael de Villiers mentioned how it involves structuring learning phases or levels by ascertaining students' intuitive understanding in order to establish a foundation for reaching higher more abstract levels. Vinner and others in Moore (1994:269) differentiate between what they refer to as the concept image, concept definition and concept usage. Their notion of concept image relates to the cognitive structure associated with the concept that exists in a person's mind. Their third aspect of concept understanding which follows concept definition, called concept usage, refers to the way in which an individual operates with the concept in doing examples and writing out proofs. Moore (1994:269) claims that students in abstract algebra have difficulty in finding or creating mental pictures of concepts and unless they have these it is difficult for them to learn formal definitions. This causes further problems in working with concepts and proving results in abstract algebra because there is a lack of intuitive understanding. Furthermore, concept images need to be followed by concept definitions because it is the latter rather than the former that suggests steps and possible justification for steps in systematising a proof. These concept levels seem to be closely linked to the four aforementioned levels of algebraic thought. Encouraging the development of concepts in abstract algebra could lead to the formation of concept images, concept definitions and concept usage. However, the narrowing down of concept image to the first level, concept definition to the second and concept usage to the third does seem to be somewhat restrictive. The development of concept image and definition seem to span the first three levels of development whilst concept usage seems to form part of any subsequent phase of development once new concepts are utilised to establish further concepts, applications, axioms or theorems at a higher level. Concept image, definition and usage will be considered in this light in the present study and possible approaches to be followed to encourage passage through the appropriate levels will be taken into consideration.

Taking levels into consideration could promote the development of intuitive knowledge in the study of concepts in an introductory course of abstract algebra. Sawyer (1959:2) advocates that instead of initially being presented with all axioms, students should be provided with the opportunity of discovering mathematics and he sees it as a subject that grows and develops in the light of new knowledge. In fact, all of the above-mentioned theories regarding levels of thought seem to have merit and share similar notions. It would seem to be worthwhile to combine the various ideas to formulate a suitable theory of levels that would be appropriate for the learning of algebra at school as well as university level. This theory would take into account various sublevels at each learning level so that a comprehensive pattern of thinking levels could be established. Here consideration will be given to the idea of a theory like Piaget's involving intra, inter and transoperational levels and sublevels. These levels seem to keep repeating themselves as each new topic is learnt. Ultimately experienced mathematics students could pass through these levels by forming their own concept images from stated definitions and axioms. However, it seems that it generally takes time and plenty of background to reach this point.

Visualisation appears to play a significant role at each sublevel. Levels also seem to depend on the way in which a topic is observed. In algebra as a whole, group theory may appear to form part of the transoperational level but when studying group theory, each new topic would seem to require passing through these levels as well. Group theory and the theory of fields as a whole or any other algebraic topic could also be subdivided into appropriate levels and sublevels. Students seem to be continually passing through these levels and sublevels in the learning process.

Consequently, the idea of a spiral of learning will be considered here. This helps to illustrate the way in which mathematics has developed over the years too. Although the spiral idea of learning is not new, the consideration of three levels at each round of the spiral will form a central theme in this study. Thompson (1996:1) observes how the “spiral curriculum” is “one of the important ideas attributed to the psychologist Jerome Bruner”. The idea of this model is that important concepts are “visited and revisited at different levels of sophistication during a child’s schooling.” This is not the same form of a spiral that is mentioned here. For in this study the same concept will not be considered to be continually revisited but established concepts will lead to new concepts at higher levels of a topic. Janet N Wineland and Larry Stephens (1995:227) conducted tests which revealed that below average grade 8 and 9 learners using the spiral testing scheme scored significantly higher marks than those not using it. Alfinio Flores (1993:152) cautions that “a curriculum intended to be spiral can end as a curriculum that repeats itself if each visit is not at a deeper level”. But he points out how, for example, Pythagoras’ theorem can be revisited at different levels in such a way that each time students do obtain a deeper insight. In this way he suggests that “The teaching of mathematics can be done in a spiral curriculum if a guide such as the van Hiele levels of development in geometry is kept in view” (Flores 1993:152).

