

**CREATING AND LEARNING ABSTRACT ALGEBRA: HISTORICAL PHASES AND  
CONCEPTUAL LEVELS**

by

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## Summary

Piaget observed that various stages involved in the construction of different forms of knowledge are sequential and that the same sequential order is evident in history. There seem to be three main stages in the development of algebra involving the independent and general solution of equations followed by the evolution of abstract algebra. Piaget referred to these as the intra, inter and transoperational stages but they are termed the levels of percepts, concepts and abstractions here. The perceptual level involves isolated forms, the conceptual level concerns correspondences and transformations amongst forms whilst the abstract level is characterised by the evolution of structures of forms.

Historically the overall perceptual level of abstract algebra lasted from antiquity to the middle of the eighteenth century. The conceptual level followed, lasting for approximately one century and the subsequent abstract level has prevailed from the middle of the nineteenth century onwards. Each of these levels involve numerous sublevels but instead of being continually broken down into more and more sublevels, in this study a spiral of learning is being considered. Each round of the spiral contains a perceptual, conceptual and abstract level. The way in which perceptual levels can arise from previous abstract levels gives an indication of how knowledge is reorganised and expanded in new unexplored directions as the spiral is climbed. The important aspects of proof and axiomatisation are also addressed here.

The historical emergence of abstract algebra reveals a significant pattern concerning the development of mathematics. The levels of thinking involved are important and reveal a general trend of algebraic thought. Hence careful consideration needs to be paid to the revelations arising from historical investigations so that these may help contribute to the encouragement of learning in students of algebra. The idea of levels of learning has been substantiated by many researchers and investigations undertaken in the past. The main characteristics of the three relevant levels and sublevels as well as insights gained from the historical emergence of algebra are being united here to form a comprehensive theory of learning algebra at both the secondary and tertiary levels of study.

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I declare that

**CREATING AND LEARNING ABSTRACT ALGEBRA: HISTORICAL PHASES AND CONCEPTUAL LEVELS** is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

.....  
(Mrs E.G.Nixon)

.....  
Date

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I wish to express my gratitude to:

*Jesus Christ the same yesterday,  
and today, and forever (Hebrews 13: 8)*

*I will love thee, O Lord, my strength.  
The Lord is my rock, and my fortress,  
and my deliverer; my God, my strength,  
in whom I will trust; my buckler,  
and the horn of my salvation,  
and my high tower.  
I will call upon the Lord,  
who is worthy to be praised. (Psalm 18: 1,2,3)*

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*Auctus tortuosus algebrae est mirificus et aeternus.*

The spiral growth of algebra is wonderful and eternal.

## Table of Contents.

<b>Chapter 1</b>	<b>Introduction and Overview</b>	<b>1</b>
1.1	Background to the study	1
1.2	Problem Statement	6
1.3	Aims and Objectives	6
1.3.1	Aims	6
1.3.2	Objectives	6
1.4	Analysis of the problem	8
1.5	Research design	17
1.6	Layout of the dissertation	17
<b>Chapter 2</b>	<b>The perceptual level or initial learning level and sublevels in the development of abstract algebra</b>	<b>19</b>
2.1	Introduction	19
2.2	Intra, inter and transoperational stages of Piaget	23
2.3	The emergence of the perceptual level in the history of abstract algebra	25
2.3.1	Introduction	25
2.3.2	The intraoperational or perceptual level in history	26
2.3.3	Sublevels of the intraoperational level in the history of algebra	36
2.3.4	Conclusion	39
2.4	The first level of various theories of stages of learning mathematics	40
2.4.1	Jean Piaget's initial level of thought	41
(i)	Introduction	41
(ii)	Piaget's original first level of development	41
(iii)	Piaget's reference to groups	43
(iv)	Mathematics at Piaget's original first level of thought (pre-operational stage)	44
(v)	The intraoperational level or Piaget's later initial learning level	47

(vi)	Sublevels of the first thinking level	49
(vii)	Freudenthal's criticism of Piaget's theories	50
(viii)	Van Hiele's criticism of Piaget	51
(ix)	Conclusion	53
2.4.2	Hans Freudenthal	55
(i)	Introduction	55
(ii)	Freudenthal's first level of learning	56
(iii)	Sublevels of Freudenthal's first level of thought	57
(iv)	Conclusion	58
2.4.3	Pierre and Dina van Hiele	59
(i)	Introduction	59
(ii)	Van Hiele's first level of thought	61
(iii)	Van Hiele's first level of thought in algebra	63
(iv)	Sublevels of the van Hiele levels	64
(v)	Some opinions expressed regarding the van Hiele's levels	65
(vi)	Conclusion	67
2.4.4	Judith Land	67
(i)	Introduction	67
(ii)	The first algebraic level of Land's theory	68
(iii)	Sublevels of Land's levels	69
(iv)	Conclusion	69
2.4.5	Glenda Nixon	70
(i)	Introduction	70
(ii)	The initial level	70
(iii)	Sublevels	71
(iv)	Visualisation	71
(v)	Exploring patterns	74