Two broad historical developments of mathematics could be described as: “a posteriori” or descriptive which involves the idea of real world experiences leading to mathematics; “a priori” or constructive which relates to mathematics constructed from other mathematics which later may be found to have application in the real world. The former method describes the way in which ancient mathematics and some modern mathematics has developed while the latter refers to a method which began to emerge in the nineteenth century and is still continuing today. Just as the “a priori” method developed in modern times, so the constructive method of approaching mathematics becomes relevant to the experienced mathematician. Parallelisms arising from the way in which mathematics has grown historically, the way in which concepts form in the minds of students and learning level theories are investigated in this study and a theory of a spiral of learning levels is developed.

1.2 Problem Statement

Could the consideration of the parallelism between the historical development of mathematics and the various established levels and sublevels of learning mathematics lead to a meaningful theory of concept development in the study of algebra at secondary level and abstract algebra at tertiary level?

1.3 Aims and Objectives

The problem statement will be investigated utilising the aims and objectives mentioned below. It is important to note that the study will be largely based on the learning theory established by Piaget and Garcia. Many of the other people whose theories are mentioned here have investigated the structure originally established by Piaget. The results seem to indicate such a unified view about teaching and learning that these two aspects are not separated in this context.

1.3.1 Aims

The aims include highlighting the stages in the historical development of abstract algebra and studying the parallelism between the historical progress, stages of teaching and development of concepts in abstract algebra. Both the "a posteriori" and the "a priori" developments of mathematics are taken into account and attention is paid to the development of axiomatisation and proof in abstract algebra. The thinking levels described by scholars such as Piaget, Freudenthal, van Hiele, Land, Nixon and Vinner are investigated as well as their application to the learning of algebra at various levels. The sublevels involved in the learning of abstract algebra are considered and a general integrated pattern for learning and teaching algebra is formulated.

1.3.2 Objectives

A learning programme is established to encourage thinking as learning involves not only learning what is in the context but developing skills of thinking. Learning and teaching are so closely linked that they are not separated in this study. The objectives for each individual chapter are listed hereafter.

Chapter 1 Introduction and Overview

The stages in the historical development of mathematics and abstract algebra in particular are investigated. Reference is made to the “a posteriori” or descriptive as well as the “a priori” or constructive methods of developing mathematics. Axiomatisation and proof in the emergence of abstract algebra are considered.

The thinking levels involved in the learning theories of people such as Piaget, Freudenthal, van Hiele, Land, Nixon and Vinner are introduced. The notion of parallelism between the historical development of abstract algebra and related learning theories is conveyed. Attention is paid to sublevels arising at the various thinking levels involved in the learning of abstract algebra.

Spirals of learning mathematics with particular reference to abstract algebra are considered. The idea of representations of sensations or percepts, concepts and linguistic formulation or abstractions in learning is introduced.

The problem statement, aims, objectives as well as the content for each chapter, the analysis of the problem and research design are all presented.

Chapter 2 The perceptual level or initial learning level and sublevels in the development of abstract algebra

The first stage of mathematics in the development of abstract algebra is studied. The initial stages in the learning levels of Piaget, Freudenthal, van Hiele, Land, Nixon and Vinner are mentioned. In addition, the sublevels involved in the first learning level of the emergence of abstract algebra are investigated. A spiral theory of initial learning levels and sublevels in abstract algebra as well as what will be termed the three levels of percepts, concepts and abstractions associated with the learning spiral are considered.

Chapter 3 The conceptual level or the intermediate learning level and sublevels in the development of abstract algebra

The intermediate stage of mathematics in the historical development of abstract algebra is studied and the intermediate stages in the learning levels of Piaget, Freudenthal, van Hiele, Land, Nixon and Vinner are considered. The sublevels involved in the intermediate learning level of the emergence of abstract algebra are investigated. A spiral theory of the intermediate levels involved in abstract algebra is presented.

Chapter 4 The abstract level or the third learning level in the development of abstract algebra

The third stage of mathematics in the development of abstract algebra is studied and the advanced stages in the learning levels of Piaget, Freudenthal, van Hiele, Land, Nixon and Vinner are considered. The development of axiomatisation and construction of proofs in abstract algebra is included here. In addition, the sublevels involved in the advanced level of the emergence of abstract algebra are investigated. A spiral theory of the advanced learning levels involved in abstract algebra is presented.