(vi)	Generalisations	74
(vii)	Conclusion	76
2.4.6	Concept understanding scheme	77
(i)	Introduction	77
(ii)	Concept image and usage	78
(iii)	The relevance of concept images for the formation of concepts, definitions and proof	79
(iv)	Concept images and usage in abstract algebra	79
(v)	Sublevels of the perceptual level involving concept images and usage	81
(vi)	Conclusion	82
2.5	Chapter Consolidation	83
<b>Chapter 3</b>	<b>The conceptual level or the intermediate thinking level and sublevels in the development of abstract algebra</b>	<b>86</b>
3.1	Introduction	86
3.2	The interoperational or conceptual level in the history of abstract algebra	87
3.2.1	Introduction	87
3.2.2	The historical emergence of the interoperational or conceptual level in abstract algebra	88
3.2.3	Sublevels of the interoperational level or rounds of the spiral at the conceptual level	98
3.2.4	Conclusion	99
3.3	Various theories of intermediate levels of learning	100
3.3.1	Piaget's intermediate level of thought	100
(i)	Introduction	100
(ii)	Mathematics at Piaget's original intermediate level of thought	101
(iv)	Piaget's later intermediate learning level termed the interoperational level	102
(v)	Sublevels of the interoperational level or rounds of the spiral at the concept level	103
(v)	Conclusion	104

3.3.2	Freudenthal's intermediate level of thought	104
(i)	Introduction	104
(ii)	Freudenthal's intermediate level of learning	105
(iii)	Freudenthal's intermediate level of learning in algebra	107
(iv)	Sublevels of Freudenthal's intermediate or conceptual level	108
(v)	Conclusion	109
3.3.3	Van Hiele's intermediate level of thought	110
(i)	Introduction	110
(ii)	Van Hiele's second level of learning	110
(iii)	Van Hiele's second level of thought in algebra	111
(iv)	Sublevels of the van Hiele levels	112
(v)	Conclusion	113
3.3.4	Land	113
(i)	Introduction	113
(ii)	The second or conceptual level of Land's theory	113
(iii)	Sublevels of Land's levels	114
(iv)	Conclusion	114
3.3.5	Nixon	115
(i)	Introduction	115
(ii)	The intermediate or conceptual level	115
(iii)	Sublevels	116
(iv)	Conclusion	117
3.4	Concept understanding scheme	117
3.4.1	Introduction	117
3.4.2	Concept image and concept usage	117
3.4.3	Sublevels of the conceptual level involving concept images and concept usage	119
3.4.4	Conclusion	121



3.5	Chapter consolidation	121
<b>Chapter 4</b>	<b>The abstract level or the third learning level in the development of abstract algebra</b>	<b>123</b>
4.1	Introduction	123
4.2	The emergence of the abstract level in the history of abstract algebra	124
4.2.1	Introduction	124
4.2.2	The transoperational or abstract level of algebra in history	125
4.2.3	Sublevels of the transoperational or abstract level in history	137
4.2.4	Conclusion	139
4.3	Axiomatisation	139
4.4	Proof	145
4.5	The third or final level of various theories of thinking levels	150
4.5.1	Piaget's original third level of development	150
(i)	Piaget's original third level of development	150
(ii)	Mathematics associated with Piaget's original third level of development	150
4.5.2	Freudenthal's third level of thinking	154
(i)	Characteristics of this level	154
(ii)	Mathematics at Freudenthal's third thinking level	154
4.5.3	Van Hiele's upper level (levels) of learning	156
(i)	Van Hiele's original upper level involving informal deduction, formal deduction and rigour	156
(ii)	Van Hiele's later third level or theoretical level	157
4.5.4	Land's third or highest level of learning	158
4.5.5	Nixon's third or highest level of learning	158
4.5.6	Concept definition and concept usage	159
4.5.7	Conclusion	161

<b>Chapter 5</b>	<b>A spiral theory of learning algebra</b>	<b>163</b>
5.1	Introduction	163
5.2	The parallelism between the historical development of mathematics and some stage theories of learning	164
5.3	The characteristics associated with the perceptual, conceptual and abstract levels	168
5.3.1	Introduction	168
5.3.2	The perceptual level	169
5.3.3	The conceptual level	173
5.3.4	The abstract level	175
5.3.5	Conclusion	176
5.4	The establishment of an integrated theory of learning algebra	177
5.4.1	The link between historical and epistemological levels of learning	177
5.4.2	The perceptual, conceptual and abstract levels	177
5.4.3	The spiral theory	178
5.4.4	Analysing the spiral of learning in the presentation of a topic	179
5.4.5	The “a posteriori” or “a priori” development of mathematics approaches	181
5.4.6	The importance of not omitting stages of rounds or rounds of a spiral	181
5.4.7	The provision of opportunities of visualisation, exploring patterns and generalisation	182
5.5	Illustrations of spirals involved in various algebraic topics	184
5.5.1	Introductory algebra	184
5.5.2	Number systems	185
5.5.3	The solution of cubic equations	186
5.5.4	Introduction to group theory	188
5.5.5	Lattice diagrams in group theory	190
5.5.6	The second homomorphism theorem for groups	192
5.6	Conclusion	194