Chapter 5 A spiral theory of learning algebra

The various learning theories are combined into a single spiral theory of thinking levels and sublevels in abstract algebra. The different levels and sublevels in the history of abstract algebra are observed. Both the “a posterion” or descriptive as well as the “a priori” or constructive approaches to mathematics are considered. In addition, the various levels and sublevels involved in a spiral theory of learning abstract algebra are investigated. Moreover, the various levels and sublevels in the learning of some school algebra as well as some university level abstract algebra are illustrated by means of examples.

Chapter 6 Conclusion and recommendations

In the final chapter the relevant points of previous chapters are summarised. Conclusions are drawn from the results of the research and recommendations are made based on the findings.

1.4 Analysis of the problem

Several people have written about how the historical emergence of group theory might typify possible levels of learning abstract algebra. Freudenthal (1973:123), referring to group theory, observed that “History moved according to these levels.” However, it has been found that group theory has application in unexpected areas. Huetinck (1996:344) has remarked how it is almost as though nature has read books on group theory. This suggests that the genetic approach may not always need to be followed slavishly. For example, Michael de Villiers, in his correspondence with the writer in 2004, observed that Shannon only found the applicability of Boolean algebra to switching circuits long after Boole had formulated the theory. Nevertheless, beginning with switching circuits and engaging in modelling could very successfully lead to a workable model and, ultimately, systemisation into an axiomatic-deductive form. Here a learning level path is followed that corresponds to how some mathematics has historically developed via mathematical modelling but it is not the actual historical one. Tall (1991) advocates that following the way in which a topic developed historically provides a useful guideline but does not necessarily always provide the most effective approach. Nevertheless, the theoretical distinction of learning levels on the basis of a historical analysis seems to still form a relevant part of the development of understanding. Piaget and Garcia (1989:8) remark that "… each stage or period begins with a reorganisation of what it has inherited from preceding ones."

After Viete (1540-1603), Piaget and Garcia (1989) noted how the study of algebra was restricted to equations where each problem required its own particular solution. This they characterised as the intraoperational period. In the second half of the eighteenth century, the new instrument of infinitesimal calculus created by Leibniz and Newton led mathematicians such as Euler, Lagrange and Gauss to a new level of development in algebra. Problems of greater generality, including the fundamental theory of algebra, were formulated and for a long time transformations dominated this period, which could be called the interoperational period of algebra. Lagrange analysed certain functions of the roots of equations and Ruffini followed up Lagrange's ideas in trying to show the impossibility of finding a solution by radicals to a general equation of degree five. Cauchy thought of functions in a more general way, calling the order of the letters representing quantities permutations. Galois inaugurated the next stage, known as the transoperational period, when he introduced the concept of a group. He lived from 1811 to 1832 and on the eve of the duel that led to his early death, he ended his letter to his friend Augusta Chevalier, with the following words:

> *Ask Jacobi or Gauss publicly to give their opinion, not as to the truth, but as to the importance of these theorems. Later there will be, I hope, some people who will find it to their advantage to decipher all this mess* (Stewart 1998:xxiii;xxiv).

Galois' work was so important that it led to the end of the period of equations and their solutions, and the beginning of a new stage in which structures predominated.

Although both Abel and Galois had explicated the concept of a group, they did not go beyond the interoperational stage of a field. Dedekind was the one who reached the transoperational stage in fields when he identified and thematised the structure of algebraic fields. This example of a transoperational level topic such as group theory also being an interoperational level regarding the topic of fields suggests the notion of sublevels arising at different levels of learning. For example, the transoperational level could be divided into three sub-levels known as the trans-intra, trans-inter and trans-trans operational stages. Piaget and Garcia refer to "… the generality of this triplet, intra; inter; and trans, and its occurrence at all sublevels." (Piaget & Garcia 1989:29). This implies the relevance of considering sublevels at each thinking level in the learning of topics such as group theory and fields and would help students to build up mental models of the topics they study.

Gentner and Stevens (1983:7) refer to mental models as "naturally evolving models" and regards the following few points as being relevant in the consideration of mental models: the target system (or the learner); the conceptual model invented by teachers or lecturers to represent the concept in an appropriate manner; the user's mental model of the concept and the scientist's conceptualisation of the mental model. As people interact with the concept, they continue to modify their mental models and it is important that these are built up properly as students' views of tasks set for them

"… depend heavily on the conceptualisations they bring to the task" while "…. these models provide predictive and explanatory power for understanding the interaction" (Gentner & Stevens 1983:7).

Hans Freudenthal (1905-1990) strongly favours methods of re-invention where he describes "… the teaching method that is built on interpreting and analysing mathematics as an activity the method of re-invention." (Freudenthal 1973:120). However, he criticises the way the idea of re-invention is often carried out in a rather superficial manner. Initial examples should be sufficiently compelling to cause the students to re-invent the concept at hand. He mentions three main levels in the acquisition of the concept of mathematical induction. On the lowest of the levels under consideration complete induction is acted out. On the next level it is made conscious as an organizing principle and can become a subject matter of reflection while on a higher level it is put into a linguistic pattern (Freudenthal 1973:122;123).

Freudenthal identifies the frequently occurring problems of omitting the lowest level or of passing too quickly from one level to the next. These steps are very important in the historical development of mathematics and, as in the teaching of mathematical induction, Freudenthal identifies three significant stages in the emergence of groups:

… in the first half of the 19th century instinctive operations with groups, in some cases up to a high degree of consciousness of principles, then the explicit formulation of the relevant group properties, and finally the axiomatic abstraction

(Freudenthal 1973:123).

Webster Family Encyclopaedia Volume 1 (1988:216) describes an axiom in the following manner:

An assumption or principle used to prove a theorem that is itself accepted as true without proof. Some mathematicians reserve the term for an assumption in logic, using postulate for an assumption made in other fields.

Freudenthal warns of the danger that when a system of knowledge is in its completed state or axiomatic form, it might appear that systemized knowledge could be reduced to merely a series of statements.

Piaget and Garcia (1989) mention the two processes of comparative tools and transformational operations involved in the setting up of instruments to be used in mathematics. However, students need to study and become aware of these tools and see how they are elaborated to form axioms within a system. Freudenthal (1973:134) remarks that "If, however, axiomatics and formalism are taught at all, axiomatizing and formalizing cannot be passed by". Sawyer (1959:1) observes that unless students know why certain axioms are stated, how the axioms are chosen, why certain axioms are chosen rather than others and what purpose they serve then they are left feeling frustrated. Since the framework is lacking, efficiency of thought is affected and "…. Students do not know where the subject fits in, and this has a paralysing effect on the mind" (Sawyer 1959:2). This is relevant in a group or a field in mathematics as they are defined through their axioms. Mariotti and Fischbein (1997:220) consider the two main types of definitions used in mathematics to be those which introduce the basic objects of a theory and those which introduce a new element within the theory itself.

Definitions form part of the third level of van Hiele's learning theory. In 1959 Pierre and Dina van Hiele analysed the problem of teaching and learning geometry and their original model consisted of five levels, which Holmes (1998:332) described in the following manner:

Level 0: Visualisation: Visual definitions are possible. Here figures may be recognised but properties not identified;

Level 1: Analysis: shapes are characterised by their properties and geometric figures recognised. Uneconomical definitions are possible.

Level 2: Informal deduction: Formal definitions are understood and "if then" thinking is possible although proofs are not yet understood.

Level 3: Formal deduction: theorems can be established within axiomatic systems.

Level 4: Rigour: students can reason formally about mathematical systems. This is an extremely abstract level which is necessary in group theory.

As far as the establishment of definitions is concerned, Michael de Villiers (1990) has pointed out that, on the basis of empirical research on van Hiele, children's understanding progresses from the informal to the formal in the following manner: Visual definitions (resulting from a drawing or physical appearance); correct but uneconomical definitions which contain superfluous properties; correct, economical definitions involving understanding of necessary and sufficient conditions.