<b>Chapter 6 Conclusion and recommendations</b>	<b>196</b>
6.1 Introduction	196
6.2 The perceptual levels and sublevels of the spiral	196
6.2.1 The perceptual level in the history of algebra	196
6.2.2 The perceptual level of learning algebra	197
6.2.3 The parallelism between the perceptual levels of the historical and learning spirals of abstract algebra	198
6.3 The conceptual levels and sublevels of the spiral	199
6.3.1 The conceptual levels in the history of algebra	199
6.3.2 The conceptual level of learning algebra	200
6.3.3 The parallelism between the conceptual levels of the historical and learning spirals of abstract algebra	201
6.4 The abstract levels and sublevels of the spiral	202
6.4.1 The abstract levels in the history of algebra	202
6.4.2 The abstract level of learning algebra	203
6.4.3 The parallelism between the abstract levels of the historical and learning spirals of abstract algebra	204
6.5 The implications for a comprehensive learning theory of algebra at all levels	204
6.5.1 The link between historical and epistemological levels of learning	205
6.5.2 Taking the perceptual, conceptual and abstract levels into consideration	205
6.5.3 The concept of an ever-growing spiral of levels	205
6.5.4 Analysing the spiral of levels of learning in the presentation of a topic	206
6.5.5 The “a posteriori” or “a priori” development of mathematics approaches	206
6.5.6 The importance of not omitting stages of rounds or rounds of a spiral	206
6.5.7 The provision of opportunities of visualisation, exploring patterns and generalisation	207
6.6 Recommendations	207
<b>Bibliography</b>	<b>209</b>

## LIST OF ILLUSTRATIONS

### CHAPTER 1

Spiral Analogy	15
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### CHAPTER 2

Visual representation of $x^2 + px = q$ (Smith 1958b:446)	32
Equivalence classes of $b$ and $b'$ (Nardi 2000:175)	72
Alternative picture of equivalence classes of $b$ and $b'$ (Nardi 2000:176)	72
Visual depiction of isomorphic groups (Skemp (1971:82)	73
Possible orientations of an equilateral triangle (Huetinck 1996:342)	80
Group table for $D_4$ (Fraleigh 1977:41)	82
Lattice diagram for $D_4$ (Fraleigh 1977:42)	82

### CHAPTER 3

Mathematical conceptualisation (Sierpinska & Kilpatrick 1988:515)	122
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### CHAPTER 4

Galois' doodles (Stewart:1998 xxii)	128
Galois' final page (Stewart:1998 xxiv)	128
Group table for $S_4$	130
Group table for subgroup $H$	131
"A priori" process of axiomatisation (de Villiers 1986:5)	143
"A posteriori" process of axiomatisation. (de Villiers 1986:6)	144
Odd + odd = even example (de Villiers 2005)	149
Concept definition to concept image (Vinner in Tall 1991: 71)	160
Interplay between concept definition and concept image (Vinner in Tall 1991: 70)	160
Deduction leading to intuitive thought (Vinner in Tall 1991: 72)	160
Omitting the abstract level of thought (Vinner in Tall 1991:73)	160
Adaptation of diagrams presented by Vinner in Tall to represent spiral of learning	161

**CHAPTER 5**

Development of the concept of number (Sfard 1991:13)	168
Hierarchy design (Bell, Costello & Küchemann 1983:177)	169
Why they fall and why they don't (Fleming & Varberg 1989:429)	171
Number line representations	171 & 172
Complex numbers in Cartesian plane	172
Exploring patterns – Pentagons (Reid in Nixon 2002:LXXVII)	174
Structure of a book (Stewart 1998:xii)	180
Spiral diagram derived from basic diagram of Vinner in Tall (1991:171)	180
Table of “a posteriori” axiomatisation	181
Table of “a priori” axiomatisation	181
Deduction lead to intuitive thought (Vinner in Tall 1991: 72)	182
Subdivision of Complex numbers	186
Possible orientations of equilateral triangle (Huetinck 1996:342)	188
It's A Snap diagrams (Huetinck 1996: 343)	189
It's A Snap result table (Huetinck 1996:344)	190
Group Table revealing $\mathbb{Z}_6$ under addition	190
Group of symmetries of a square	191
Group table for $D_4$ (Fraleigh 1977:41)	191
Lattice diagram for $D_4$ (Fraleigh 1977:42)	192
Visualising the second homomorphism theorem for groups (Heidema 2004:29)	192