According to van Hiele's theory, it is at the fourth level that proofs can be constructed and understood. Birchman's translation of the Russian manuscript by Orevkov (1993:2) holds that:

> *In proof theory one usually supplies applications of rules and axioms with analysis and a sequence of analysis of axioms and inference rules in an axiomatised theory is called a deduction schema. A deduction schema can be considered as an economical and rather convenient code of a proof.*

Patrick Suppes (1957:128) points out how a formal proof in logic requires a rule for everything whereas in ordinary mathematics an informal proof states enough about the argument so that anyone who knows the subject could easily follow the line of thought. He gives all the axioms of a field and then asks the reader to note that each one may be recognised as a familiar truth in arithmetic. De Villiers (1990:1) claims that "…traditionally the function of proof has been seen almost exclusively in terms of the verification (conviction or justification) of mathematical statements". However, he points out that rather than proof being a prerequisite for conviction, it is conviction that is in fact a prerequisite for proof. Judith Segal (2000:193) mentions the private and public aspects of proof while Fischbein (Segal 2000:193) notes that normally conviction increases with each new concrete positive instance but proof does not lie within the mainstream of human behaviour. If students consider everyday reason they are likely not to consider every possible case and appreciate that just one counter-example cancels out the proposition. Nevertheless proof is very important in mathematics and needs to be understood because it is "…. an indispensable tool in the systematisation of various known results into a deductive system of axioms, definitions and theorems" (de Villiers 1990:4).

Proof also forms part of Judith Land's fourth level of the development of algebraic thought. Her four levels correspond to van Hiele's first four levels of thought. She omitted the fifth level because she felt that it was inappropriate for the teaching of functions to her students. Her first two levels were called pre-descriptive and descriptive while the last two levels were called theoretical informal and theoretical formal. These four levels were also utilised by Glenda Nixon in 2000 for teaching sequences and series to grade 12 students. At Level 0 visual representations were provided to students in order to enable them to recognise a sequence in various contexts. At the second level or Level 1, after the provision of numerous number patterns, students recognised the algebraic properties of a sequence or series, were able to state these accurately and could formulate expressions involving symbols as well as perform elementary manipulations of symbols. At the third or theoretical informal level, students were able to give concise definitions, follow a deductive argument, follow the derivation of a formula and use these formulae in various contexts to solve problems. At the fourth or the theoretical formal level students could derive formulae from formulae, solve more complex problems as well as understand and initiate a deductive proof. However, in the present study, once the third level involving definitions has been reached, it will be considered that a round of the spiral is complete. Definitions subsequently form part of the perceptual level leading towards new conceptual and abstract levels involving theorems and further applications. Furthermore, even the formulation of algebraic expressions previously associated with the second level will be regarded as forming part of the abstract level low down the spiral.

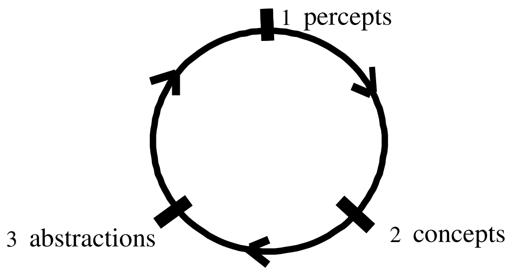
In Nixon's study of sequences and series, lessons were designed to take students through the levels as they progressed through the various topics. In addition, the role of visualisation and the parts played by the processes of patterning and generalisation were considered in the advancement from one level to the next. Proof by mathematical induction was included as a higher level activity. The use of visualisation, exploring patterns and generalisation to promote progress from one level to the next is reminiscent of the three sublevels mentioned by Piaget and Garcia (1989). Visualisation is associated with the intraoperational level at which forms are studied in isolation and become recognised. Exploring patterns encourages the interoperational stage where common properties are established. Generalisation leads to the formation of structure at the transoperational level. In this way it seems that visualisation, exploring patterns and generalisation helped to lead students through three sublevels in order to reach the appropriate subsequent levels in their learning.

The distinction between the definition of a mathematical concept and the concept image drawn by Vinner and others (Moore 1994:252) is relevant. It relates to the cognitive structure or mental picture associated with the concept in the mind of individuals. The third aspect of their concept-understanding scheme, called concept usage, refers to the way in which the concept is utilised in doing examples and proofs. Once again, these three aspects of concept understanding seem to relate to Piaget and Garcia's intra, inter and transoperational stages. However, as the spiral is traversed, concept image will be considered as forming part of the two lower levels of a particular spiral of learning, leading to definitions and subsequently new concept images. Thus established concepts may be extended or united to form new concepts at a higher level.

In this study various types of representations will be considered. The names initially given to these various phases are sensations, percepts, concepts and linguistic formulations and are those used by Johannes Heidema (Personal communication, 2004). The first type includes sensations and is of a pre-conscious nature. It is closely linked to the first conscious representation which involves percepts, resulting from the brain synthesising sensations and forming unified pictures of the images acquired. This in turn leads to concepts which may be of both a simple or a more involved nature. At the level of percepts and concepts, concept images are formed. Higher up the spiral, concept images are formed from other algebraic concepts or concept usage and may be of an abstract nature. The final level here involves linguistic representations which include definitions, axioms and theorems. This could be associated with both concept definition and concept usage. The names of the three main levels will be shortened to percepts, concepts and abstractions. The suggested levels of Freudenthal (1973:122,123) seem to bear a very close resemblance to the suggested levels of sensations and percepts followed first by concepts and then by linguistic formulation or abstractions at the third level. The three stages in the acquisition of definitions described by de Villiers (1990) involving visual, uneconomical and economical definitions also seem to be closely related to the levels of percepts, concepts and abstractions. Moreover, there are notable similarities to be found between visualisation, exploring patterns and generalisation and the levels of percepts, concepts and abstractions too. In fact, visualisation, patterning and generalisation are further names which could be used to describe the levels referred to as percepts, concepts and abstractions or Piaget and Garcia's intra, inter and transoperational levels.

Each of the levels of representations has its own dimension of complexity. Concepts lie between percepts and linguistic representations or abstractions and need to be acquired to promote understanding at the third level. Once the linguistic representation is reached in a particular topic, it may form part of the initial level of the next round of the spiral of learning.

The three main levels could then be pictured in the following way:



The spiral analogy suggests the possibility of continuous growth from third levels into new unexplored directions. This approach does not seem to be quite the same as the previously mentioned type of spiral approach mentioned by Bell (1978) which is used in the curriculum for mathematics teaching. One problem associated with the original spiral approach to the curriculum is that the same concept may keep on being repeated year after year. To make it more worthwhile, at secondary or tertiary level it could be possible to keep on dealing with the same topic while passing through a hierarchy of concepts each time. In this study once a concept or concepts have been established at an abstract level, they form the perceptual level of a new round or new rounds of the spiral and can branch out in many ways. The present type of spiral image seems to have a more three dimensional aspect to it than the one in which one continually "returns to concepts and procedures" (Breetzke 1988:1). Although these concepts may be handled with more sophistication each time, the impression does not seem to be created that old concepts keep on leading to new concepts at a higher level of the spiral. Furthermore it seems that attention is not paid to levels such as the perceptual, conceptual and abstract levels that occur at each round of the spiral.

However, Bell (1978:278) points out that

... concepts and principles can be learned by students at various stages of development provided each concept is defined and represented in a manner which is concrete and specific enough to be consistent with students' intellectual development and mathematical maturity.

This seems to suggest the importance of the perceptual, conceptual and abstract levels at each round of the spiral. Bell (1978:279) also alludes to the parallelism in the historical development of mathematics and the formation of concepts by students:

This sequential nature of concept development in mathematics is apparent in the structure and historical development of mathematics. The sequential nature of concept learning in students results from the chronological development of human mental processes.

The spiral representations involving percepts, concepts and abstractions depict articulated spiral levels or types of conceptual realisations or manifestations. Once the highest level is reached, one returns to the lowest level but further up the spiral. In the spiral theory, definitions, axioms and theorems all form part of the highest level but at different cycles of the spiral. This would be different from van Hiele's idea that definition forms part of the third level, theorems the fourth and mathematical structures the fifth level. If the historical development of the solution of algebraic equations is considered as a whole, group theory does form part of the final level. However, as different topics of group theory are studied, the three levels are passed through again and again, suggesting the idea of sublevels or different levels of a spiral.

The spiral may be viewed in many different ways and even small concepts could be identified as being built up as a cycle of the spiral. As the second level approaches the third one, one cycle of the spiral is completed and level 1 is reached once again. Thereafter very simple examples could be utilised to illustrate concepts defined at the highest level of the previous cycle. These would build up to more challenging, high-level types of examples. In addition, definitions could be utilised to build up simple and, at a later stage, more complex theorems. Thus the whole spiral could be viewed in a more macro fashion or a small part of it could be viewed in finer detail, suggesting the idea of sublevels of learning.

The three above-mentioned levels can have meaning in the various contexts: dimension of abstraction (where iconic progresses to symbolic or simple and then to complex); historical phases (in the development of concepts either in history or in the individual learner's life); learning levels (linking it to some theory of learning). In subsequent chapters the historical and conceptual stages in the emergence of abstract algebra are studied in detail and a combined spiral theory of learning levels and sublevels is formulated and applied to the development of concepts concerning various levels of algebra.

1.5 Research design

The research design is comprised of the following: A literature research regarding the stages in the historical development of abstract algebra; the theories of Piaget, Freudenthal, van Hiele, Land and Nixon as well as the concept – understanding scheme of Vinner and others; axiomatisation and proof in abstract algebra. The various theories are consolidated into one theory of learning levels and sublevels which is appropriate for and applied to the formation of concepts of various levels of algebra culminating in abstract algebra. Since learning and teaching could be regarded as two sides of the same coin, the implications of the learning theory for teaching algebraic topics is taken into consideration.

1.6 Layout of the dissertation

Chapter 1: Introduction and overview

Here the motivation for this dissertation is presented, the problem statement given, aims and objectives outlined, the problem analysed, research design mentioned and layout provided.

Chapter 2: The perceptual level or initial learning level and sublevels in the development of abstract algebra

In this chapter the initial stage of the emergence of abstract algebra in history is studied. The first stages in the learning theories of Piaget, Freudenthal, van Hiele, Land and Nixon, as well as concept image and concept usage are considered. The intra, inter and transoperational stages of Piaget are specified and the various sublevels involved at level one are also investigated. Furthermore, the idea of a spiral is utilised and the idea of sensations and percepts (abbreviated as percepts), concepts and linguistic formation (which are called abstractions) are included as well.

Chapter 3: The conceptual level or the intermediate learning level and sublevels in the development of abstract algebra

The intermediate stage of the development of abstract algebra in history is explored. The intermediate stages in the learning theories of Piaget, Freudenthal, van Hiele, Land, Nixon as well as further implications of concept image and progress towards definitions are researched. The various sublevels involved at the intermediate level are observed. In addition, the idea of a spiral of learning involving the intermediate phase of concept formation is incorporated.

Chapter 4: The abstract level or the third learning level in the development of abstract algebra

The developments of abstract algebra as well as the advanced level in learning algebra are researched. In addition, the advanced stage in the theories of Piaget, Freudenthal, van Hiele, Land and Nixon as well as concept definition are considered. The various sublevels arising at the advanced level are studied. Once again the spiral illustration is used where the third linguistic level termed abstractions is highlighted. Axiomatisation and the construction of proofs in abstract algebra are investigated. Both the “a posteriori” or descriptive as well as the “a priori” or constructive approaches are considered.

Chapter 5: A spiral theory of learning algebra

The various learning theories are consolidated into a single theory of learning levels and sublevels in abstract algebra and algebra in general. The spiral level approach plays a significant role in this regard. This is illustrated by historical references and application to methods of learning various levels of algebraic topics.

Chapter 6: Conclusion and recommendations

The main points of findings are summarised, conclusions are drawn from the results of the investigation and recommendations are made for future application and research.